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RESULTS AND PROBLEMS IN GAMES OF TIMING

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Abstract. This paper presents the history of investigations concerning a subclass of zero-sum two-person games called duels, which were initiated by David Blackwell and other mathematicians in the reports of the RAND Corporation in 1948-52. The second part of the paper discusses mutual relationships between discrete and non-discrete duels, and gives a review of recent more general results. The paper also discusses some open problems in the general theory and makes hypothesis on them (strongly suggested by previous results).

1. INTRODUCTION

In 1948 the RAND Corporation collected a team of mathematicians, statisticians, economists, and social scientists to analyze "the uncertainties" in the global world situation and to construct a blue-print for the optimal operation against that. One of the results of this study, achieved within a component of the program, was the solution of many problems formulated in the form of some zero-sum two-person games, called duels, or games of timing when considered in a more general sense. Particularly, David Blackwell, one of the members of that team, was very instrumental in formulating and solving several versions of such games. He, together with M. Shiffman, M. A. Girshick, L. S. Shapley, R. Bellman, I. Glicksberg and others, initiated a new topic within zero-sum games at that time, and recognized the wide scope of possible applications of games of timing, particularly in the description and explaining of some conflict situations in economics. Since then many new general problems in games of timing have been formulated and many important and interesting results have been achieved. However, to say more in detail about it, at first, we must give a definition of games of timing in a sufficiently general form to include the whole rich collection of all different duel-models studied in the literature. We will do it in a slightly different (but equivalent) convention in comparison to that adopted by the pioneers of this topic mentioned above.

Consider the following model of a zero-sum game: There are two Players 1 and 2 with initial amounts M_1 and M_2 of some homogeneous resources, respectively. It is assumed that they should distribute some or all of their

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resources over a common one-dimensional continuum, say $[0,1]$ as an time-interval, so that each player's distribution progresses over the time. As the consequence of such a game will already be a pair (μ_1, μ_2) of measures on $[0,1]$ that describes the way in which the players have distributed their resources. If (μ_1, μ_2) is such a resulting pair, Player 1 wins from Player 2 a value $K(\mu_1, \mu_2)$, where K is some fixed payoff function. The purpose of Player 1 is to maximize his winnings $K(\mu_1, \mu_2)$, while Player 2 wishes to minimize it.

In specifying and analyzing such a game, the next possible feature is whether the opponent of Player i, $i = 1, 2$, constantly knows the exact history of player i's expenditures so far. If this is the case, Player i is called a *noisy* player, while at the opposite extreme (his opponent is completely ignorant about that) Player i is called *silent*. Thus a situation of the indicated type actually gives rise to three possible types of games: Γ_{nn} (*noisy*) in which both players are noisy, Γ_{ss} (*silent*) in which both are silent, and Γ_{ns} (*mixed*) in which Player 1 is noisy and Player 2 is silent.

Obviously, the possible strategies of a player have much more complicated structure when his opponent is noisy than when he is silent. Anyway, independently of the possible types of the players, any such a *generalized game of timing* has, as its starting game, the following *basic game* in normal form

$$\Gamma = < \mathcal{M}_1, \mathcal{M}_2, K >,$$

where for $i = 1, 2$,

(A1) *the strategy space $\mathcal{M}_i$ of Player i is a specified subset of the set of all measures μ_i on (the Borel sets) of $[0,1]$ satisfying $\mu_i([0,1]) \le M_i$*; (Here $\mu_i([0,t))$ is interpreted as the quantity of Player i's resource expended up to time t);

(A2) $K : \mathcal{M}_1 \times \mathcal{M}_2 \to R$ *is the payoff kernel (from Player 2 to 1).*

Obviously, the normal form of the game Γ_{ss} coincides with that of the basic game Γ, but this is not longer true for games Γ_{nn} and Γ_{ns}. It was shown in Radzik & Goldman (1996) that the normal form of Γ_{ns} is completely determined by its basic game Γ, and the same is true only for some cases of Γ_{nn}. In general, the problem of mutual relationships between Γ and Γ_{nn} is very complex, and rather far from being completely solved.

Now, let us consider the next possible feature of the players' resources which can be in one of the two states: "*indivisible*" - when player's resources consist of only some finite number of indivisible "*actions*" of the same amount 1 each, and each of them can be distributed only at single moments of $[0,1]$; and "*divisible*" - when a player is able to distribute his resources quite arbitrarily (in a continuous or discontinuous way) over the time interval. This gives rise to the next three types of games of timing: *discrete* when both the players have only resources of indivisible type, *non-discrete* when both have only divisible ones, and *mixed* - in the situation where one of the players possesses only divisible while his opponent indivisible resources. So the games of timing may be considered with various configurations of "noisy", "silent", "discrete" and "non-discrete". In the paper, we shall use the notation $\Gamma_{nn}(k,l)$, $\Gamma_{ss}(k,l)$ or $\Gamma_{ns}(k,l)$ to denote a discrete game of timing with k and l indivisible actions at Player 1 and 2, respectively, and with an appropriate type of actions (noisy, silent). Analogously, by $\Gamma_{nn}(\hat{M_1},\hat{M_2})$, $\Gamma_{ss}(\hat{M_1},\hat{M_2})$ or $\Gamma_{ns}(\hat{M_1},\hat{M_2})$ such non-discrete games of timing will be denoted, where M_1 and M_2 are the amounts of resources possessed by the players. Finally, $\Gamma_{nn}(k,\hat{M})$ and $\Gamma_{ss}(k,\hat{M})$ denote mixed games, where Player 1 is in possession of k indivisible actions and Player 2 has divisible resources of amount M.

Historically, the notion of classical *duels* (*discrete* and *non-discrete*) is reserved to a subclass of generalized games of timing such that the payoff kernels K of their basic games Γ (as expected payoffs of Player 1) are consistent with the following five additional assumptions on a game. Namely, for $i = 1, 2$,

(B1) *Player i taking one his actions (an indivisible part of resources of amount 1) at a single moment t, succeeds with a probability* $P_i(t)$; here $P_i(t), 0 \le t \le 1$, the so-called *accuracy function* associated with that player and known to both, is (usually) assumed to be nondecreasing and continuous with $P_i(0) = 0$ and $P_i(1) = 1$;

(B2) *the players act independently of each other in the game*;

(B3) *events that a player will not succeed in any disjoint subintervals of* $[0, 1]$, *respectively, are always independent*;

(B4) *the game ends at the moment of the first success of any of the players, or at* $t = 1$, *otherwise*;

(B5) *the payoff of Player 1 amounts +1, -1 or 0, respectively in the three cases: (a) the game ends with Player 1's success only; (b) Player 2 succeeds alone; (c) both the players succeed or the game ends without a success of either player.*

For discrete duels of any of type $\Gamma_{nn}(k,l)$, $\Gamma_{ss}(k,l)$ or $\Gamma_{ns}(k,l)$, the five additional assumptions introduced above uniquely determine the payoff kernel K in their basic games Γ. It can be given in an equivalent form as a function $K(\bar{x}_k, \bar{y}_l)$ of $k + l$ variables, defined on the product of the sets $X = \{\bar{x}_k \in [0,1]^k : 0 \le x_1 \le x_2 \le \ldots \le x_k \le 1\}$ and $Y = \{\bar{y}_l \in [0,1]^l : 0 \le y_1 \le y_2 \le \ldots \le y_l \le 1\}$; here x_i and y_j describe the moments at which Player 1 takes his i-th action and Player 2 takes his j-th action, respectively. Now, after identifying vectors $\bar{x}_k$ with the measures μ_1 of total mass k and concentrated equally at all points x_i of $\bar{x}_k$ with mass 1 each (and analogously for $\bar{y}_l$), these discrete duels are quite in accordance with the convention of the basic game Γ.

As far as a non-discrete duel is concerned, the payoff function K of its basic game Γ is not uniquely determined by the additional assumptions (B1) - (B5) given above. But after adding some other ones (connecting very closely discrete and non-discrete duels), the uniqueness of K can also be ensured. This will be widely discussed in Section 3. In Section 2, we present the history of investigations and achieved results in games of timing. Section 4 contains a review and discussion of some recent more general results related to discrete and non-discrete duels. Section 5 is devoted to open problems and some conjectures on them in the general theory of games of timing.

2. History of achieved results

Since 1948, when the first models of discrete duels were formulated, many interesting results have been achieved. The basic difficulty in studying such games is that neither of the general theorems (known in the game theory) answers the question about the possible existence of their value and optimal strategies for the players. Even in the simplest cases of discrete silent duels, the payoff kernel is discontinuous. On the other hand, the structure of duels with noisy actions is much more complex, since in general, the players' strategies must depend on the information about the behavior of the opponent within the time interval. In such a way, these games are in fact extensive games with discontinuous payoff functions, and with continuum possible alternatives in each position. This short explanation sufficiently puts across the complexity of studying games of timing. Below, in seven parts, we present the history of problems and achieved results in this field.

2.1. *The beginning of the theory.* Let us denote by $\Gamma(k,l)$ the class of discrete duels (games of timing) involving such games with silent or noisy actions, that is, $\Gamma_{nn}(k,l)$, $\Gamma_{ss}(k,l)$ and $\Gamma_{ns}(k,l)$. The first cases of games $\Gamma(1,1)$ were formulated and studied in 1948-53, in the RAND Corporation reports of American mathematicians. Particularly Blackwell, Shiffman, Girshick, Bellman, Glicksberg and Shapley were very instrumental in formulating and solving several various duels. As far as the first more important papers on duels of type $\Gamma(1,1)$ are concerned, we can list here Blackwell (1949*a*, 1949*b*), Blackwell & Girshick (1949) and Bellman & Girshick (1949). The first two papers study duels under arbitrary accuracy functions $P_1(t) \neq P_2(t)$. In the first of them a solution of a general noisy duel $\Gamma_{nn}(1,1)$ is given, permitting the possibility of nonmonotonic accuracy functions, while the second one solves the general silent duel $\Gamma_{ss}(1,1)$, by employing a suitable extension of the equalizer strategy technique. The next two papers deal with slightly enriched models of duels, under the assumption that $P_1(t) = P_2(t) = t$. Namely, the first of them studies $\Gamma_{nn}(1,1)$ assuming the random actions at the players, while the last one analyses the duel $\Gamma_{ss}(1,1)$ on a constrained time interval $[b,1]$.

2.2. *Games of timing of class I and II.* The complete results on general silent games of timing of type $\Gamma_{ss}(1,1)$ (defined by the general form of payoff kernel $K(x_1, y_1)$ as a function on the unit square, increasing in x_1 and decreasing in y_1 below and over the diagonal, and with possible discontinuities only on it) belong toShiffman (1953) - the symmetric case ($K(x_1, y_1) = -K(y_1, x_1)$), and to Karlin (1953) - the unsymmetric case. In the literature, such games are called *games of timing of class II.* The first author showed that the solution in the symmetric case , which is the same for each player, occurs in four categories and can be obtained by solving a single integral equation of the second kind. Karlin (1953) extended Shiffman's result to unsymmetric case, and showed that this leads to fourteen possible categories of solutions. The method of solving games of timing in both cases appeals fundamentally to the theory of positive integral transformations. It is worth mentioning here that games of timing of class II in a non-zero sum version also were studied, and some partial results can be found in Sudżute (1983).

The second general result concerning games $\Gamma(1,1)$ belongs to Glicksberg (1950). He found a solution of a general discrete noisy game of timing of type $\Gamma_{nn}(1,1)$. These games, called in literature as *games of timing of class I*, differs from that of class II in the payoff kernel that it is constant in each variable over the diagonal. One should add here that an independent and more complete determination of the solution was given by Karlin (1953), where he solved those games by approximating a solution uniformly by a sequence of games of timing of class II and then invoking a standard limiting process. (For a non-zero-sum version, see Pitchik (1981)).

2.3. *Noisy discrete duels.* Now let us pass to more complicated discrete noisy games of timing $\Gamma_{nn}(k,l)$ with numbers of actions greater than 1 for both players. Unfortunately, there are no general results here, comparable (with respect to degree of generality) to the results on games of timing of classes I and II, found by Glicksberg and Karlin. In spite this, many interesting and very difficult problems have been solved in this topic. The first result belongs here to Blackwell & Girshick (1954), where they found the solution of noisy duels $\Gamma_{nn}(k,l)$ with general number of actions, $k, l \geq 1$, and with equal accuracy functions $P_1(t) = P_2(t) = t$. These authors showed that then all the duels $\Gamma_{nn}(k,l)$ possessed values, and they constructed ϵ-optimal strategies with a recursive structure for both players. Unfortunately, their method does not allow to solve that duel under $P_1(t) \neq P_2(t)$. The further essential generalization of the last result is due to Fox & Kimeldorf (1969), who solved the noisy duel $\Gamma_{nn}(k,l)$ with arbitrary continuous and nondecreasing accuracy functions, satisfying only the slight restriction, $P_i(0) = 0, P_i(1) = 1$ for $i = 1,2$. The ϵ-optimal strategies found there are also of a recursive type, but much more complex in comparison to the ones of the previous case. However their method, although based on very subtle considerations, does not enable them to find whether the duels $\Gamma_{nn}(k,l)$ have values in the general case, $0 \leq P_i(0) < P_i(1) \leq 1$, $i = 1,2$. Żadan (1976) obtained the next important result in this direction. He showed with the help of a largely complex theory, especially constructed for the need of the problem, that under only the continuity of accuracy functions with $P_i(0) = 0$, $P_i(1) < 1$ for $i = 1, 2$, noisy discrete duels $\Gamma_{nn}(k, l)$ still have values, and he found the form of ϵ-optimal strategies for the players. The assumption made by Żadan has appeared to be necessary there.

In the last three papers discussed above and concerning the duel $\Gamma_{nn}(k, l)$, the condition, $P_i(0) = 0, i = 1, 2$, is necessary because of the method adopted there. ϵ-optimal strategies found there have such a recursive structure which cannot be transferred to duels $\Gamma_{nn}(k, l)$ under $P_i(0) > 0$, $i = 1, 2$. This restriction was finally overcome in Radzik (1991), where a complete solution of duels $\Gamma_{nn}(k, l)$ with $k, l \geq 1$ was found, under the general assumption that the accuracy functions are continuous and nondecreasing with $0 \leq P_i(0) \leq P_i(1) \leq 1$, $i = 1, 2$. To this end, a special theory studying properties of optimal strategies and equilibria in some matrix games with restricted set of admissible pairs of the players' pure strategies has been developed. In the case, $P_i(0) > 0$ for $i = 1, 2$, the structure of ϵ-optimal strategies has appeared much more complex in comparison to the ones considered in the three papers mentioned above. At the end, it is worth mentioning that some asymptotic properties of noisy discrete duels were discussed in Fox & Kimeldorf (1970) and of all types discrete duels in Kimeldorf & Lang (1978). However, in spite of strong results found in this topic, we still do not know how to extend the result of Glicksberg and Karlin for games of timing $\Gamma_{nn}(1, 1)$ of class I to that of $\Gamma_{nn}(k, l)$ with $k, l \geq 1$.

2.4. *Silent discrete duels.* Now we shall treat of a general silent discrete duels $\Gamma_{ss}(k, l)$ with $k, l \geq 1$. The most famous result in this topic we find in Restrepo (1957). It is shown there that for all k, l, the silent duel $\Gamma_{ss}(k, l)$, with continuously differentiable accuracy functions satisfying $P_i(0) = 0, P_i(1) = 1$ and $P_i'(t) > 0$ for $i = 1, 2$ and $0 < t < 1$, always has a value, and optimal strategies for the players have been found. In spite of large complexity of the analysis adopted there, Restrepo's method is insufficient to derive a theorem involving a full extension of the general game of timing of class II, analogous to that of Karlin for $\Gamma_{ss}(1, 1)$. Summarizing, the questions how to extend Karlin's results to general games of timing $\Gamma_{ss}(k, l)$ with $k, l > 1$, is still an open problem. As yet Restrepo's result has been extended only to two somewhat modified models of $\Gamma_{ss}(k, l)$ in Ciegielski (1986*a*, 1986*b*). In the first paper the assumption on a the model is weakened to a random number of actions at each player, while the second one deals with $P_i(1) < 1$ for $i = 1, 2$, but still with the restriction $P_i(0) = 0$ for $i = 1, 2$ in both papers. It is also worth mentioning here two other modifications of $\Gamma_{ss}(k, l)$, where by assumption, a duel ends with a "small delay" (Orłowski & Radzik (1985)), and, where, a possibility of a "retreat after the shots" is admitted to the players (Trybuła (1990, 1993)), 2.5. *Mixed discrete duels.* In the topic of mixed discrete duels $\Gamma_{ns}(k,l)$, there are not more general results. It seems that this kind of duel is the most difficult to study. It is rather surprising that, for instance, even the duel $\Gamma_{ns}(2,1)$ with $P_1(t) \neq P_2(t)$, still remains an open problem. Only partial and rather particular problems (though mostly often very hard) have been solved here, though the number of published papers is rather large. We can list only some of them, representative in some sense. They concern different modifications of duels with, mostly often, arbitrary accuracy functions. Kurisu (1991) solved a class of duels $\Gamma_{sn}(1,1)$ with such a modification that the action of Player 2 is noisy with a constant "time lag"; so that class contains the standard discrete duels $\Gamma_{sn}(1,1)$ and $\Gamma_{ss}(1,1)$ as its two extremes. In the next interesting paper, Kurisu (1983) solved (with the help of computer calculations and graphs) a solution of duel $\Gamma_{ns}(2,1)$ under $P_1(t) = P_2(t) = t$, but the adopted method cannot be used to in a general analysis of such games. Smith (1967) solved the duel, where Player 1 has one silent and one noisy action while Player 2 has only one noisy action. Styszyński (1974) found a solution of mixed duel $\Gamma_{sn}(k,1)$ with $k > 1$. The last two results were generalized in Radzik & Orłowski (1982*a*) and Radzik & Orłowski (1982*b*), where such a duel is studied, in which Player 2 has only one noisy action while Player 1 is in possession of any fixed number of noisy and silent actions, and by assumption, he must take them in an arbitrarily fixed order (known to Player 2). Another modification of duels was studied in in Sakaguchi (1984) considering a duel of type $\Gamma(1,1)$ with the uncertain knowledge about the existence of players' actions. Here one can also mention models of duels of type $\Gamma(1,1)$, considered with a "random termination" of a duel in Teraoka (1983, 1986), and with a "player detection probability" in Sweat (1971).

2.6. *Silent non-discrete duels.* The next subclass of games of timing studied in the literature, are silent non-discrete duels of type $\Gamma_{ss}(\hat{M}_1, \hat{M}_2)$. They differ from the silent discrete ones in the possibility of quite arbitrary way of distributing the players' resources over the time interval. Early formulation of such a game (called the two machine-gun duel) with a partial solution was given by Danskin & Gillman (1953), but the first rigorous solution was due to Karlin (1959). In that model the players were allowed to distribute his resources in a continuous way over $[0,\infty)$, but only with a bounded intensity. Yanovskaya (1969) studied a more general version considered by Karlin. But the construction of the payoff kernel in the non-discrete duels discussed in the last three papers, was rather far from that of discrete duels. The first close relation between these two kinds of duels was noticed by Lang & Kimeldorf (1975, 1976) who formulated a slightly another model of $\Gamma_{ss}(\hat{M}_1, \hat{M}_2)$ and solved it in two versions. In the first paper the authors considered the silent non-discrete duel with $P_1(t) = P_2(t)$, under the restriction that the players have the possibility of only continuous distributing of their resources. The second paper contains a solution of the same duel without that restriction, under $P_1(t) \neq P_2(t)$. The optimal strategies found there are independent of each other and, consequently, they remain optimal also in duels $\Gamma_{sn}(\hat{M}_1, \hat{M}_2)$ and $\Gamma_{nn}(\hat{M}_1, \hat{M}_2)$. Next, in Positielskaya (1984), one can find an optimal strategy for Player 2 in a duel $\Gamma_{ns}(\hat{M}_1, \hat{M}_2)$, that is an equalizer against almost all continuous strategies of Player 1. It is also worth mentioning that asymptotic properties of non-discrete duels were discussed in Kimeldorf & Lang (1977).

As far as the two models of $\Gamma_{ss}(\hat{M}_1, \hat{M}_2)$ discussed above are concerned, they belong (when considered on $[0,1]$) to the same class of non-discrete silent duels. The only difference is that the accuracy functions satisfy $P_i(1) = 1-e^{-1}$ in Karlin's model, and $P_i(1) = 1$ in the model of Lang and Kimeldorf, $i = 1,2$.(The equalities $P_i(0) = 0$ are common to both.) This fact is widely discussed in Section 3, where payoff kernel for $\Gamma_{ss}(\hat{M}_1, \hat{M}_2)$ is constructed in a new axiomatic way. The last result concerning silent non-discrete duels was obtained in Radzik (1988*a*), where such games were solved under total general assumptions: $P_i(0) \geq 0, P_i(1) \leq 1$ for $i = 1,2$. A complete characterization of optimal strategies found there is analogous to that of Karlin for games of timing of class II. It is rather surprising that in the case, $P_1(1) < 1, P_2(1) = 1$, Player 2 has only ϵ-optimal strategies. The method exploited there is a strong extension of that one from Lang & Kimeldorf (1976). At the end, it is worth adding here that some new modification of these duels was analyzed in Radzik & Orłowski (1985).

2.7. *Silent mixed duels.* The next very natural subclass of games of timing are silent mixed duels of type $\Gamma_{ss}(1, \hat{M})$, where Player 1 possesses one indivisible action and behaves as in a discrete duel, while Player 2 with his divisible resources of amount M acts as in a non-discrete duel. There are rather few results here, and this topic lacks any coherent theory. The first authors who were instrumental in formulating and solving some different examples of such games were Gillman, Blackwell, Shiffman, Bellman and Karlin. They all studied the first version of this game called the fighter-bomber duel, where in the model Player 2 was allowed to distribute his resources only in a continuous way, and with a bounded intensity. Its connection with the classical discrete duels was rather loose. At first, that problem was studied and solved in Blackwell & Shiffman (1949*a*, 1949*b*) and in other unpublished papers of Weiss, Bellman and Blackwell. The interpretation of it as an advertising campaign was conceived by Gillman (1950). It was also studied in Karlin (1959), where the author applied a new method, relying heavily on the Neyman-Pearson lemma, to find the optimal strategy for Player 1. Besides, there are only two results in this topic, and they belong to Radzik (1988*b*, 1989). Both these papers study mixed silent duels of type $\Gamma_{ss}(k, \hat{M})$, but in a slightly different version to that discussed above, more closely related to discrete duels. The first of them gives the complete characterization of solutions for the duel $\Gamma_{ss}(1, \hat{M})$, where Player 2 can distribute his resources without any restrictions. A more general duel of type

$\Gamma_{ss}(k, \hat{M})$ with $k > 1$ is analyzed in the second paper. It is shown there that such duels always have values and the form of an optimal strategy for Player 2 was found.

2.8. *Final comments.* It follows from the above discussion that no general coherent theory has been found in the topic of games of timing during the last 35 years. In this period a lot of problems have been solved, but we must critically admit that very many of them concern rather various particular and strongly detailed models. It has been caused by not only a large degree of complexity of models and huge theoretical difficulties with information that arise there. We are still lacking general and effective methods in this topic. To find here a solution of any general problem, most often one has to construct a special theory devoted only to it. As yet there is no homogeneous theory for more general games of timing that would be satisfactory. Practically, there is such a theory only for the very narrow case of Karlin's games of timing of classes I and II, where the payoff kernel is determined not by accuracy functions but in more general form of a function $K(x, y)$ on the unit square. Unfortunately, this beautiful theory is quite powerless in more general problems, where payoff kernels are defined on multi-dimensional spaces or on a set of pairs of measures. Hence the question: are there any chances to change this situation? It seems that we can risk the answer YES!, and this is strongly motivated by all the more general results achieved just during the last 35 years. We shall say more about it in Section 5.

3. Relationships between discrete and non-discrete duels

Let us consider any duel, that is, any game of timing with its basic game Γ satisfying assumption (A1)-(A2) and (B1)-(B5) from Introduction. Let $P_i(t), 0 \leq t \leq 1$, be an accuracy function of Player i. By definition, $P_i(t)$ describes the probability that Player i succeeds at a moment t, in the situation when he expends one unit of his resources (an action) exactly at t.

For any vector $\bar{z}_m = (z_1, z_2, \dots, z_m)$ with $0 \leq z_1 \leq z_2 \leq \dots \leq z_m \leq 1$, let $I(\bar{z}_m)$ define the measure on $[0, 1]$ of total mass m and concentrated exactly at points $z_1, z_2, \dots, z_m$ with masses 1 at each.

Further, for any strategy μ_i of Player i and for any subinterval $D \subseteq [0, 1]$, we define $Q^{\mu_i}(D)$ as the probability that Player i, distributing his resources according to μ_i, succeeds in D.

It appears that in general, $Q^{\mu_i}(D)$ is not uniquely determined by the accuracy function $P_i(t)$ and assumptions (B1)-(B4) mentioned above. On the other hand, $Q^{\mu_i}(D)$ is basic in constructing of the payoff kernel $K(\mu_1, \mu_2)$ of the basic game Γ. Namely, under the notation

$$(1) \qquad Q_i(t) \stackrel{def}{=} Q^{\mu_i}([0, t]), \qquad \bar{Q}_i(t) \stackrel{def}{=} 1 - Q_i(t),$$

the payoff kernel can be easily derived as

$$K(\mu_1,\mu_2)=\int_{[0,1]}\bar{Q}_2dQ_1-\int_{[0,1]}\bar{Q}_1dQ_2. \tag{2}$$

To justify this formula, notice that $\bar{Q}_i(t)$ represents the probability that Player i does not succeed in $[0,t]$. Therefore, the quantity $\bar{Q}_2(t)dQ_1(t)$ is the probability that Player 1 succeeds in the interval $(t,t+dt)$, given neither succeed by the time t. Then the limit sum of these probabilities (equal to the first integral in (2)) is the probability that Player 1 succeeds before Player 2, ensuring himself payoff +1 (by assumption (B5)). Similar arguments apply to the second integral in (2).

By definition, in a discrete duel the players can distribute their resources only according to measures of the form $I(\bar{z}_m)$, and it is easy to show with the help of assumptions (B1)-(B4) from the introduction that

$$Q^{I(\bar{z}_m)}([0,t])=1-\prod_{s\le t}[1-P_i(z_s)],\qquad 0\le t\le 1.$$

Thus, in view of (1) and (2), assumptions (B1)-(B5) uniquely determine the basic game Γ of any discrete duel. But this is not the case for non-discrete duels, and to determine $Q^{\mu_i}(t)$ for all μ_i, some new assumption must be considered. Below we analyze three approaches to $Q^{\mu_i}(t)$.

3.1. *Model I.* This model was considered in two equivalent versions, on $[0,1]$ and on $[0,\infty)$, as time intervals, in Blackwell & Shiffman (1949*a*, 1949*b*) and in Karlin (1959). Here we present it on $[0,1]$ in a slightly more general form.

It is assumed that $Q^{\mu_i}(t)$ satisfies:

(3)
$$Q^{\mu_i}([t,t+h])=\mu_i([t,t+h])\cdot A_i(t)+o(h),\qquad a.e.,\ 0\le t\le 1,$$

for any absolutely continuous measure μ_i on $[0,1]$, where $A_i(t)$ is a continuously monotone function with

$$A_i(0)=0,\quad A_i(1)=1. \tag{4}$$

(The function $A_i(t)$ does not coincide with the accuracy function $P_i(t)$, and is called a *modified accuracy function.*)

Condition (3), by the standard limit procedure, leads to

$$Q^{\mu_i}(t)=1-\exp\left(-\int_{[0,t]}A_i(u)d\mu_i(u)\right),\qquad 0\le t\le 1, \tag{5}$$

for any absolutely continuous measure μ_i. Now, if we formally extend formula (5) to the set of all measures μ_i, then after substituting $\mu_i=I(t)$, we shall obtain

$$P_i(t)=1-\exp[-A_i(t)],\qquad 0\le t\le 1, \tag{6}$$

because of the obvious equality, $P_i(t) = Q^{I(t)}(t)$. Therefore (see (6) and (4)), this model is consistent with such discrete duels, where the accuracy functions satisfy: $P_i(0) = 0$ and $P_i(1) = 1 - e^{-1}$.

3.2. *Model II.* Lang & Kimeldorf (1975) proposed to change the function $A_i(t)$ in (3) for another one, defined on $[0, 1)$ and satisfying

$$A_i(0) = 0, \qquad A_i(1-) = \infty.$$

Now we can repeat the considerations of Model I to get the conclusion that Model II is consistent with discrete duels, with $P_i(0) = 0$ and $P_i(1) = 1$.

3.3. *Model III.* Both the approaches in Models I and II have some deficiency. Namely, though formula (5) on $Q^{\mu_i}(t)$ can be well defined for all measures μ_i, the starting condition (3) is consistent with (5) only for absolutely continuous μ_i. It is not difficult to check that for discontinuous measures, (3) may contradict (5), as for instance, for $\mu_i = I(t)$ with $h \to 0$. On the other hand, in Model II, $Q^{\mu_i}(t)$ and thereby non-discrete duels, are defined on $[0, 1)$ instead of $[0, 1]$. In Radzik (1988*b*) another model was proposed without these deficiencies, and we present it below.

Let $P_i(t)$ be any accuracy function on $[0, 1]$ with $0 \le P_i(0),\ P_i(1) \le 1$, not necessarily monotone or continuous. It is very natural to require that the function

$$\bar{Q}_i^{\mu_i}(D) \stackrel{def}{=} 1 - Q_i^{\mu_i}(D) \tag{7}$$

satisfy the following four conditions:

(C1) *for any measurable set $D \subseteq [0, 1]$ and for all $t \in D$ and $\alpha \ge 0$,*

$$0 \le \bar{Q}_i^{\alpha I(t)}(D) \le 1, \qquad \bar{Q}_i^{I(t)}(D) = 1 - P_i(t)\ ;$$

(C2) *for any measurable set $D \subseteq [0, 1]$ and for all $t \in D$ and $\alpha, \beta \ge 0$,*

$$\bar{Q}_i^{(\alpha+\beta)I(t)}(D) = \bar{Q}_i^{\alpha I(t)}(D) \cdot \bar{Q}_i^{\beta I(t)}(D)\ ;$$

(C3) *for any measure μ on $[0, 1]$ and for all nonempty measurable sets $D \subseteq [0, 1]$,*

$$\inf_{t \in D} \bar{Q}_i^{\alpha I(t)}(D) \le \bar{Q}_i^{\mu}(D) \le \sup_{t \in D} \bar{Q}_i^{\alpha I(t)}(D)\ ,$$

where $\alpha = \mu(D)$;

(C4) *for any sequence $\{D_m\}$ of disjoint measurable subsets of $[0, 1]$, and for any measure μ on $[0, 1]$,*

$$\bar{Q}_i^{\mu}\Big(\bigcup_m D_m\Big) = \prod_m \bar{Q}_i^{\mu}(D_m)\ .$$

Notice that in fact, conditions (C1) and (C4) are a repetition of assumptions (B1) and (B3) from the introduction. The remaining two are new. These four conditions have a very natural interpretation. Now we present the main result related to them that gives a rigorous answer about the possible form of the function $Q^\mu(t)$.

THEOREM 3.1. *For any measurable accuracy function $P_i(t)$ on $[0,1]$, conditions* (C1)-(C4) *uniquely determine the function $Q^\mu(t)$ in the form*

$$Q^\mu(t) = 1 - \exp\left(\int_{[0,t]} \log[1 - P_i(u)]d\mu(u)\right), \qquad 0 \le t \le 1, \tag{8}$$

for all measures μ on $[0,1]$. (Here, by definition: $\exp(-\infty) = 0$, $\log 0 = -\infty$, $0 \cdot (-\infty) = 0$.*)*

PROOF. In view of (7), to show (8) it suffices to verify that conditions (C1)-(C4) are equivalent to the following: for all measures μ on $[0,1]$ and for all measurable sets $D \subseteq [0,1]$,

$$\bar{Q}^\mu(D) = \exp\left(\int_D \log[1 - P_i(u)]d\mu(u)\right). \tag{9}$$

Namely, it is not difficult to check that (C1) and (C2) imply (9) for all measures of the form $\mu = \alpha I(t)$ with $\alpha \ge 0$, $0 \le t \le 1$. After this, we can show by the definition of Lebesgue integral that conditions (C3) and (C4) are sufficient for (9) to hold for all measures. The converse implication is immediate. ∎

REMARK 3.1. If $P_i(t)$ is monotone, Theorem 3.1 holds after replacing measurable sets D and D_m in (C1)-(C4) by intervals.

This ends our construction of Model III, since we have shown that formula (5), extended to the set of all measures, is the unique solution determined by conditions (C1)-(C4). This fact is an immediate consequence of (8), (7) and (6).

4. RECENT GENERAL RESULTS

In this section we present some recent general results about discrete and non-discrete duels. They are fundamental in our discussion about open problems related to general theory of games of timing (in Sect. 5), where also some hypothesis about them are made. Here we present four results, two for discrete and two for other types of duels.

4.1. *A basic system for $\Gamma_{nn}(k,l)$.* Here we present the results of Radzik (1991), where the noisy discrete duel $\Gamma_{nn}(k,l)$ was solved, with $k,l \ge 1$ and with continuous nondecreasing accuracy functions, satisfying

$$0 \le P_i(0) \le P_i(1) \le 1, \qquad i = 1,2. \tag{10}$$

That paper generalizes earlier results of Blackwell & Girshick (1954) and Fox & Kimeldorf (1969). The first of these papers treats the problem under $P_i(t) = t$, while the second solved it under $P_i(0) = 0$ and $P_i(1) = 1$ for $i = 1, 2$.

At first, introduce the notation of the modified noisy duel $\Gamma^a_{nn}(k,l)$ to describe the noisy duel $\Gamma_{nn}(k,l)$ played on the time interval $[a, 1]$ instead of $[0, 1]$. Now, it is not difficult to see that the solution of $\Gamma_{nn}(k,l)$ under (10), can be easily concluded from the solution of some modified noisy duel $\Gamma^a_{nn}(k,l)$ with some $0 \leq a \leq 1$, where (10) is replaced by

$$P_i(0) = 0, \quad P_i(1) \leq 1, \qquad i = 1, 2. \tag{11}$$

As it will appear, this way of studying of noisy discrete duels is more convenient, so we fix (11) as our assumption instead of (10). The solution of duel $\Gamma^a_{nn}(k,l)$ will be presented in two stages, with $a = 0$ and next, with a general $0 \leq a \leq 1$. At first, we must introduce the following constant

$$t_* = \max\{t : 0 \leq t \leq 1 \text{ and } (P_1(t) = 0 \text{ or } P_2(t) = 0)\},$$

which is very essential for duels $\Gamma^0_{nn}(k,l)$. This is seen in the context of the next theorem which determines some constants v_{ij} and t_{ij}, basic for $\Gamma^0_{nn}(k,l)$.

THEOREM 4.1. *Under (11), there exist a set $\{t_{ij}\}$ of numbers from $[t_*, 1]$ and a unique set $\{v_{ij}\}$, $i, j = 0, 1, \ldots$, such that*

$$\begin{array}{ll} t_{i0} = t_{0j} = 1 \text{ and } v_{00} = 0, & i, j \geq 1, \\ v_{ij} = P_1(t_{ij}) + [1 - P_1(t_{ij})]v_{i-1,j}, & i \geq 1, j \geq 0, \\ v_{ij} = -P_2(t_{ij}) + [1 - P_2(t_{ij})]v_{i,j-1}, & i \geq 0, j \geq 1. \end{array}$$

Furthermore, for $i, j \geq 1$ we always have

$$t_{ij} \begin{cases} < \min(t_{i-1,j}, t_{i,j-1}) & \text{if } \min(t_{i-1,j}, t_{i,j-1}) > t_* , \\ = \min(t_{i-1,j}, t_{i,j-1}) & \text{if } \min(t_{i-1,j}, t_{i,j-1}) = t_* . \end{cases}$$

REMARK 4.1. The above theorem generalizes and slightly corrects a corollary of Fox & Kimeldorf (1969). The solutions t_{ij} are basic in constructing optimal strategies in $\Gamma^0_{nn}(i,j)$. We shall show later that the equality $t_{ij} = \min(t_{i-1,j}, t_{i,j-1})$ can occur even in non-trivial cases. Just this possibility was overlooked in that paper.

4.2. *Optimal strategies in $\Gamma^0_{nn}(i,j)$.* Let $\{v_{ij}\}$ and $\{t_{ij}\}$ be any solution of Theorem 4.1. We shall define two sets $\{\xi_{ij}\}$ and $\{\eta_{ij}\}$, $i, j = 0, 1, \ldots$, of ϵ-optimal strategies for Players 1 and 2, respectively, in the games $\Gamma^0_{nn}(i,j)$. At first, we define the set $\{\xi_{ij}\}$ as follows:

(1) for all $i \geq 1$, the strategy ξ_{i0} prescribes Player 1 to take all his i actions at time $t = 1$ with probability 1;

(2) for $i \geq 1$ and $j \geq 1$, the strategy ξ_{ij} has the following structure: At the beginning, if $t_{ij} = t_*$, Player 1 should take s of his actions with probability 1 at moment t_*, where $s = \max(r : t_{i-r,j} = t_*)$, and next, apply $\xi_{i-s,j}$ in the time interval $(t_*, 1]$. If $t_{ij} > t_*$, ξ_{ij} consists of the following steps;

(a) Player 1 chooses t in any time interval $(t_{ij}, t_{ij} + \tau_{ij})$ according to the uniform distribution such that $0 < \tau_{ij} < \min(t_{i-1,j}, t_{i,j-1}) - t_{ij}$;

(b) if Player 2 has taken s his actions at a moment $t_1 < t$, Player 1 changes ξ_{ij} for $\xi_{i,j-s}$ in $(t_1, 1]$;

(c) if Player 2 has not taken any of his actions before time t, Player 1 should take one of his actions at this moment, and next apply $\xi_{i-1,j}$ in $(t, 1]$.
In the analogous way, the strategies η_{ij} of Player 2 are defined.

Now, we can formulate the main theorem on the games $\Gamma^0_{nn}(i,j)$.

THEOREM 4.2. *Let $\epsilon > 0, i \geq 0$ and $j \geq 0$. Then, under (11), the game $\Gamma^0_{nn}(i,j)$ has the value v_{ij}, and for sufficiently small τ_{rs} for $r \leq i$ and $s \leq j$, ξ_{ij} and η_{ij} are ϵ-optimal strategies for Players 1 and 2, respectively, in this game.*

EXAMPLE 4.1. Consider the games $\Gamma^0_{nn}(i,j)$, $ij \geq 1$, where accuracy functions are of the following form:

$$P_1(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq 1/4 \\ 1/2 & \text{if } 1/4 < t < 1/2 \\ t & \text{if } 1/2 \leq t \leq 1, \end{cases} \quad P_2(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1/4 \\ 2t - 1/2 & \text{if } 1/4 < t < 1/2 \\ t & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Here $t_* = 1/4$, and by Theorem 4.1, we can easily get:

$$\begin{aligned} t_{i1} &= 1/4 + 1/2^{i+1}, & & i \geq 1, \\ t_{ij} &= t_*, & & i \geq 1, j > 1, \\ v_{ij} &= 1 - 1/2^{i-1}, & & i \geq 1, j \geq 1. \end{aligned}$$

According to a corollary in Fox & Kimeldorf (1969), the strict inequalities, $t_{ij} < \min(t_{i-1,j}, t_{i,j-1})$ for all $i, j > 1$, should hold. But our example contradicts that. On the other hand, it is a very surprising fact that the values v_{ij} of duels $\Gamma^0_{nn}(i,j)$, $ij \geq 1$, do not depend on the number j of actions of Player 2.

4.3. *Optimal strategies in $\Gamma^a_{nn}(k,l)$.* Here, for fixed $0 \leq a < 1$, $k, l \geq 1$, and for all $0 \leq i \leq k$ and $0 \leq j \leq l$, we construct some strategies ξ^a_i and η^a_j for Players 1 and 2, respectively, in the game $\Gamma^a_{nn}(k,l)$. Namely, strategy ξ^a_i has the following structure:

(1) At first, Player 1 should take i of his actions at the starting moment $t = a$, and next (if Player 2 has taken s of his actions at a),

(a) (if $t_{k-i,l-s} \geq a$) Player 1 acts according to $\xi_{k-i,l-s}$ in the time interval $(a, 1]$;

(b) (if $t_{k-i,l-s} < a$) Player 1 chooses a moment t in the time interval $(a, a+\tau)$ according to the uniform distribution and takes r of his actions at this moment, where

$$r = \min\{u \geq 0 : t_{k-i-u,l-s-p} \geq t\}$$

and p is the number of Player 2's actions taken by him in the open interval (a, t). Next, Player 1 adopts $\xi_{k-i-r,l-s-p}$ as his strategy in $(t, 1]$.
In the analogous way, for all $0 \leq j \leq l$, the strategies η_j^a for Player 2 in the game $\Gamma_{nn}^a(k, l)$ are defined.

To formulate our main theorem in this subsection, we need the successive notation. For any probability vector $(\lambda, 1-\lambda)$ and for two strategies ξ_1 and ξ_2 of a player, $\lambda\xi_1 + (1-\lambda)\xi_2$ denotes such his strategy which prescribes him to adopt ξ_1 with probability λ and ξ_2 with probability $1-\lambda$.

THEOREM 4.3. *Let $\epsilon > 0$, $k \geq 0$ and $l \geq 0$ with $(k, l) \neq (0, 0)$ and $0 \leq a < 1$. Then, under (11), the game $\Gamma_{nn}^a(k, l)$ has a value and there exist $0 \leq p \leq k$, $0 \leq q \leq l$, $0 \leq \lambda \leq 1$ and $0 \leq \gamma \leq 1$ such that for sufficiently small parameters τ_{ij} and τ, the strategies*

$$\xi^* = \lambda\xi_p^a + (1-\lambda)\xi_{p+1}^a \qquad \text{and} \qquad \eta^* = \gamma\eta_q^a + (1-\gamma)\eta_{q+1}^a,$$

are ϵ-optimal for Players 1 and 2, respectively, in the game $\Gamma_{nn}^a(k, l)$.
For p and q, the inequality: $t_{k-p,l-q} \geq a$ holds.

REMARK 4.2. In Radzik (1991),a procedure is found which allows to find the values of parameters p, q, λ and γ, basic for the last theorem. To this end, a special theory of equilibria in constrained matrices was constructed there.

EXAMPLE 4.2. Consider the duel $\Gamma_{nn}^{0.3}(3, 1)$ for which $P_1(t) = P_2(t) = t$. By Theorem 4.1, we easily get

$$t_{i1} = \frac{1}{(i+1)}, \quad v_{i1} = \frac{(i-1)}{(i+1)} \text{ for } i \geq 0, \quad \text{and} \quad t_{i0} = 1, \; v_{i0} = 1 \text{ for } i \geq 1.$$

Since $t_{31} = 1/4 < 0.3$, the games $\Gamma_{nn}^{0.3}(3, 1)$ and $\Gamma_{nn}^{0}(3, 1)$ are different. By the procedure related to Theorem 4.3, one can find the ϵ-optimal strategies $\xi^* = 0.500\xi_1^{0.3} + 0.500\xi_2^{0.3}$ and $\eta^* = 0.733\eta_0^{0.3} + 0.267\eta_1^{0.3}$ for Players 1 and 2, respectively. The value of our game $\Gamma_{nn}^{0.3}(3, 1)$ is equal to 0.5215.

4.4. *General silent discrete duel.* Here we present the result of Ciegielski (1986*b*) (generalizing slightly the famous paper of Restrepo (1957)), where the silent discrete duel $\Gamma_{ss}(k, l)$ was solved under (11), with $k, l \geq 1$. (Unfortunately, the solution under (10) is still not known.) We begin with some notation and definition.

Denote the mixed strategies of Players 1 and 2 in the duel $\Gamma_{ss}(k, l)$ by F and G, respectively. So F and G are probability distributions over the sets

X and Y of the form

$$X = \{\bar{x} \in [0,1]^k : 0 \le x_1 \le x_2 \le \ldots \le x_k \le 1\}, \tag{12}$$

$$Y = \{\bar{y} \in [0,1]^l : 0 \le y_1 \le y_2 \le \ldots \le y_l \le 1\}. \tag{13}$$

DEFINITION 4.1. *We say that the strategy F belongs to the class $\mathcal{F}_k$ if:*
(i) $F(\bar{x}) = \prod_{i=1}^k F_i(x_i)$, where F_i is a probability distribution of x_i;
(ii) there are some $0 \le a_1 < a_2 < \ldots < a_q < a_{q+1} = a_{q+2} = \ldots = a_{k+1} = 1$ for some $q, 1 \le q \le k$, such that $\text{supp}F_i = [a_i, a_{i+1}]$ *for $i = 1, 2 \ldots, k$;*
(iii) all F_i are absolutely continuous on $(0,1)$.
In a similar way, we define the class $\mathcal{G}_l$ with l, G_j, b_j and r instead of k, F_i, a_i and q, respectively.

So, for instance, a mixed strategy F from the class $\mathcal{F}_k$ prescribes Player 1 to behave as follows. He should take his q first actions (for some q) independently, in a continuous way in the intervals $(a_1, a_2), \ldots, (a_q, 1)$, respectively, with the possible positive probability at $t = 1$ for the last action in this group. The rest of his actions Player 1 should take at $t = 1$ with probability 1.

THEOREM 4.4. *Assume that (11) holds. Then for any $k, l \ge 1$ the silent discrete duel $\Gamma_{ss}(k,l)$ has a value, and there are optimal mixed strategies $F \in \mathcal{F}_k$ and $G \in \mathcal{G}_l$ for the players in this game such that $a_1 = b_1$ and F_k or G_l is continuous at $t = 1$.*

REMARK 4.3. It follows from the above theorem that in some duels $\Gamma_{ss}(k,l)$ with (11), the optimal strategy for one of the players may be such that more than one action is concentrated at the point $t = 1$. This is contrary to the result of Restrepo (1957) because of the stronger assumption $P_1(1) = P_2(1) = 1$ considered there. On the other hand, one can expect that under (10), it may happen that an optimal strategy of a player in $\Gamma_{ss}(k,l)$ prescribes him to take some of his actions also at moment $t = 0$.

4.5. *General silent non-discrete duels.* The most general result in this topic was achieved in Radzik (1988*a*). It studies the duel $\Gamma_{ss}(\hat{M}_1, \hat{M}_2)$ under very weak restrictions. Namely, it is assumed on the accuracy functions $P_i(t)$ that they are absolutely continuous and strictly increasing on $[0,1]$, and are restricted only by condition (10). On the other hand, both the players in this game are allowed to distribute their divisible resources quite in an arbitrary way over the time interval $[0,1]$. To present that result, we must begin with some notation and definitions.

For fixed $M_1, M_2 > 0$ (quantities of the players' resources), let $\mathcal{M}_i^*$ denote the set of all measures μ_i on $[0,1]$ satisfying $\mu_i([0,1]) \le M_i$, $i = 1,2$. Therefore, the normal form of the considered silent non-discrete duel is

$$\Gamma_{ss}(\hat{M}_1, \hat{M}_2) = < \mathcal{M}_1^*, \mathcal{M}_2^*, K >, \tag{14}$$

where K is the payoff kernel defined by (8) and (2).

Further, for any $\alpha, \beta \geq 0$ and $0 \leq a < 1$, let $(\alpha I_0\ , f^a, \beta I_1)$ denote the measure on $[0,1]$ described by two masses α and β concentrated at points 0 and 1, respectively, and by its continuous part on the subinterval $(a, 1)$ determined by density function f^a.

Now we are ready to formulate the result of Radzik(1988*a*).

THEOREM 4.5. *Assume that (10) holds. Then the silent non-discrete duel $\Gamma_{ss}(\hat{M}_1, \hat{M}_2)$ of the form (14) has a value, and both players have optimal pure strategies $\mu_1^* \in \mathcal{M}_1^*$ and $\mu_2^* \in \mathcal{M}_2^*$ of the form $\mu_i^* = (\alpha_i I_0\ , f_i^a, \beta_i I_1)$, $i = 1, 2$, with the exception that Player 1 or 2 has only ϵ-optimal strategy of that form when $[P_1(1) = 1,\ P_2(1) < 1]$ or $[P_1(1) < 1,\ P_2(1) = 1]$, respectively. The parameters β_1 and β_2 satisfy: $\beta_1\beta_2 = 0$.*

REMARK 4.4. In Radzik(1988*a*) a complete characterization of optimal strategies is given, and it is presented in 18 cases in a similar way to that of given in Karlin(1959) for games of timing of class II on the unit square. For one of these cases, $P_i(0) = 0,\ P_i(1) = 1$ for $i = 1, 2$, we get $\alpha_i = \beta_i = 0$ for $i = 1, 2$, which coincides with the result of Lang & Kimeldorf(1976).

4.6. *General silent mixed duels.* Here we present the last result. It concerns the duel $\Gamma_{ss}(1, \hat{M}_2)$ with one indivisible action for Player 1 and divisible resources of amount M_2 for Player 2. In Radzik (1988*b*) that game was completely solved. We preserve all the notation and assumptions made in the previous subsection. So the normal form of our duel is

$$\Gamma_{ss}(1, \hat{M}_2) = < [0,1], \mathcal{M}_2^*\ , K >, \tag{15}$$

where the payoff kernel K is defined by $K(t, \mu_2) = K(I(t), \mu_2)$, and next,by (8) and (2).

THEOREM 4.6. *Assume (10) holds. Then the silent mixed duel $\Gamma_{ss}(1, \hat{M}_2)$ of the form (15) has a value. Player 1 has an optimal mixed strategy μ_1^* (a probability measure on $[0,1]$) and Player 2 has an optimal pure strategy $\mu_2^* \in \mathcal{M}_2^*$ of the form $\mu_i^* = (\alpha_i I_0\ , f_i^a, \beta_i I_1)$, $i = 1, 2$, with the exception that Player 2 has only ϵ-optimal strategy of that form when $P_2(1) = 1$. The parameters β_1 and β_2 satisfy: $\beta_1\beta_2 = 0$.*

REMARK 4.5. In Radzik(1988*b*) a complete characterization of optimal strategies is given, and the problem of their uniqueness is discussed.

5. OPEN PROBLEMS AND HYPOTHESIS IN THE GENERAL THEORY

The theory of games of timing is still lacking a coherent theory involving rather a rich set of different, more or less detailed models of duels studied in the literature. In this section we discuss some open problems related to such a theory.

In connection with the results presented in the previous section, some questions about the possibility of their further generalization arise. Namely, it is natural to ask in case of those four theorems how wide is the class of games of timing, for which the type of optimal strategies for the players found there still remains valid. It seems that those results are sufficiently general and rich to make some real hypothesis that could be seen a multi-dimensional extension of Glicksberg and Karlin's theorems on games of timing of classes I and II on the unit square (chapts 5 and 6 in Karlin (1959)).

At the beginning, we must introduce some notation and definitions. For fixed natural k and l, let the sets X and Y be of the form (12) and (13). Further, for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l$, let $Q_i(t)$, $R_i(t)$, $S_i(t)$ and $T_i(t)$ be functions defined on $[0, 1]$ with $S_i(t), T_i(t) \geq 0$.

For $\bar{x} \in X$ and $\bar{y} \in Y$ with $x_i \neq y_j$ for all i, j, let $\bar{z} = (z_1, z_2, \ldots, z_{k+l})$ denotes the vector whose components are $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_l$, rearranged in increasing order. Now, let

$$r(z_u) = \begin{cases} Q_i(x_i) & if \quad z_u = x_i \\ R_j(y_j) & if \quad z_u = y_j \end{cases} \qquad s(z_u) = \begin{cases} S_i(x_i) & if \quad z_u = x_i \\ T_j(y_j) & if \quad z_u = y_j \end{cases}$$

Finally, $\psi(\bar{z})$ is defined recursively as follows:

$$\begin{aligned} &\psi(z_1, z_2, \ldots, z_u) = r(z_1) + s(z_1)\psi(z_2, \ldots, z_u), \\ &\psi(z_2, \ldots, z_u) = r(z_2) + s(z_2)\psi(z_3, \ldots, z_u), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\psi(z_u) = r(z_u), \end{aligned}$$

for $u = 1, 2, \ldots, m + n$.

Now, let us define a basic game $\Gamma^* = < X,\ Y,\ K >$, with the payoff kernel $K = K(\bar{x}, \bar{y})$ satisfying the following three assumptions (D1)-(D3):

(D1) *For any $\bar{x} \in X$ and $\bar{y} \in Y$ with $x_i \neq y_j$ for all i and j, $K(\bar{x}, \bar{y}) = \psi(\bar{z}_{k+l})$, where*

(a) for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l$, the functions $Q_i(t)$, $R_j(t)$, $S_i(t)$ and $T_j(t)$ are continuous on $[0, 1]$, and $S_i(t)$ and $T_j(t)$ are nonnegative;

(b) for $u = 1, 2, \ldots, k + l$ the functions $\psi(z_1, z_2, \ldots, z_u)$ are strictly increasing or strictly decreasing in variable z_i in the interval (z_{i-1}, z_{i+1}) if $z_i = x_t$ or $z_i = y_t$ for some t, respectively, $i = 1, 2, \ldots, u$;

(D2) *for any $\bar{x} \in X$ and $\bar{y} \in Y$ with $x_1 = \ldots = x_s = 0$ and $y_1 = \ldots = y_u = 0$, there hold the inequalities:*

$$(16) \quad K(\bar{x}, \bar{y}) \geq \lim_{x_s \to 0+} K(\bar{x}, \bar{y}) \geq \lim_{(x_{s-1}, x_s) \to (0+, 0+)} K(\bar{x}, \bar{y}) \geq \\ \ldots \geq \lim_{(x_1, x_2, \ldots, x_s) \to (0+, 0+, \ldots, 0+)} K(\bar{x}, \bar{y}),$$

$$(17)\quad K(\bar{x},\bar{y}) \le \lim_{y_u\to 0^+} K(\bar{x},\bar{y}) \le \lim_{(y_{u-1},y_u)\to(0^+,0^+)} K(\bar{x},\bar{y}) \le \dots \le \lim_{(y_1,y_2,\dots,y_u)\to(0^+,0^+,\dots,0^+)} K(\bar{x},\bar{y});$$

(D3) *for any $\bar{x} \in X$ and $\bar{y} \in Y$ with $x_a = \ldots = x_k = 1$ and $y_b = \ldots = y_l = 1$, there hold the inequalities:*

$$(18)\quad K(\bar{x},\bar{y}) \le \lim_{x_a\to 1^-} K(\bar{x},\bar{y}) \le \lim_{(x_a,x_{a+1})\to(1^-,1^-)} K(\bar{x},\bar{y}) \le \dots \le \lim_{(x_a,x_{a+1},\dots,x_k)\to(1^-,1^-,\dots,1^-)} K(\bar{x},\bar{y}),$$

$$(19)\quad K(\bar{x},\bar{y}) \ge \lim_{y_b\to 1^-} K(\bar{x},\bar{y}) \ge \lim_{(y_b,y_{b+1})\to(1^-,1^-)} K(\bar{x},\bar{y}) \ge \dots \ge \lim_{(y_b,y_{b+1},\dots,y_l)\to(1^-,1^-,\dots,1^-)} K(\bar{x},\bar{y}).$$

In spite of the fact that the payoff kernel of the our basic game Γ^* does not have the most general form (it is of a recursive form), almost all the models of discrete duels studied in the literature, have their basic games as detailed subcases of Γ^*. On the other hand, assumptions (D1)-(D3) can be seen as only a slightly simplified version of multi-dimensional generalization of assumptions (a)-(c) from Sect. 5.2 of Karlin (1959)), given for games of timing of class II on the unit square (after adding the differentiability of $Q_i(t)$, $R_j(t)$, $S_i(t)$ and $T_j(t)$). So, these two arguments sufficiently motivate the importance of game Γ^* defined above.

Now, we are ready to present the first two problems that generalize all the studied (hitherto) models of discrete duels with more than 1 action for each player. Namely, at first, consider the general game of timing $\Gamma^*_{nn}(k,l)$ with Γ^* as its basic game. The results of paper Radzik (1991)), cited in the previous section, strongly suggest the validity of the following conjecture.

CONJECTURE 5.1. *Under assumptions (D1)-(D3), the noisy discrete game of timing $\Gamma^*_{nn}(k,l)$ has a value, and there are ϵ-optimal strategies for the players in this game that are of the form constructed in Theorem 4.3.*

The second conjecture concerns the game of timing $\Gamma^*_{ss}(k,l)$ with Γ^* as its basic game. It can be expected by the results of the papers Restrepo (1957) and Ciegielski (1986*b*), that optimal strategies for the players will be of the form slightly extended in comparison to that of found there. To describe it, we need some new notation.

Denote the mixed strategies of Players 1 and 2 in $\Gamma^*_{ss}(k,l)$ by F and G, respectively.

DEFINITION 5.1. *We say that the strategy F belongs to the class $\mathcal{F}^*_k$ if:*
(i) *$F(\bar{x}) = \prod_{i=1}^{k} F_i(x_i)$, where F_i is a probability distribution of x_i;*

(ii) *there are some* $p, 0 \leq p \leq k$ *and* $0 \leq a_1 < a_2 < \ldots < a_q \leq a_{q+1} = a_{q+2} = \ldots = a_{k-p+1} = 1$ *for some* $q, 1 \leq q \leq k-p+1$, *such that*

(1) $suppF_i = \{0\}$ *for* $i = 1, 2, \ldots, p$,

(2) $suppF_{p+1} \supseteq [a_1, a_2]$ *and* $suppF_{p+1} \subseteq \{0\} \cup [a_1, a_2]$,

(3) $suppF_{p+j} = [a_j, a_{j+1}], j = 2, 3, \ldots, k-p$;

(iii) *all* F_i *are absolutely continuous on* $(0,1)$.

In a similar way, we define the class $\mathcal{G}_l^*$ *with* l, r, s *and* $b_j's$ *instead of* k, p, q *and* $a_i's$, *respectively.*

So, any mixed strategy F from the class $\mathcal{F}_k^*$ prescribes Player 1 to behave in the following way. Some number p of his first actions should be taken at moment $t = 0$ with probability 1. His $(p+1)$-th action should be taken with some probability at $t = 0$ while with the rest probability in the time interval $[a_1, a_2]$ in a continuous way. Next, Player 1 takes $q - 1$ (for some q) his successive actions continuously in intervals $[a_2, a_3], \ldots, [a_q, a_{q+1}]$, respectively, with the possible jump at $t = 1$ for the last action in this group. The rest of his actions Player 1 should take at $t = 1$ with probability 1.

CONJECTURE 5.2. *Assume that (D1)-(D3) are satisfied and additionally, let all the functions* $Q_i(t)$, $R_i(t)$, $S_i(t)$ *and* $T_i(t)$ *be absolutely continuous. Then, the silent discrete game of timing* $\Gamma_{ss}^*(k,l)$ *has a value, and there are optimal mixed strategies* $F \in \mathcal{F}_k^*$ *and* $G \in \mathcal{G}_l^*$ *for the players in this game such that* $a_1 = b_1$ *and* F *or* G *is continuous at* $t = 1$.

The third conjecture concerns a generalization of silent non-discrete games of timing of type $\Gamma_{ss}(\hat{M}_1, \hat{M}_2)$ and is strongly suggested by the result of Theorem 4.5. To express it rigorously, we must introduce the next definition.

DEFINITION 5.2. *Let* $F(\mu)$ *be a function defined on a set measures* μ *on* $[0,1]$, *and let* S *be a subinterval of* $[0,1]$. *The function* F *is said to be be* S*-increasing (*S*-decreasing) if for all measures* μ *and for every increasing function* $f : [0,1] \mapsto [0,1]$ *such that* $f : S \mapsto S, f(x) \geq x$ *for* $x \in S$, *and* $f(x) = x$ *for* $x \notin S$,

$$F(\mu) > F(\mu f^{-1}) \qquad (F(\mu) < F(\mu f^{-1})),$$

if only $\mu \neq \mu f^{-1}$. :

Now, for fixed $M_1, M_2 > 0$, let us consider a new basic game $\hat{\Gamma}^* = < \hat{\mathcal{M}}_1, \hat{\mathcal{M}}_2, K >$, where, for $i = 1, 2$, $\hat{\mathcal{M}}_i$ is the set of all measures μ_i with $\mu([0,1]) \leq M_i$, and the payoff kernel $K = K(\mu_1, \mu_2)$ satisfies the following five assumptions (E1)-(E5):

(E1) *for any two measures* $\mu_1 \in \hat{\mathcal{M}}_1$ *and* $\mu_2 \in \hat{\mathcal{M}}_2$ *having no common discontinuity points in* $[0,1]$, *the functions* $K(\,\cdot\,, \mu_2)$ *and* $K(\mu_1, \,\cdot\,)$ *are continuous at the points* μ_1 *and* μ_2, *respectively (in the topology of weak convergence);*

(E2) *Let* $\gamma > 0$*,* $\mu_2 \in \hat{\mathcal{M}}_2$ *and* $\nu, \nu_m \in \hat{\mathcal{M}}_1$ *with* $\nu + \nu_m \in \hat{\mathcal{M}}_1$*, for* $m = 1, 2, \ldots$. *Then*

$$\lim_{m\to\infty} K(\nu + \nu_m, \mu_2) \leq K(\nu + \gamma I_0, \mu_2) \qquad \text{as } \nu_m \to \gamma I_0,$$
$$\lim_{m\to\infty} K(\nu + \nu_m, \mu_2) \geq K(\nu + \gamma I_1, \mu_2) \qquad \text{as } \nu_m \to \gamma I_1;$$

(E3) *Let* $\gamma > 0$*,* $\mu_1 \in \hat{\mathcal{M}}_1$ *and* $\nu, \nu_m \in \hat{\mathcal{M}}_2$ *with* $\nu + \nu_m \in \hat{\mathcal{M}}_2$*, for* $m = 1, 2, \ldots$. *Then*

$$\lim_{m\to\infty} K(\mu_1, \nu + \nu_m) \geq K(\mu_1, \nu + \gamma I_0) \qquad \text{as } \nu_m \to \gamma I_0,$$
$$\lim_{m\to\infty} K(\mu_1, \nu + \nu_m) \leq K(\mu_1, \nu + \gamma I_1) \qquad \text{as } \nu_m \to \gamma I_1;$$

(E4) *for any* $\mu_1 \in \hat{\mathcal{M}}_1$ *and* $\mu_2 \in \hat{\mathcal{M}}_2$*, and for all* $[a, b] \subset [0, 1] \setminus \mathrm{supp}\mu_2$ *and* $[c, d] \subset [0, 1] \setminus \mathrm{supp}\mu_1$*, the functions* $K(\,\cdot\,, \mu_2)$ *and* $K(\mu_1, \,\cdot\,)$ *are* $[a, b]$*-increasing and* $[c, d]$*-decreasing, respectively;*

(E5) *the payoff kernel* $K(\mu_1, \mu_2)$ *is concave in* μ_1 *and convex in* μ_2*.*

All the assumptions (E1)-(E5) proposed above are satisfied by the silent non-discrete duel $\Gamma_{ss}(\hat{M}_1, \hat{M}_2)$ considered in Theorem 4.5 under $P_i(1) < 1$ for $i = 1, 2$, and they are essential there. On the other hand, these assumptions can be seen as a counterpart of the assumptions made for games of timing of class II in Sect. 5.2 in Karlin (1959). Therefore, one can expect that the result of that theorem will also be true under more general assumptions.

CONJECTURE 5.3. *Assume that* $\hat{\Gamma}^* = < \hat{\mathcal{M}}_1,\ \hat{\mathcal{M}}_2,\ K >$ *satisfies (E1)-(E5) and is the basic game of a silent non-discrete game of timing* $\Gamma^*_{ss}(\hat{M}_1, \hat{M}_2)$*. Then* $\Gamma^*_{ss}(\hat{M}_1, \hat{M}_2)$ *has a value, and there are pure optimal strategies* μ_1^* *for Player 1 and* μ_2^* *for Player 2 of the form* $\mu_i^* = (\alpha_i I_0\ , f_i^a, \beta_i I_1)$*,* $i = 1, 2$*, in this game. The parameters* β_1 *and* β_2 *satisfy:* $\beta_1 \beta_2 = 0$*.*

The last conjecture concerns silent mixed games of type $\Gamma_{ss}(1, \hat{M}_2)$, and is suggested by Theorem 4.6 under $P_2(1) < 1$. It is quite analogous to Conjecture 3. We preserve all the notation introduced before.

For a fixed $M_2 > 0$, consider a new basic game $\bar{\Gamma}^* = < [0, 1],\ \hat{\mathcal{M}}_2,\ K >$, where $\hat{\mathcal{M}}_2$ is the set of all measures μ_2 with $\mu_2([0, 1]) \leq M_2$, and the payoff kernel $K = K(x, \mu_2)$ satisfies the following assumptions (F1)-(F5):

(F1) *for any* $x \in [0, 1]$ *and measure* $\mu_2 \in \hat{\mathcal{M}}_2$ *continuous at* x*, the functions* $K(\,\cdot\,, \mu_2)$ *and* $K(x, \,\cdot\,)$ *are continuous at point* x *and measure* μ_2*, respectively;*

(F2) *For all* $\mu_2 \in \hat{\mathcal{M}}_2$

$$\lim_{x\to\, 0^+} K(x, \mu_2) \leq K(0, \mu_2),$$
$$\lim_{x\to\, 1^-} K(x, \mu_2) \geq K(1, \mu_2);$$

(F3) *Let* $\gamma > 0$ *and* $\nu, \nu_m \in \hat{\mathcal{M}}_2$ *with* $\nu + \nu_m \in \hat{\mathcal{M}}_2$ *for* $m = 1, 2, \ldots$. *Then*

$$\lim_{m\to\infty} K(0, \nu + \nu_m) \geq K(0, \nu + \gamma I_0), \qquad \text{as } \nu_m \to \gamma I_0,$$
$$\lim_{m\to\infty} K(1, \nu + \nu_m) \leq K(1, \nu + \gamma I_1), \qquad \text{as } \nu_m \to \gamma I_1;$$

(**F4**) *for any measure* $\mu_2 \in \hat{\mathcal{M}}_2$ *the function* $K(\,\cdot\,, \mu_2)$ *is strictly increasing in any subinterval of* $[0,1]$ *disjoint with supp*μ_2*;*
for any $0 \le t \le 1$*, the function* $K(t,\,\cdot\,)$ *is* $[0,t)$ *and* $(t,1]$*-decreasing;*

(**F5**) *the payoff kernel* $K(t,\mu_2)$ *is convex in* μ_2.

CONJECTURE 5.4. *Assume that* $\bar{\Gamma}^* = < [0,1],\ \hat{\mathcal{M}}_2,\ K >$ *satisfies (F1)-(F5) and is the basic game of a silent mixed game of timing* $\bar{\Gamma}^*_{ss}(1,\hat{M}_2)$*. Then* $\bar{\Gamma}^*_{ss}(1,\hat{M}_2)$ *has a value, and there is a mixed optimal strategy* μ_1^* *for Player 1 and a pure optimal strategy* μ_2^* *for Player 2 of the form* $\mu_i^* = (\alpha_i I_0\ , f_i^a, \beta_i I_1)$*,* $i = 1,2$*, in this game. The parameters* β_1 *and* β_2 *satisfy:* $\beta_1\beta_2 = 0$.

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