PARTIAL ORDERINGS ON PERMUTATIONS

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Results concerning partial orderings defined on permutations and their applications in statistics are surveyed.

1. Introduction. There are a variety of partial orderings on S_n, the set of all permutations of $\{1, \ldots, n\}$, which have applications in statistics. These orderings have been considered among others by Sobel (1955), Savage (1957), Lehmann (1966), Yanagimoto and Okamoto (1969), Hăjek (1969), Hollander, Proschan, and Sethuraman (1977), hereafter referred to as HPS, and Schriever (1985), (1987a), (1987b). These partial orderings have been defined in various ways with different names and notations. They have been used to define monotone functions on S_n, and the monotonicity properties of such functions have been used in many areas including stochastic comparisons of rank order statistics. For example, using a particular ordering HPS introduced an important class of nondecreasing functions, called decreasing in transposition (DT) functions, and discussed their importance in statistics. Recently, Block, Chhetry, Fang, and Sampson (1990), hereafter referred to as BCFS(a), considered three well-known and one new partial orderings on S_n through a unified approach and characterized them by four important positive dependence orderings on bivariate empirical rank distributions.

In addition to partial orderings the notion of metrics on $S_n \times S_n$ plays an important role in many areas. The metrics arise naturally in a variety of situations, such as in the analysis of sorting algorithms (Knuth (1973)), in measuring association in bivariate rank data (Kendall (1970) and Diaconis and Graham (1977)), and in the study of ranking models (Fligner and Verducci (1986)). Recently, Block, Chhetry, Fang, and Sampson (1987) considered in a unified fashion three well-known and two new metrics on $S_n \times S_n$, and discussed their applications.

The main objective of this paper is to survey important results and applications of partial orderings defined over permutations. In Section 2 we review basic results concerning five partial orderings. Their applications are considered in Section 3 and Section 4. Also in Section 3 we introduce five classes of nondecreasing or arrangement increasing functions on S_n and discuss their basic implications.

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Throughout this paper, permutations on S_n are denoted by lower case bold face letters, such as $\mathbf{i}, \mathbf{j}, \mathbf{r}, \mathbf{s}$, etc., and their components are denoted by the corresponding lower case letters. The permutations $(1, \ldots, n)$ and $(n, \ldots, 1)$ are denoted by $\mathbf{e}$ and $\mathbf{e}^*$, respectively. The composition or product of $\mathbf{r} = (r(1), \ldots, r(n))$ and $\mathbf{s} = (s(1), \cdots, s(n))$ is denoted by $\mathbf{r} \cdot \mathbf{s}$, where $\mathbf{r} \cdot \mathbf{s} = (r(s(1)), \cdots, r(s(n)))$. The inverse of $\mathbf{r}$ is denoted by $\mathbf{r}^{-1}$, where $\mathbf{r}^{-1} = (r^{-1}(1), \ldots, r^{-1}(n))$. An inversion of $\mathbf{r} = (r(1), \cdots, r(n))$ is a pair $(r(k), r(\ell))$ such that $(k - \ell)(r(k) - r(\ell)) < 0$.

2. Partial Orderings On S_n: In order to define several partial orderings on S_n through a unified approach, BCFS(a) considered the following definition.

DEFINITION 2.1. An interchange of two components $i(k)$ and $i(\ell)$ of $\mathbf{i}$ is said to be: (1) a correction of type 1 inversion if $\Delta_{k\ell} < 0$; (2) a correction of type 2 inversion if $\Delta_{k\ell} < 0$ and $|i(k) - i(\ell)| = 1$; (3) a correction of type 3 inversion if $\Delta_{k\ell} < 0$ and $|k - \ell| = 1$, where $\Delta_{k\ell} = (k - \ell)(i(k) - i(\ell))$.

Definition 2.1 is used to define three well-known ordering relations for permutations as follows.

DEFINITION 2.2. For $t = 1, 2, 3$ a permutation $\mathbf{i}$ is said to be better ordered in the sense of ordering b_t than $\mathbf{j}$, denoted by $\mathbf{i} \overset{b_t}{\geq} \mathbf{j}$, if $\mathbf{i} = \mathbf{j}$ or if $\mathbf{i}$ is obtainable from $\mathbf{j}$ in a number of steps each of which consists of correcting a type t inversion.

The following ordering relation is described by BCFS(a).

DEFINITION 2.3. A permutation $\mathbf{i}$ is said to be better ordered in the sense of ordering b_4 than $\mathbf{j}$, denoted by $\mathbf{i} \overset{b_4}{\geq} \mathbf{j}$, if $\mathbf{i} = \mathbf{j}$ or if $\mathbf{i}$ is obtainable from $\mathbf{j}$ in a number of steps each of which consists of correcting a type 2 or type 3 inversion.

One more ordering relation whose application has been considered by Savage (1957) is as follows.

DEFINITION 2.4. A permutation $\mathbf{i}$ is said to be better ordered in the sense of ordering b_0 than $\mathbf{j}$, denoted by $\mathbf{i} \overset{b_0}{\geq} \mathbf{j}$, if $\sum_{k=1}^{m} i(k) \leq \sum_{k=1}^{m} j(k)$ for all $m = 1, \ldots, n$.

Each of the above ordering relations is a partial ordering on S_n, in the sense that they are reflexive, transitive, and anti-symmetric. The implications among these orderings are as follows:

$$\begin{array}{l} \mathbf{i} \overset{b_2}{\geq} \mathbf{j} \;\searrow \\ \qquad\qquad \mathbf{i} \overset{b_4}{\geq} \mathbf{j} \Rightarrow \mathbf{i} \overset{b_1}{\geq} \mathbf{j} \Rightarrow \mathbf{i} \overset{b_0}{\geq} \mathbf{j}. \\ \mathbf{i} \overset{b_3}{\geq} \mathbf{j} \;\nearrow \end{array}$$

The above implications are strict (BCFS(a) and Savage (1957)), and the ordering b_2 neither implies nor is implied by the ordering b_3 (Lehmann (1966)).

For theoretical as well as practical purposes it is natural to seek one-to-one functions f: $S_n \to S_n$ such that the order relation between $f(\mathbf{i})$ and $f(\mathbf{j})$ would be easily demonstrable once we know the order relation between $\mathbf{i}$ and $\mathbf{j}$. The results for two such functions are as follows.

THEOREM 2.5. *Let* $\mathbf{i}$ *and* $\mathbf{j}$ *be two permutations in* S_n. *Then*

(a) $\mathbf{i} \overset{b_1}{\geq} \mathbf{j} \Longleftrightarrow \mathbf{i}^{-1} \overset{b_1}{\geq} \mathbf{j}^{-1}$

(b) $\mathbf{i} \overset{b_2}{\geq} \mathbf{j} \Longleftrightarrow \mathbf{i}^{-1} \overset{b_3}{\geq} \mathbf{j}^{-1}$

(c) $\mathbf{i} \overset{b_4}{\geq} \mathbf{j} \Longleftrightarrow \mathbf{i}^{-1} \overset{b_4}{\geq} \mathbf{j}^{-1}$

(d) $\mathbf{i} \overset{b_t}{\geq} \mathbf{j} \Longleftrightarrow \bar{\mathbf{i}} \overset{b_t}{\geq} \bar{\mathbf{j}}$, *for* $t = 0, 1, 2, 3, 4$,

where for an arbitrary permutation $\mathbf{r} \in S_n$, $\bar{\mathbf{r}} = \mathbf{e}^* \cdot \mathbf{r} \cdot \mathbf{e}^*$ *and* $\bar{\mathbf{r}}$ *is known as the complement of* $\mathbf{r}$.

Note that $\mathbf{i} \overset{b_0}{\geq} \mathbf{j}$ does not necessarily imply that $\mathbf{i}^{-1} \overset{b_0}{\geq} \mathbf{j}^{-1}$. To see this choose $\mathbf{i} = (1432)$ and $\mathbf{j} = (3241)$. Consequently the comparison $\mathbf{i}^{-1} \overset{b_0}{\geq} \mathbf{j}^{-1}$ leads to an ordering different than $\mathbf{i} \overset{b_0}{\geq} \mathbf{j}$. One might label this companion ordering $\mathbf{i} \overset{b_0'}{\geq} \mathbf{j}$.

In the remainder of this section we state several theorems concerning the formulations of the above five orderings. These formulations are easier to handle than the original definitions. The following theorem characterizes the b_0 ordering.

THEOREM 2.6. $\mathbf{i} \overset{b_0}{\geq} \mathbf{j}$ *if and only if* $\sum_{k=1}^n a_k i(k) \geq \sum_{k=1}^n a_k j(k)$ *for every choice of* $a_1, \ldots, a_n$ *provided* $a_1 \leq \cdots \leq a_n$.

The following theorem which characterizes the b_1-ordering is stated in Yanagimoto and Okamoto (1969). A simple intuitive proof is given in Metry and Sampson (1988a).

THEOREM 2.7. *For any positive integer* $m \leq n$, *let* $i(1,m) < \cdots < i(m,m)$ *be the increasing rearrangement of the last* m *components of* $\mathbf{i}$, *and* $j(1,m) < \cdots < j(m,m)$ *be the increasing rearrangement of the last* m *components of* $\mathbf{j}$. *Then* $\mathbf{i} \overset{b_1}{\geq} \mathbf{j}$ *if and only if* $i(k,m) \geq j(k,m)$ *for any* k *and* m *such that* $1 \leq k \leq m \leq n$.

The following theorem is due to Hăjek (1969).

THEOREM 2.8. $\mathbf{i} \overset{b_2}{\geq} \mathbf{j}$ *if and only if the following holds*

$$k < \ell,\ j(k) < j(\ell) \Rightarrow i(k) < i(\ell).$$

The following two theorems are due to BCFS(a).

THEOREM 2.9. $\mathbf{i} \overset{b_3}{\geq} \mathbf{j}$ *if and only if the following holds*

$$k < \ell,\ j(k) < j(\ell) \Rightarrow i^{-1}(j(k)) < i^{-1}(j(\ell))$$

$$i^{-1}(j(k)) < i^{-1}(j(\ell)),\ j(k) > j(\ell) \Rightarrow k < \ell.$$

THEOREM 2.10. $\mathbf{i} \overset{b_4}{\geq} \mathbf{j}$ *if and only if there exists a permutation* $\mathbf{r}$ *such that*

$$k < \ell,\ j(k) < j(\ell) \Rightarrow r(k) < r(\ell),\ i(r(k)) < i(r(\ell)),$$

$$r(k) < r(\ell),\ i(r(k)) > i(r(\ell)) \Rightarrow k < \ell,\ j(k) > j(\ell).$$

An important consequence of Theorem 2.10 is as follows: $\mathbf{i} \overset{b_4}{\geq} \mathbf{j}$ if and only if there exists a permutation $\mathbf{t}$ such that $\mathbf{i} \overset{b_2}{\geq} \mathbf{t}$ and $\mathbf{t} \overset{b_3}{\geq} \mathbf{j}$ or $\mathbf{i} \overset{b_3}{\geq} \mathbf{t}$ and $\mathbf{t} \overset{b_2}{\geq} \mathbf{j}$.

3. Partial Orderings and Monotone Functions. Throughout the paper $x_1, \ldots, x_n$ denote n distinct observations and $x_{(1)} < \cdots < x_{(n)}$ denote the increasing rearrangement of $x_1, \ldots, x_n$. Also, $\mathbf{y} \cdot \mathbf{r}$ denotes the vector $(y_{r(1)}, \ldots, y_{r(n)})$ in R^n, for all $\mathbf{y} \in R^n$ and $\mathbf{r} \in S_n$. Unless otherwise stated we adopt the following definition of rank orders.

DEFINITION 3.1. The rank order of $x_1, \ldots, x_n$ is the permutation $\mathbf{r}$ in S_n such that

$$(x_{(1)}, \ldots, x_{(n)}) \cdot \mathbf{r} = (x_1, \ldots, x_n).$$

Another definition of rank order mentioned in Savage (1957) is as follows.

DEFINITION 3.2. The rank order of $x_1, \ldots, x_n$ is the permutation $\mathbf{r}'$ in S_n such that

$$(x_1, \ldots, x_n) \cdot \mathbf{r}' = (x_{(1)}, \ldots, x_{(n)}).$$

From the above two definitions it is clear that $\mathbf{r}' = \mathbf{r}^{-1}$, and in the literature $\mathbf{r}^{-1}$ is called the antirank of the rank order $\mathbf{r}$. In view of this fact throughout this paper antiranks may be viewed as the rank orders in the sense of Definition 3.2. In the above definition when the x values are replaced by the corresponding continuous random variables we replace $\mathbf{r}(\mathbf{r}')$ by the random vector $\mathbf{R}$ ($\mathbf{R}^*$) of ranks. We now present an example of Savage (1957) to show that some results are valid only in terms of the rank orders of Definition 3.2.

EXAMPLE 3.3. Let $X_1, \ldots, X_n$ be n independent random variables such that each X_k has continuous density $g(k\lambda)h(x)e^{k\lambda x}$, where g and h are nonnegative functions and $\lambda > 0$. Assume that the possible rank orders $\mathbf{r}'$, $\mathbf{s}'$, etc. and the random vector $\mathbf{R}^*$ of ranks of $X_1, \ldots, X_n$ are defined as in the Definition 3.2. Then, Savage (1957, Theorem 3) proved that

$$\text{(1)} \qquad \mathbf{r}' \overset{b_0}{\geq} \mathbf{s}' \Rightarrow \operatorname{Prob}(\mathbf{R}^* = \mathbf{r}') \geq \operatorname{Prob}(\mathbf{R}^* = \mathbf{s}').$$

We also present an example of Savage (1957) to show that there are situations where it does not matter which definition of rank orders have been used.

EXAMPLE 3.4. Let $X_1, \ldots, X_n$ be n independent random variables such that each X_k has continuous TP_2 density $f(x; \lambda_k)$ where $\lambda_1 \leq \cdots \leq \lambda_n$. Assume that the rank orders $\mathbf{r}$, $\mathbf{s}$, etc. and the random vector $\mathbf{R}$ of ranks of $X_1, \ldots, X_n$ are defined as in Definition 3.1. Then, Savage (1957, Corollary 1.1) proved that

$$\text{(2)} \qquad \mathbf{r} \overset{b_1}{\geq} \mathbf{s} \Rightarrow \operatorname{Prob}(\mathbf{R} = \mathbf{r}) \geq \operatorname{Prob}(\mathbf{R} = \mathbf{s})$$

provided $\lambda_{r(k)}$ corresponding to these k for which $\mathbf{r}(k) \neq \mathbf{s}(k)$ are not all equal. The result in (2) remains valid even if we replace $\mathbf{r}$, $\mathbf{s}$ and $\mathbf{R}$ by $\mathbf{r}'$, $\mathbf{s}'$ and $\mathbf{R}^*$, respectively, where $\mathbf{r}'$, $\mathbf{s}'$ and $\mathbf{R}^*$ are defined in Example 3.3.

In the above examples two ordering relations of Section 2 were used to define nondecreasing functions in S_n. Such functions are desirable in statistics since their monotonicity properties may be used to establish important results concerning rank statistics. For example, Savage (1957) used the monotonicity property (1) in testing the hypothesis H_0: there exists a c.d.f. $F(x)$ such that Prob $(X_k \leq x) \equiv F_k(x) = F(x)$ for $k = 1, \ldots, n$ against $H_1 : X_k$ has density as described in Example 3.3 for $k = 1, \ldots, n$. Savage (1957, Corollary 1.1) gives a necessary criterion for a rank test of H_0 against H_1 to be admissible. For the application of the monotonicity property in (2) see Savage (1957, Corollary 1.2).

To study the properties and application of a class of monotone functions on S_n, HPS introduced the notion of DT (decreasing in transposition) functions as follows.

DEFINITION 3.5. A function $f : S_n \to R^1$ is said to be DT on S_n if $\mathbf{r} \overset{b_1}{\geq} \mathbf{s} \Rightarrow f(\mathbf{r}) \geq f(\mathbf{s})$.

The notion of DT functions on $M^n \times N^n$, where M and N are subsets of R^1, due to HPS, is as follows.

DEFINITION 3.6. Let $g : M^n \times N^n \to R^1$ be a function. Then g is said to be DT on $M^n \times N^n$ if

(a) $g(\boldsymbol{\lambda}, \mathbf{x}) = g(\boldsymbol{\lambda} \cdot \mathbf{r},\ \mathbf{x} \cdot \mathbf{r})$ for all $\boldsymbol{\lambda} \in M^n$, $\mathbf{x} \in N^m$ and $\mathbf{r} \in S_n$;

(b) $\mathbf{r} \overset{b_1}{\geq} \mathbf{s} \Rightarrow g(\boldsymbol{\lambda},\ \mathbf{x} \cdot \mathbf{r}) \geq g(\boldsymbol{\lambda},\ \mathbf{x} \cdot \mathbf{s})$ whenever $\lambda_1 \leq \cdots \leq \lambda_n$ and $x_1 \leq \cdots \leq x_n$.

HPS in Lemma 2.2 proved that the DT class of functions includes as special cases other well-known classes of functions, such as Schur convex (concave) functions, TP_2-functions, and L-superadditive functions. Also HPS (Section 3) considered the preservation properties under mixtures, compositions, products and integral transformations of DT functions, and in their Example 3.10, proved that many multivariate density functions are DT functions. One of their important results, which has direct applications in statistics, is as follows.

THEOREM 3.7. *Let $X_1, \ldots, X_n$ have joint density function $h(\boldsymbol{\lambda}, \mathbf{x})$, where h is a DT function on $R^n \times R^n$ with vector parameter $\boldsymbol{\lambda}$. Let $\mathbf{R} = (R_1, \ldots, R_n)$ be the random vector of ranks of $X_1, \ldots, X_n$ and $g(\boldsymbol{\lambda}, \mathbf{r}) = \text{Prob}_{\boldsymbol{\lambda}}(\mathbf{R} = \mathbf{r})$. Then $g(\boldsymbol{\lambda}, \mathbf{r})$ is a DT function on $R^n \times S_n$.*

The above theorem with a slight modification in the definition of rank order (see HPS) is applicable even when the underlying multivariate density is discrete.

A DT function is essentially a nondecreasing function defined on S_n when the partial ordering on the domain of the function is b_1. Besides the b_1 ordering, other orderings have been employed to define nondecreasing functions on S_n. For example, see Savage (1957, Theorem 3, or Example 3.3) for the b_0 ordering; Lehmann (1966, Sections 6 and 7); and Hăjek (1969, Section 5) for the b_2 ordering; Yanagimoto and Okamoto (1969, Section 6) and Boland, El-Neweihi and Proschan (1989) for the b_3 ordering. Some of these results will be considered in the next section.

In view of the above facts the following definition which is a simple extension of Definition 3.5, seems imperative.

DEFINITION 3.8. Let $\overset{b}{\geq}$ be a partial ordering on S_n. Then, a function $f : S_n \to R^1$ is said to be arrangement increasing with respect to b on S_n, denoted AI(b) if $\mathbf{r} \overset{b}{\geq} \mathbf{s} \Rightarrow f(\mathbf{r}) \geq f(\mathbf{s})$.

4. Partial Orderings and Dependence Orderings. Several dependence orderings for bivariate distributions were considered among others by Yanagimoto and Okamoto (1969), Cambanis, Simons and Stout (1976), Ahmed, Langberg, Léon and Proschan (1979), Tchen (1980), Karlin and Rinott (1980), Whitt (1982), Block and Sampson (1988), Schriever (1985), (1987a), Kimeldorf and Sampson (1987), and Yanagimoto (1990). We describe some of these. Let the bivariate random variable $(X^{(1)}, Y^{(1)})$ and $(X^{(2)}, Y^{(2)})$ have joint c.d.f.'s $H_1(x, y)$ and $H_2(x, y)$, respectively. Denote the corresponding pairs of marginals by $F_1(x)$, $G_1(y)$ and $F_2(x)$, $G_2(y)$. Denote the supports of $X^{(1)}$, $Y^{(1)}$, $X^{(2)}$ and $Y^{(2)}$ by $D_x^{(1)}$, $D_y^{(1)}$, $D_x^{(2)}$ and $D_y^{(2)}$, respectively. By convention, we have all the marginal distributions are strictly increasing on their corresponding supports. The definitions of our orderings of positive dependence are as follows:

DEFINITION 4.1. (Tchen (1980)): H_2 is said to be more concordant than H_1, denoted by $H_2 \overset{c}{\geq} H_1$, if

(i) $F_1(x) = F_2(x)$ and $G_1(y) = G_2(y)$ for all x, y;

(ii) $H_2(x,y) \geq H_1(x,y)$ for all x, y.

DEFINITION 4.2. (Schriever (1987a)): H_2 is said to be more associated than H_1, denoted by $H_2 \overset{a}{\geq} H_1$, if there exist functions

$$\phi_1 : D_x^{(1)} \times D_y^{(1)} \to D_x^{(2)} \text{ and } \phi_2 : D_x^{(1)} \times D_y^{(1)} \to D_y^{(2)}$$

such that for all x_1, $x_2 \in D_x^{(1)}$ and y_1, $y_2 \in D_y^{(1)}$.

(i) $x_1 \leq x_2$, $y_1 \leq y_2 \Rightarrow \phi_1(x_1, y_1) \leq \phi_1(x_2, y_2)$, $\phi_2(x_1, y_1) \leq \phi_2(x_2, y_2)$

(ii) $\phi_1(x_1, y_1) < \phi_1(x_2, y_2)$, $\phi_2(x_1, y_1) > \phi_2(x_2, y_2) \Rightarrow x_1 < x_2$, $y_1 > y_2$,

(iii) $(X^{(2)}, Y^{(2)}) \sim (\phi_1(X^{(1)}, Y^{(1)}), \phi_2(X^{(1)}, Y^{(1)}))$,
where $\sim$ means "distributed as."

DEFINITION 4.3. (Schriever (1985)): H_2 is said to be more row regression dependent than H_1, denoted by $H_2 \overset{rr}{\geq} H_1$, if there exists a function $\phi_2 : D_x^{(1)} \times D_y^{(1)} \to D_y^{(2)}$ such that

(i) $x_1 \leq x_2$, $y_1 \leq y_2 \Rightarrow \phi_2(x_1, y_1) \leq \phi_2(x_2, y_2)$;

(ii) $(X^{(2)}, Y^{(2)}) \sim (X^{(1)}, \phi_2(X^{(1)}, Y^{(1)}))$.

DEFINITION 4.4. (Schriever (1985)): H_2 is said to be more column regression dependent than H_1, denoted by $H_2 \overset{cr}{\geq} H_1$, if there exists a function $\phi_1 : D_x^{(1)} \times D_y^{(1)} \to D_x^{(2)}$ such that

(i) $x_1 \leq x_2$, $y_1 \leq y_2 \Rightarrow \phi_1(x_1, y_1) \leq \phi_1(x_2, y_2)$;

(ii) $(X^{(2)}, Y^{(2)}) \sim (\phi_1(X^{(1)}, Y^{(1)}), Y^{(1)})$.

The orderings "more concordant" and "more row regression dependent" were introduced by Yanagimoto and Okamoto (1969) under the additional assumptions that the distributions of both pairs have the same continuous marginals.

DEFINITION 4.4(A). H_2 is said to be more weakly column ordered than H_1, denoted by $H_2 \overset{w-c}{\geq} H_1$, if

(i) $F_1(x) = F_2(x)$ and $G_1(y) = G_2(y)$ for all x, y;

(ii) $E_{H_2}[\phi(X)Y] \geq E_{H_1}[\phi(X)Y]$ for all non-decreasing functions ϕ.

The notion of *more weakly row ordered*, denoted by $\overset{w-r}{\geq}$, is analogously defined. We summarize some of the basic consequences of these orderings below.

THEOREM 4.5.

(1) $H_2 \overset{c}{\geq} F_2 \cdot G_2$ if and only if $X^{(2)}$ and $Y^{(2)}$ are PQD in the sense of Lehmann (1966).

(2) $H_2 \overset{a}{\geq} F_2 \cdot G_2$ implies $X^{(2)}$ and $Y^{(2)}$ are associated.

(3) $H_2 \overset{rr}{\geq} F_2 \cdot G_2$ if and only if $Y^{(2)}$ is positively regression dependent on $X^{(2)}$ in the sense of Lehmann (1966), provided F_2 and G_2 are continuous.

(4) $H_2 \overset{cr}{\geq} F_2 \cdot G_2$ if and only if $X^{(2)}$ is positively regression dependent on $Y^{(2)}$ in the sense of Lehmann (1966), provided F_2 and G_2 are continuous.

(5)
$$\begin{tikzcd} H_2 \overset{rr}{\geq} H_1 \arrow[dr, Rightarrow] & \\ & H_2 \overset{a}{\geq} H_1. \\ H_2 \overset{cr}{\geq} H_1 \arrow[ur, Rightarrow] & \end{tikzcd}$$

(6) $H_2 \overset{a}{\geq} H_1 \Rightarrow H_2 \overset{c}{\geq} H_1$, provided $F_1(x) = F_2(x)$ and $G_1(y) = G_2(y)$ for all x, y.

(7) $H_2 \overset{c}{\geq} H_1 \Rightarrow H_2 \overset{w-r}{\geq} H_1$ and $H_2 \overset{w-c}{\geq} H_1$.

PROOF. For the proof of (1) to (4) see Schriever (1985, Proposition 4.1.2). The proof of (5) is obvious. For the proof of (6) see Schriever (1987a, Proposition 2.1). The proof of (7) is also obvious.

We adopt the following definition and notation of rank order for n bivariate observations $(X_1, Y_1), \ldots, (X_n, Y_n)$ where we assume that there are no ties among the x-values and among the y-values.

DEFINITION 4.6. Let $(j(1), \ldots, j(n))$ be the permutation such that $y_{j(1)} < \ldots < y_{j(n)}$. Then, $\mathbf{r}$ is said to be the row rank order of $(x_1, y_1), \ldots, (x_n, y_n)$ if $\mathbf{r}$ is the rank order of $x_{j(1)}, \ldots, x_{j(n)}$.

Appropriately interchanging the roles of x's and y's in Definition 4.6 one could define the column rank order, denoted by $\mathbf{s}$, of $(x_1, y_1), \ldots, (x_n, y_n)$. Note that $\mathbf{s} = \mathbf{r}^{-1}$.

Let $\hat{H}$ denote the empirical rank distribution based on $(X_1, Y_1), \ldots, (X_n, Y_n)$, i.e., $\hat{H}$ is the c.d.f. of a bivariate discrete random vector (R,S) putting mass n^{-1} at the points $(r(1), 1), \ldots, (r(n), n)$ (or equivalently $(1, s(1)), \ldots, (n, s(n))$) where $\mathbf{r}$ and $\mathbf{s}$ are the row and column rank orders of $(X_1, Y_1), \ldots, (X_n, Y_n)$. To compare, according to a positive dependence ordering, two bivariate c.d.f.'s H_1 and H_2 based upon samples of size n from each, it is natural to compare $\hat{H}_1$ and $\hat{H}_2$, where $\hat{H}_1$ and $\hat{H}_2$ are empirical rank distributions based on samples of size n from H_1 and H_2, respectively. For $k = 1, 2$, let $\mathbf{s}^{(k)}$ and $\mathbf{r}^{(k)}$ correspondingly denote the column and row rank order of the sample from H_k. With this notation, BCFS(a) proved essentially the following theorem.

THEOREM 4.7.

(1) $\hat{H}_1 \overset{c}{\geq} \hat{H}_1 \Leftrightarrow \mathbf{s}^{(2)} \overset{b_1}{\geq} \mathbf{s}^{(1)} \Leftrightarrow \mathbf{r}^{(2)} \overset{b_1}{\geq} \mathbf{r}^{(1)}$.

(2) $\hat{H}_2 \overset{rr}{\geq} \hat{H}_1 \Leftrightarrow \mathbf{s}^{(2)} \overset{b_2}{\geq} \mathbf{s}^{(1)} \Leftrightarrow \mathbf{r}^{(2)} \overset{b_3}{\geq} \mathbf{r}^{(1)}$.

(3) $\hat{H}_2 \overset{cr}{\geq} \hat{H}_1 \Leftrightarrow \mathbf{s}^{(2)} \overset{b_3}{\geq} \mathbf{s}^{(1)} \Leftrightarrow \mathbf{r}^{(2)} \overset{b_2}{\geq} \mathbf{r}^{(1)}$.

(4) $\hat{H}_2 \overset{a}{\geq} \hat{H}_1 \Leftrightarrow \mathbf{s}^{(2)} \overset{b_4}{\geq} \mathbf{s}^{(1)} \Leftrightarrow \mathbf{r}^{(2)} \overset{b_4}{\geq} \mathbf{r}^{(1)}$.

(5) $\hat{H}_2 \overset{w-c}{\geq} \hat{H}_1 \Leftrightarrow \mathbf{s}^{(2)} \overset{b_0}{\geq} \mathbf{s}^{(1)} \Leftrightarrow (\mathbf{r}^{(2)})^{-1} \overset{b_0}{\geq} (\mathbf{r}^{(1)})^{-1}$.

(6) $\hat{H}_2 \overset{w-r}{\geq} \hat{H}_1 \Leftrightarrow \mathbf{r}^{(2)} \overset{b_0}{\geq} \mathbf{r}^{(1)} \Leftrightarrow (\mathbf{s}^{(2)})^{-1} \overset{b_0}{\geq} (\mathbf{s}^{(1)})^{-1}$.

PROOF. See Theorems 3.6, 3.7, and 3.8 of BCFS(a) for the proof of parts (1) through (4). Parts (5) and (6) are obvious proofs.

BCFS(a) also proved that $\hat{H}_2 \overset{a}{\geq} \hat{H}_1$ if and only if there exist an empirical rank distribution $\hat{H}^*$ such that $\hat{H}_2 \overset{rr}{\geq} \hat{H}^* \overset{cr}{\geq} \hat{H}_1$ or there exists an empirical rank distribution $\hat{H}_*$ such that $\hat{H}_2 \overset{cr}{\geq} \hat{H}_* \overset{rr}{\geq} \hat{H}_1$.

We close this section by mentioning an important result of Yanagimoto and Okamoto (1969). For this we need the following definition of Yanagimoto and Okamoto.

DEFINITION 4.8. Let $\mathbf{r}$ and $\mathbf{s}$ correspondingly denote the row and column rank orders of $(X_1, Y_1), \ldots, (X_n, Y_n)$. A statistic T is said to be a rank statistic if there exists a function f (or equivalently g) from S_n to R^1 such that

$$T((X_1, Y_1), \ldots, (X_n, Y_n)) = f(\mathbf{r}) = g(\mathbf{s})$$

Moreover, T is said to be AI with respect to the ordering b_3 if f preserves the b_3 ordering.

THEOREM 4.9. *Assume that H_1 and H_2 have continuous marginals with $F_1(x) = F_2(x)$ and $G_1(y) = G_2(y)$. If (a) $H_2 \overset{rr}{\geq} H_1$ and (b) the rank statistic T is AI with respect to the ordering b_3, then T is stochastically not smaller under H_2 than under H_1, i.e.,*

$$Prob_{H_2}(T \geq c) \geq Prob_{H_1}(T \geq c) \textit{ for all } c.$$

PROOF. See Yanagimoto and Okamoto (1969, Theorem 6.1).

Lehmann (1966) considered the rank statistic $T((X_1, Y_1, \ldots, X_n, Y_n)) = \sum_{k=1}^{n} A(k)B(\mathrm{s}(k)) \equiv g(\mathrm{s})$, where A and B are nondecreasing. It follows that $g \in \mathrm{AI}(b_3)$. Hence, T is AI with respect to the ordering b_3. Theorem 4 of Lehmann (1966), therefore, follows from Theorem 4.9.

REFERENCES

AHMED, A.N., LANGBERG, N.A., LÉON, R., and PROSCHAN, F. (1979). Partial orderings of positive quadrant dependence, with applications. Technical Report No. 78–3. Department of Statistics, Florida State University.

BLOCK, H.W., CHHETRY, D., FANG, Z., and SAMPSON, A.R. (1987). Metrics on permutations based on partial orderings. Technical Report No. 87–11. University of Pittsburgh.

BLOCK, H.W., CHHETRY, D., FANG, Z., and SAMPSON, A.R. (1990). Partial orders on permutations and dependence orderings on bivariate empirical distributions. *Ann. Statist.*, to appear.

BLOCK, H.W. and SAMPSON, A.R. (1988). Conditionally ordered distributions. *J. Multi. Anal.* **27** 91–104.

BOLAND, P.J., EL-NEWEIHI, E., and PROSCHAN, F. (1989). Stochastic order for inspection and repair policies. Technical Report. Florida State University.

CAMBANIS, S., SIMONS, G., and STOUT, W. (1976). Inequalities for EK(x, y) when marginals are fixed. Z. WAHR. **36** 285–294.

DIACONIS, P. and GRAHAM, R.L. (1977). Spearman's foot rule as a measure of disarray. *J. Roy. Statist. Soc. B* **39** 262–268.

FLIGNER, M.A. and VERDUCCI, J.S. (1986). Distance based ranking models. *J. Roy. Statist. Soc. B* **48** 359–369.

HÁJEK, J. (1969). Miscellaneous problems of rank testing. *Proc. Symp. Nonparametric Tech.*, Bloomington, IN.

HOLLANDER, M., PROSCHAN, F., and SETHURAMAN, J. (1977). Functions decreasing in transposition and their applications in ranking problems. *Ann. Statist.* **5** 722–733.

KARLIN, S. and RINOTT, Y. (1980). Classes of orderings of measures and related correlation inequalities I. Multivariate totally positive distributions. *J. Multi. Anal.* **10** 476–498.

KENDALL, M.G. (1970). *Rank Correlation Methods, 4th Ed.* Griffin, London.

KIMELDORF, G. and SAMPSON, A.R. (1987). Positive dependence orderings. *Ann. Inst. Statist. Math. A* **39** 113–128.

KNUTH, D. (1973). *The Art of Computer Programming, Vol. 2.* Addison-Wesley, Reading, MA.

LEHMAN, E.L. (1966). Some concepts of dependence. *Ann. Math. Statist.* **37** 1137–1153.

METRY, M. and SAMPSON, A.R. (1988a). Characterizing and generating bivariate empirical rank distributions satisfying certain positive dependence concepts. Technical Report No. 88–04. University of Pittsburgh.

METRY, M. and SAMPSON, A.R. (1988b). Positive dependence concepts for multivariate empirical rank distributions. Technical Report No. 88–09. University of Pittsburgh.

SAVAGE, I.R. (1957). Contributions to the theory of rank order statistics—the 'trend' case. *Ann. Math. Statist.* **28** 968–977.

SCHRIEVER, B.F. (1985). *Order Dependence.* Ph.D. Dissertation, Free University of Amsterdam.

SCHRIEVER, B.F. (1987a). An ordering for positive dependence. *Ann. Statist.* **15** 1208–1214.

SCHRIEVER, B.F. (1987b). Monotonity of rank statistics in some non-parametric testing problems. *Statistica Neerlandica* **41** 91–109.

SOBEL, M. (1955). On a generalization of an inequality of Hardy, Littlewood, and Polya. *Proc. Amer. Math. Soc.* **5** 596–602.

TCHEN, A.H.T. (1980). Inequalities for distributions with given marginals. *Ann. Prob.* **8** 814–927.

WHITT, W. (1982). Multivariate monotone likelihood ratio and uniform conditional stochastic order. *J. Appl. Prob.* **19** 695–701.

YANAGOMOTO, T. (1990). Dependence ordering in statistical models and other notions. To appear in *Topics in Statistical Dependence*, Institute of Mathematical Statistics Lectures Notes—Monograph Series.

YANAGIMOTO, T. and OKAMOTO, M. (1969). Partial orderings of permutations and monotonicity of a rank correlation statistics. *Ann. Inst. Statist. Math.* **21** 489–506.

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