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THE GAMBLER AND THE STOPPER[1]

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Abstract

A gambler (or stochastic controller) selects the distribution for the stochastic process $x, X_1, X_2, \ldots$ from those available in a given gambling house. An optimal stopper selects a stop rule t and pays the gambler the expected value of $u(X_t)$, where u is a bounded, real-valued function. Under certain measurability assumptions, this game has a value and there is a transfinite algorithm for calculating it.

1 Introduction

Suppose a gambler begins play with fortune x in the state space S. The gambler selects a strategy σ from those available in the gambling house Γ and thereby determines the distribution of the process of fortunes $x, X_1, X_2, \ldots$ on S. In the classical Dubins and Savage theory, the gambler would also select a stop rule t and receive as reward the expected value of $u(X_t)$, where u is a bounded, real-valued utility function. However, we assume that the stop rule is chosen by a second player, called the stopper, who seeks to minimize the gambler's reward.

Under measurability conditions on S, Γ, u, σ, and t which are specified in the next section, we show that this two-person, zero-sum game has a value and we give a transfinite algorithm for calculating the value. Technical difficulties arise in the proof largely because the set of stop rules is a complicated set for which there seems to be no nice measurable structure when S is uncountable. These difficulties are surmounted by the use of effective descriptive set theory. The effective theory allows us to replace the set of stop rules at each state x by a countable set of recursive stop rules.

The gambler and stopper game is related to the "leavable games" studied in [9], [10], and [11]. In the special case where S is countable, the fact that the gambler and the stopper game has a value follows from Theorem 4.7 of [11].

The next section is devoted to definitions and preliminaries. Section 3 presents the effective theory we need to prove the main results which are in Section 4. In Section 5 an application is given to gambling problems in which the gambler's reward is the expected value of $\liminf_n u(X_n)$.

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2 Definitions and Notation

Let S be a nonempty Borel subset of a Polish space and let $\mathbf{P}(S)$ be the collection of countably additive probability measures defined on the Borel subsets of S. Give $\mathbf{P}(S)$ its usual weak topology so that it too has the structure of a Borel subset of a Polish space (see Parthasarathy [14] for information about the weak topology on $\mathbf{P}(S)$). An *analytic gambling house* Γ is a mapping which assigns to each $x \in S$ a nonempty subset $\Gamma(x)$ of $\mathbf{P}(S)$ in such a way that the set

$$\Gamma = \{(x, \gamma) \in S \times \mathbf{P}(S) : \gamma \in \Gamma(x)\}$$

is analytic.

For each $n \geq 1$, equip S^n with the σ-field generated by the analytic subsets of S^n. Functions measurable with respect to this σ-field will be called *analytically measurable*. Starting at some initial state $x \in S$, a gambler in the house Γ chooses an *analytically measurable strategy σ available at x*, which means a sequence $\sigma = (\sigma_0, \sigma_1, \ldots)$, where $\sigma_0 \in \Gamma(x)$ and, for $n \geq 1$, σ_n is an analytically measurable function from S^n to $\mathbf{P}(S)$ such that $\sigma_n(x_1, x_2, \ldots, x_n) \in \Gamma(x_n)$ for every $(x_1, x_2, \ldots, x_n) \in S^n$. Every analytically measurable strategy σ determines a probability measure, also denoted by σ, on the Borel subsets of the space H of histories:

$$H = S \times S \times \cdots .$$

The probability measure σ can be regarded as the distribution of the coordinate process $h = (h_1, h_2, \ldots)$, where h_1 has distribution σ_0 and h_{n+1} has conditional distribution $\sigma_n(x_1, x_2, \ldots, x_n)$ given that $h_1 = x_1, h_2 = x_2, \ldots, h_n = x_n$. For $x \in S$, let $\Sigma(x)$ be the set of all analytically measurable strategies available at x. Set

$$\Sigma = \{(x, \sigma) \in S \times \mathbf{P}(H) : \sigma \in \Sigma(x)\}.$$

It is known that Σ is an analytic subset of $S \times \mathbf{P}(H)$, where H is given the product of copies of the topology on S and $\mathbf{P}(H)$ the usual weak topology. (See Dellacherie [1] and Sudderth [15].) The reason for considering analytically measurable strategies is that there may not be a Borel measurable selector for Γ, so that it is possible for the set of Borel measurable strategies to be vacuous. (A *selector for* Γ is a function $\phi : S \to \mathbf{P}(S)$ such that $\phi(x) \in \Gamma(x)$ for all $x \in S$.) In case Γ does admit a Borel measurable selector, we can restrict the gambler to Borel measurable strategies without changing Σ.

Suppose that $\sigma \in \Sigma(x)$ and $p \in S^m$. We define the *conditional strategy* $\sigma[p]$ as follows:

$$\sigma[p]_0 = \sigma_m(p)$$

and, for $n \geq 1$,

$$\sigma[p]_n(x_1, x_2, \ldots, x_n) = \sigma_{m+n}(px_1x_2 \ldots x_n),$$

where $px_1x_2 \ldots x_n$ is the element of S^{m+n} obtained by catenating p and $(x_1, x_2, \ldots, x_n)$. It is easily checked that $\sigma[p] \in \Sigma((p)_m)$, where $(p)_m$ is the mth coordinate of p. Furthermore, the measures (induced by) $\sigma[p]$, $p \in S^m$, are versions of the conditional σ-distributions of $(h_{m+1}, h_{m+2}, \ldots)$ given that $(h_1, h_2, \ldots, h_m) = p$. If g is a bounded, analytically measurable function on H, gp will denote the section of g at p, that is

$$gp(h) = g(ph), \quad h \in H,$$

where ph is the history obtained by catenating p and $h = (h_1, h_2, \ldots)$. Plainly, gp is an analytically measurable function on H. The usual formula stating that the expectation of a conditional expectation is the expectation takes the form

$$\int g d\sigma = \int [\int (gp_m(h))(h')\, \sigma[p_m(h)](dh')]\, \sigma(dh), \tag{2.1}$$

where $p_m(h) = (h_1, h_2, \ldots, h_m)$.

A function t on H to ω, the set of natural numbers, is a *stop rule* if

$$t(h) = n \text{ and } h \equiv_n h' \rightarrow t(h') = n,$$

where $h \equiv_n h'$ means that the histories h and h' agree through the first n coordinates. If t is a stop rule such that $t(h) = 0$ for some h, then it follows from the definition that t is identically zero. Stop rules which are not identically zero will be called *proper* stop rules. Let $\mathcal{T}$ be the set of Borel measurable stop rules and $\mathcal{T}_1$ the set of Borel measurable stop rules that are proper.

If $t \in \mathcal{T}_1$, we define p_t as the function on H whose value at h is the finite sequence $(h_1, h_2, \ldots, h_{t(h)})$. Next we define, for an analytically measurable strategy σ, $\sigma[p_t]$ as the function whose value at h is $\sigma[p_t(h)]$. Formula (2.1) extends to stop rules t as follows:

$$\int g d\sigma = \int [\int (gp_t(h))(h')\, \sigma[p_t(h)](dh')]\, \sigma(dh). \tag{2.2}$$

Formula (2.2) remains true for $t \equiv 0$ if we define $p_t(h)$ to be the empty sequence and $\sigma[p_t(h)] = \sigma$ for each $h \in H$.

If $p = (x_1, x_2, \ldots, x_m) \in S^m$ and $t \in \mathcal{T}$, we define $t[p]$ on H by

$$t[p](h) = t(x_1, x_2, \ldots, x_m, h_1, h_2, \ldots) - m.$$

Note that, if $t(x_1, x_2, \ldots, x_m, \ldots) \geq m$, then $t[p] \in \mathcal{T}$. When $p = (x)$, we write $t[x]$ for $t[p]$.

There is a natural way to associate with every $t \in \mathcal{T}$ an ordinal number $j(t)$, called the *index* of t, by setting $j(t) = 0$ if t is the identically zero stop rule and requiring that

$$j(t) = \sup\{j(t[x]) + 1 : x \in S\}$$

for every $t \in \mathcal{T}_1$. The concept of index was introduced by Dellacherie and Meyer [3].

We say that $\sigma(x)$, $x \in S$, is a *measurable family of strategies* if $\sigma(x) \in \Sigma(x)$ for every $x \in S$ and, for every $n \geq 0$, $\sigma(x)_n(x_1, x_2, \ldots, x_n)$ is an analytically measurable function from $S \times S^n$ to $\mathbb{P}(S)$. A measurable family $\sigma(x)$, $x \in S$, of strategies is said to be *Markov* if for each $n \geq 0$ there is an analytically measurable selector $\tilde{\gamma}_n$ for Γ such that

$$\sigma(x)_n(x_1, x_2, \ldots, x_n) = \tilde{\gamma}_n(x_n)$$

for all $n \geq 0$ and $x, x_1, x_2, \ldots x_n \in S$.

We define the operator Γ^1 on bounded, analytically measurable functions v on S by setting

$$(\Gamma^1 v)(x) = \sup_{\gamma \in \Gamma(x)} \int v d\gamma, \quad x \in S.$$

Say that v is *excessive* (*deficient*) if $\Gamma^1 v \leq v$ ($\Gamma^1 v \geq v$). A function v is *invariant* if it is both excessive and deficient.

A real-valued function v on S is *upper analytic* if $\{x \in S : v(x) \geq a\}$ is analytic for every real number a. It is not hard to prove that v is upper analytic iff the set $\{(x, a) \in S \times \mathbb{R} : v(x) \geq a\}$ is analytic.

If v is a bounded, upper analytic function on S, $\sigma \in \Sigma(x_0)$ and $t \in \mathcal{T}_1$, formula (2.1) in the special case $g = v(h_{t(h)})$ and $m = 1$ becomes

$$\int v(h_t)\, d\sigma = \int [\int v(h'_{t[x]})\, \sigma[x](dh')]\, d\sigma_0(x) \tag{2.3}$$

where we write h_t for $h_{t(h)}$ and define $h'_{t[x]} = x$ when $t[x] \equiv 0$.

3 Effective Descriptive Set Theory

Effective descriptive set theory takes place in Polish spaces which admit a smooth recursion theory. This is made precise in the next definition.

We say that a topological space Z is Δ^1_1*-recursively presented* if Z admits a complete metric d and a dense sequence $(r_n)_{n \in \omega}$ such that the relations

$$d(r_n, r_m) \leq \frac{p}{q+1} \text{ and } d(r_n, r_m) < \frac{p}{q+1}$$

are Δ_1^1 in ω^4. Examples of such spaces are $\{0,1\}^\omega$, ω^ω, $[0,1]$, $[0,1]^\omega$, $\mathbb{R}$, etc. (See Moschovakis ([13], pp. 128–135).)

Suppose now that Z_1, Z_2 are Δ_1^1-recursively presented spaces. Then $Z_1 \times Z_2$ is Δ_1^1-recursively presented. If Z is a Δ_1^1-recursively presented compact metric space, then so is $\mathbb{P}(Z)$, the set of probability measures on Z. In what follows, our terminology and notation, pertaining to concepts in effective descriptive set theory, are taken from Moschovakis [13].

For the rest of this section, the state space S will be $\{0,1\}^\omega$ and the history space H is therefore $(\{0,1\}^\omega)^N$. Both these spaces are Δ_1^1-recursively presented, compact metric spaces, as are the spaces of probability measures on S and H.

We now need to describe a coding of Borel measurable functions on H to ω. To do this, fix a coding (W, C) of the Borel subsets of $H \times \omega$, that is,

(a) W is a Π_1^1 subset of $\omega^\omega \times \omega$;

(b) C is a Π_1^1 subset of $\omega^\omega \times \omega \times H \times \omega$;

(c) the set $\{(\alpha, n, h, m) \in \omega^\omega \times \omega \times H \times \omega : (\alpha, n) \in W \text{ and } (\alpha, n, h, m) \notin C\}$ is Π_1^1;

(d) for fixed $(\alpha, n) \in \omega^\omega \times \omega$, the section $C_{\alpha,n} = \{(h, m) \in H \times \omega : (\alpha, n, h, m) \in C\}$ is $\Delta_1^1(\alpha)$; and

(e) if $B \subseteq H \times \omega$ is $\Delta_1^1(\alpha)$, then there is $n \in \omega$ such that $B = C_{\alpha,n}$.

See Louveau (5, p. 13) for this coding.

We now define a partial function on $\omega^\omega \times \omega \times H$ as follows: U is defined at (α, n, h) (we write $U(\alpha, n, h) \downarrow$) iff there exists a unique m such that $(\alpha, n) \in W$ and $(\alpha, n, h, m) \in C$. If $U(\alpha, n, h) \downarrow$, set $U(\alpha, n, h) = m$ (described above). Consequently,

$$U(\alpha, n, h) \downarrow \leftrightarrow (\exists m)[(\alpha, n) \in W \text{ and } (\alpha, n, h, m) \in C \\ \text{and } (\forall k)(k \neq m \to (\alpha, n) \in W \text{ and } (\alpha, n, h, k) \notin C)]$$

Hence, the domain of U is Π_1^1. Moreover, on its domain, U is "computed" (in the sense of [13, p.175]) by C, so that U is a Π_1^1-recursive partial function. The following properties of U are easy to verify by using the analogous properties of the coding (W, C):

(i) for fixed $(\alpha, n) \in W$, $U(\alpha, n, \bullet)$ is a $\Delta_1^1(\alpha)$-recursive partial function; and

(ii) if g is a $\Delta_1^1(\alpha)$-recursive partial function from (a subset of) H to ω, then there is $n \in \omega$ such that $g = U(\alpha, n, \bullet)$. In this case, we say that (α, n) "codes" g.

Let $\underline{\underline{N}}$ be a gambling house on S and let $\underline{\underline{M}}$ be a mapping that assigns to each $x \in S$ a nonempty subset $\underline{\underline{M}}(x)$ of $\mathbb{P}(H)$. Assume that $\{(x, \gamma) \in S \times \mathbb{P}(S) : \gamma \in \underline{\underline{N}}(x)\}$ and $\{(x, \mu) \in S \times \mathbb{P}(H) : \mu \in \underline{\underline{M}}(x)\}$ are Σ_1^1 subsets of $S \times \mathbb{P}(S)$ and $S \times \mathbb{P}(H)$, respectively. Further, suppose that if $\mu \in \underline{\underline{M}}(x_0)$, then

1. $\mu_0 \in \underline{\underline{N}}(x_0)$, where μ_0 is the distribution of h_1 under μ; and
2. $\mu_0(\{x \in S : \mu[x] \in \underline{\underline{M}}(x)\}) = 1$, where $\mu[x]$ is a version of the regular conditional distribution of $(h_2, h_3, \ldots)$ given $h_1 = x$ under μ such that the function $(\mu, x) \to \mu[x]$ on $\mathbb{P}(H) \times S$ to $\mathbb{P}(H)$ is Borel measurable (see [6, Lemma 22]).

Let v be a bounded, nonnegative function on S such that the set

$$\{(x, a) \in S \times \mathbb{R}_+ : v(x) \geq a\}$$

is Σ_1^1.

Define, for $x \in S$,

$$R(x) = \inf \sup_{\mu \in \underline{\underline{M}}(x)} \int v(h_t)\, d\mu(h),$$

where the infimum is over all $\Delta_1^1(x)$-recursive stop rules on H; and

$$R'(x) = \inf \sup_{\mu \in \underline{\underline{M}}(x)} \int v(h_t)\, d\mu(h),$$

where the infimum is over all $\Delta_1^1(x)$-recursive, proper stop rules.

We will now establish some properties of the functions R and R'. In the sequel, we will use without explicit mention results of Kechris [4] on the evaluation of the level of the analytical hierarchy to which definable sets of probability measures belong. Also in what follows we think of S as being a Π_1^0 subset of ω^ω.

Let T be the set of codes of Borel measurable stop rules, that is,

$$T = \{(x, i) \in S \times \omega : U(x, i, \bullet) \text{ is a stop rule}\}.$$

Similarly, let

$$T' = \{(x, i) \in S \times \omega : U(x, i, \bullet) \text{ is a proper stop rule}\}.$$

Lemma 3.1 *T and T' are Π_1^1 subsets of $S \times \omega$.*

Proof. Observe that

$$(x,i) \in T \leftrightarrow x \in \omega^\omega \text{ and } (\forall h)(U(x,i,h)\downarrow)$$
$$\text{and } (\forall h)(\forall h')(\forall j)(U(x,i,h) = j \text{ and } h \equiv_j h' \to U(x,i,h') = j).$$

It now follows from the closure properties of the pointclass Π^1_1 and the properties of the coding U that T is a Π^1_1 set. One proves that T' is Π^1_1 similarly. □

Lemma 3.2 *The sets* $\{(x,a) \in S \times \mathbb{R}_+ : R(x) < a\}$ *and* $\{(x,a) \in S \times \mathbb{R}_+ : R'(x) < a\}$ *are both* Π^1_1.

Proof: Note that

$$R(x) < a \leftrightarrow (\exists r)(\exists n)[r \in \underline{\underline{Q}} \text{ and } r < a \text{ and } (x,n) \in T$$
$$\text{and } (\forall \mu)((x,\mu) \in \underline{\underline{M}} \to \int v(h_{U(x,n,\bullet)})\, d\mu(h) \leq r)],$$

where $\underline{\underline{Q}}$ is the set of nonnegative rationals. Once again it follows from the closure properties of the pointclass Π^1_1 and the equivalence above that the hypograph of R is Π^1_1.

A similar proof, with the set T replaced by T', works to show that the hypograph of R' is Π^1_1. □

The next result will be critical in the proof that the game defined in the Introduction has a value.

Theorem 3.3 *For each* $x \in S$,

$$R'(x) \leq \sup_{\gamma \in \underline{\underline{N}}(x)} \int R(x')\, d\gamma(x').$$

Proof. Let $E = \{(x,a) \in S \times \mathbb{R}_+ : R(x) \geq a\}$. By Lemma 3.2, E is Σ^1_1.

Fix $x_0 \in S$ and $\epsilon > 0$ rational. Since the set $\{\gamma \times \lambda : \gamma \in \underline{\underline{N}}(x_0)\}$, where λ is Lebesgue measure, is $\Sigma^1_1(x_0)$, it follows from [7, Lemma 4.3] that there is a $\Delta^1_1(x_0)$ subset B of $S \times \mathbb{R}_+$ such that $B \supseteq E$ and

$$\sup_{\gamma \in \underline{\underline{N}}(x_0)} (\gamma \times \lambda)(B) \leq \sup_{\gamma \in \underline{\underline{N}}(x_0)} (\gamma \times \lambda)(E) + \frac{\epsilon}{2} \tag{3.4}$$
$$= \sup_{\gamma \in \underline{\underline{N}}(x_0)} \int R(x)\, d\gamma(x) + \frac{\epsilon}{2}.$$

Set

$$g(x) = \lambda(B_x),\ x \in X.$$

Then

$$g(x) \geq \lambda(E_x) = R(x).$$

Moreover, g is a $\Delta_1^1(x_0)$-recursive function. From (3.4), we have

$$\sup_{\gamma \in \underline{\underline{N}}(x_0)} \int g(x)\, d\gamma(x) \leq \sup_{\gamma \in \underline{\underline{N}}(x_0)} \int R(x)\, d\gamma(x) + \frac{\epsilon}{2}. \tag{3.5}$$

Next, define a set $P \subseteq S \times \omega$ as follows:

$$(x, n) \in P \leftrightarrow$$
$$(x, n) \in T \text{ and } (\forall \mu \in \underline{\underline{M}}(x)) \left(\int v\left(h_{U(x,n,\bullet)}\right) d\mu(h) \leq g(x) + \frac{\epsilon}{2} \right).$$

Then P is $\Pi_1^1(x_0)$. It follows from the definition of R and the fact that $g \geq R$ that

$$(\forall x)(\exists n)((x, n) \in P).$$

By the Kreisel selection theorem ([13, p.203]), there is a $\Delta_1^1(x_0)$-recursive function $f : S \to \omega$ such that

$$(\forall x)((x, f(x)) \in P).$$

Define a stop rule t such that

$$t[x] = U(x, f(x), \bullet). \tag{3.6}$$

Then t is a $\Delta_1^1(x_0)$-recursive, proper stop rule. It follows that

$$R'(x_0) \leq \sup_{\mu \in \underline{\underline{M}}(x_0)} \int v(h_t)\, d\mu(h). \tag{3.7}$$

Now, for any $\mu \in \underline{\underline{M}}(x_0)$,

$$\begin{aligned} \int v(h_t)\, d\mu(h) &= \int \left[\int v\left(h'_{t[x]}\right) \mu[x](dh') \right] \mu_0(dx) \\ &\leq \int g(x)\, \mu_0(dx) + \frac{\epsilon}{2} \\ &\leq \sup_{\gamma \in \underline{\underline{N}}(x_0)} \int g(x)\, \gamma(dx) + \frac{\epsilon}{2} \\ &\leq \sup_{\gamma \in \underline{\underline{N}}(x_0)} \int R(x)\, \gamma(dx) + \epsilon, \end{aligned} \tag{3.8}$$

where the first inequality holds by virtue of the definition of t and the fact that $\mu[x] \in \underline{\underline{M}}(x)$ for almost all $(\mu_0)x$, the second inequality follows from the fact that $\mu_0 \in \underline{\underline{N}}(x_0)$ and the last inequality is by virtue of (3.5). Since (3.8) holds for every $\mu \in \underline{\underline{M}}(x_0)$ and ϵ is arbitrary, we have

$$\sup_{\mu \in \underline{\underline{M}}(x_0)} \int v(h_t)\, d\mu(h) \leq \sup_{\gamma \in \underline{\underline{N}}(x_0)} \int R(x)\, d\gamma(x). \tag{3.9}$$

Hence, from (3.7),

$$R'(x_0) \leq \sup_{\gamma \in \underline{\underline{N}}(x_0)} \int R(x)\, d\gamma(x).$$

□

4 The Gambler and Stopper Game

Let Γ be an analytic gambling house on a Borel subset S of a Polish space. Suppose that u is a bounded, upper analytic function on S. We now formulate the game described in the Introduction in precise terms. There are two versions of the game.

For each $x \in S$, $G(x)$ is the zero sum, two person game where the gambler (player I) chooses a strategy $\sigma \in \Sigma(x)$ and, simultaneously, the stopper (player II) chooses a stop rule $t \in \mathcal{T}$; the payoff from II to I is $\int u(h_t)\, d\sigma(h)$. The game $G'(x)$ is similar, except that the stopper is allowed to choose only proper stop rules.

Theorem 4.1 *Suppose that Q is a bounded, upper analytic, deficient (w.r.t. Γ) function such that $Q \leq u$. Then, for each $x \in S$*

(a) $\sup_{\sigma \in \Sigma(x)} \inf_{t \in \mathcal{T}} \int u(h_t)\, d\sigma(h) \geq Q(x)$,

(b) $\sup_{\sigma \in \Sigma(x)} \inf_{t \in \mathcal{T}_1} \int u(h_t)\, d\sigma(h) \geq (\Gamma^1 Q)(x)$.

Proof. Fix $\epsilon > 0$ and choose $\delta_n > 0$, $n \geq 0$, such that $\sum_{n=0}^{\infty} \delta_n < \epsilon$. By a well-known selection theorem (see, for example, [7, Lemma 2.1]), we can choose, for each $n \geq 0$, an analytically measurable selector $\tilde{\gamma}_n : S \to \mathbb{P}(S)$ for Γ such that

$$\int Q\, d\tilde{\gamma}_n(x) \geq (\Gamma^1 Q)(x) - \delta_n \tag{4.2}$$

for each $x \in S$.

For each $m \geq 0$, let

$$(\sigma^m(x))_0 = \tilde{\gamma}_m(x),$$

and, for $n \geq 1$,

$$(\sigma^m(x))_n(x_1, x_2, \dots, x_n) = \tilde{\gamma}_{m+n}(x_n).$$

Then, for each $m \geq 0$, $\sigma^m(x)$, $x \in S$, is a Markov family of strategies. We will now prove by induction on the index $j(t)$ that

$$\int u(h_t)\,\sigma^m(x)(dh) \geq Q(x) - \sum_{n=m}^{\infty} \delta_n \tag{4.3}$$

for all $m \geq 0$, $x \in S$ and $t \in \mathcal{T}$.

If $j(t) = 0$, t is the improper stop rule, so that

$$\begin{aligned}\int u(h_t)\,\sigma^m(x)(dh) &= u(x)\\ &\geq Q(x)\\ &\geq Q(x) - \sum_{n=m}^{\infty} \delta_n.\end{aligned}$$

Now suppose that $j(t) > 0$ and that (4.3) is true for all $s \in \mathcal{T}$ such that $j(s) < j(t)$. Then

$$\begin{aligned}\int u(h_t)\,\sigma^m(x)(dh) &= \int \left[\int u\left(h'_{t[x_1]}\right) \sigma^m(x)[x_1](dh')\right] \tilde{\gamma}_m(x)(dx_1)\\ &= \int \left[\int u\left(h'_{t[x_1]}\right) \sigma^{m+1}(x_1)(dh')\right] \tilde{\gamma}_m(x)(dx_1)\\ &\geq \int Q(x_1)\,\tilde{\gamma}_m(x)(dx_1) - \sum_{n=m+1}^{\infty} \delta_n\\ &\geq (\Gamma^1 Q)(x) - \sum_{n=m}^{\infty} \delta_n\\ &\geq Q(x) - \sum_{n=m}^{\infty} \delta_n,\end{aligned} \tag{4.4}$$

where the first inequality is by virtue of the inductive hypothesis, the second is by (4.2) and the final equality follows from the fact that Q is deficient. This proves (4.3).

Let $\sigma(x) = \sigma^0(x)$, $x \in S$. Then, by (4.3),

$$\inf_{t \in \mathcal{T}} \int u(h_t)\,\sigma(x)(dh) \geq Q(x) - \epsilon$$

for every $x \in S$. As ϵ is arbitrary, we have established assertion (a).

An inspection of (4.4) shows that

$$\inf_{t \in \mathcal{T}_1} \int u(h_t)\, \sigma(x)(dh) \geq (\Gamma^1 Q)(x) - \epsilon$$

for every $x \in S$. As ϵ is arbitrary, this proves (b). □

Theorem 4.5 *(a) For each $x \in S$, the game $G(x)$ has a value $L(x)$.*

(b) The function L is upper analytic.

(c) For each $\epsilon > 0$, the gambler (player I) has an ϵ-optimal Markov family of strategies in the games $G(x)$, $x \in S$.

(d) L is the largest bounded, upper analytic, deficient function Q such that $Q \leq u$.

Proof. The case where X is countable is easy and we omit the proof. So suppose that S is uncountable. By the Borel isomorphism theorem, we may further assume that $S = \{0,1\}^\omega$. Also without loss of generality, we can assume that $u \geq 0$.

Since Γ, Σ and the epigraph of u are all analytic sets, we can choose $\alpha \in \omega^\omega$ such that these sets are $\Sigma_1^1(\alpha)$ (see [13, 3E.4]). Let $v = u$, $\underline{\underline{N}} = \Gamma$, $\underline{\underline{M}} = \Sigma$ and let

$$R(x) = \inf \sup_{\mu \in \underline{\underline{M}}(x)} \int v(h_t)\, d\mu(h), \quad x \in S,$$

where the infimum is over all $\Delta_1^1(\alpha, x)$-recursive stop rules on H.

We now relativize Lemma 3.2 and Theorem 3.3 to α. It follows that R is a bounded, upper analytic, deficient function such that $R \leq u$. Consequently, for each $x \in S$,

$$\begin{aligned} \inf_{t \in \mathcal{T}} \sup_{\sigma \in \Sigma(x)} \int u(h_t)\, d\sigma(h) &\leq R(x) \\ &\leq \sup_{\sigma \in \Sigma(x)} \inf_{t \in \mathcal{T}} \int u(h_t)\, d\sigma(h), \end{aligned}$$

where the first inequality holds because every $\Delta_1^1(x, \alpha)$-recursive stop rule is Borel measurable, and the second is by virtue of Theorem 4.1. This establishes that the game $G(x)$ has value $R(x)$, so that assertions (a) and (b) are proved. Assertion (c) is true because, as in the proof of Theorem 4.1, a Markov family of strategies $\sigma(x)$, $x \in S$, can be constructed, for each $\epsilon > 0$, such that

$$\inf_{t \in \mathcal{T}} \int u(h_t)\, \sigma(x)(dh) \geq R(x) - \epsilon$$

for each $x \in S$. Finally, (d) follows from Theorem 4.1 and the fact that R is the value function of the games $G(x)$, $x \in S$. □

The analogous result for the games $G'(x)$, $x \in S$, is as follows.

Theorem 4.6 *(a) For each $x \in S$, the game $G'(x)$ has value $(\Gamma^1 L)(x)$.*

(b) For each $\epsilon > 0$, the gambler (player I) has an ϵ-optimal Markov family of strategies in the games $G'(x)$, $x \in S$.

(c) $L = (\Gamma^1 L) \wedge u$, where the right side is the pointwise minimum of the functions $\Gamma^1 L$ and u.

Proof. Assertions (a) and (b) are proved in a manner similar to the proof of Theorem 4.5. To prove (c), use Theorem 4.5(d) to see that $L \leq (\Gamma^1 L) \wedge u$. Suppose now that $L(x) < (\Gamma^1 L)(x)$ for some $x \in S$. In this case, the stop rule $t \equiv 0$ is clearly optimal for player II in $G(x)$, so $L(x) = u(x)$. On the other hand, if $L(x) < u(x)$, then there is no incentive for player II to choose the stop rule $t \equiv 0$ in the game $G(x)$. Indeed, he is better off choosing a stop rule $t \geq 1$. But then $L(x) = (\Gamma^1 L)(x)$, which completes the proof. □

Theorem 4.6(c) suggests how we may calculate the functions L and $\Gamma^1 L$. Towards this end, define an operator T on bounded, analytically measurable functions on S as follows:

$$Tv = (\Gamma^1 v) \wedge u.$$

Note that L is the largest bounded, upper analytic fixed point of T. It follows, courtesy of a result of Moschovakis in the theory of inductive definability [13, 7C.8], that L can be calculated in accordance with the following transfinite scheme:

Let

$$L_0 = u$$

and for each ordinal $\xi > 0$,

$$L_\xi = T\left(\inf_{\eta<\xi} L_\eta\right).$$

Then

$$L = L_{\omega_1},$$

where ω_1 is the first uncountable ordinal.

Similarly, one can define an operator T' on bounded, analytically measurable functions on S, thus

$$T'v = \Gamma^1(v \wedge u).$$

It is not difficult to see that $\Gamma^1 L$ is the largest, bounded, upper analytic fixed point of T'. So, by the result of Moschovakis cited above, $\Gamma^1 L$ can be calculated as follows:

$$L'_0 = T'u$$

and for each ordinal $\xi > 0$,

$$L'_\xi = T'\left(\inf_{\eta<\xi} L'_\eta\right).$$

Then

$$\Gamma^1 L = L'_{\omega_1}.$$

We conclude this section with some remarks.

First, it is easy to make up examples of games $G(x)$ (or $G'(x)$) where neither player has an optimal strategy.

Second, there are gambling and dynamic programming problems for which the optimal reward function can be realized as the value function of the games $G'(x)$, $x \in S$. In gambling, an example is provided by the problem where the gambler wishes to maximize the probability of staying in a set forever (see [7] for details). The optimal reward function for negative dynamic programming can also be realized as the value function of the games $G'(x)$, $x \in X$, though to do so we have to relax the condition that u is bounded below. This can be done because of the special nature of u in negative dynamic programming (see [8]).

5 The Liminf Gambling Problem

In this section, results of the previous section will be used to calculate the optimal reward function of the gambler whose aim is to maximize the expected value of the liminf of utilities evaluated along histories. The liminf gambling problem was introduced by Sudderth [17].

Let, then, S, Γ and u be as described in the first paragraph of the previous section. Define u_* on H by

$$u_*(h) = \liminf_n u(h_n).$$

Regard $u_*(h)$ as the payoff to the gambler when he experiences the history h. The aim of the gambler is to choose a strategy $\sigma \in \Sigma(x)$ so as to maximize his expected payoff $\int u_* d\sigma$. The optimal reward function for this problem is therefore

$$W(x) = \sup_{\sigma\in\Sigma(x)} \int u_* d\sigma, \quad x \in S.$$

Since Σ is analytic, W is an upper analytic function. Our calculation of W will be based on the following result. Recall that $\Gamma^1 L$ is the value function of the games $G'(x)$, $x \in S$.

Theorem 5.1 *For each $x \in S$,*

$$W(x) = \sup_{\pi} \int (\Gamma^1 L)(h_t)\, d\sigma(h), \tag{5.2}$$

where the supremum is taken over all measurable policies $\pi = (\sigma, t)$ available at x, that is, over all (σ, t) such that $\sigma \in \Sigma(x)$ and $t \in \mathcal{T}$.

Proof. Fix $x_0 \in X$ and $\epsilon > 0$. Choose a policy (σ^0, s) available at x_0 such that

$$\int (\Gamma^1 L)(h_s)\, d\sigma^0(h) \geq Q(x_0) - \frac{\epsilon}{2}, \tag{5.3}$$

where Q denotes the function on the right side of (5.2). Let $\sigma^1(x)$, $x \in X$, be an $\epsilon/2$-optimal family of strategies for the gambler in the games $G'(x)$, $x \in X$. Define $\sigma \in \Sigma(x_0)$ as follows:

$$\sigma_0 = \sigma_0^0$$

and for $n \geq 1$

$$\begin{aligned} \sigma_n(h_1, h_2, \ldots, h_n) &= \sigma_n^0(h_1, h_2, \ldots, h_n), \quad \text{if } n < s(h_1, h_2, \ldots, k_n, \ldots) \\ &= \sigma_{n-s}^1(h_{s+1}, h_{s+2}, \ldots, h_n), \quad \text{otherwise.} \end{aligned}$$

Thus, σ is the strategy that starts out by following σ^0 and then switches to σ^1 at time s. Now calculate as follows:

$$\begin{aligned} \int u_* d\sigma &= \int [\int u_*(h')\, \sigma[p_s(h)](dh')]\, d\sigma(h) \\ &= \int [\int u_*(h')\, \sigma^1(h_s)(dh')]\, d\sigma^0(h). \\ &\geq \int \inf_{t \in \mathcal{T}'} [\int u(h'_t)\, \sigma^1(h_s)(dh')]\, d\sigma^0(h) \\ &\geq \int (\Gamma^1 L)(h_s)\, d\sigma^0(h) - \frac{\epsilon}{2} \\ &\geq Q(x_0) - \epsilon, \end{aligned}$$

where the first inequality is a consequence of the Fatou equation ([16]), the second inequality holds by virtue of the fact that $\sigma^1(h_s)$ is $\epsilon/2$-optimal for the gambler (player I) in the game $G'(h_s)$, and the final equality holds because of the choice of σ^0 and s. Hence

$$\begin{aligned} W(x_0) &\geq \int u_* d\sigma \\ &\geq Q(x_0) - \epsilon. \end{aligned}$$

Since ϵ is arbitrary, this proves that $W \geq Q$.

For the reverse inequality, let

$$w_n(h) = \inf_{m \geq n+1} u(h_m), \quad n = 0, 1, 2, \ldots .$$

For any Borel measurable stop rule t, let

$$w_t(h) = w_{t(h)}(h).$$

The section $w_t p_t(h)$ of the function w_t can be calculated as follows:

$$\begin{aligned} w_t p_t(h)(h') &= w_t(p_t(h)h') \\ &= w_{t(h)}(p_t(h)h') \\ &= \inf_{m \geq 1} u(h'_m). \end{aligned} \tag{5.4}$$

Now let $\sigma \in \Sigma(x_0)$. Then

$$\begin{aligned} \int u_* d\sigma &= \sup_{t \in T'} \int w_t d\sigma \\ &= \sup_{t \in T'} \int\int w_t p_t(h)(h')\, \sigma[p_t(h)](dh')\, \sigma(dh) \\ &= \sup_{t \in T'} \int\int \left\{ \inf_{m \geq 1} u(h'_m) \right\} \sigma[p_t(h)](dh')\, \sigma(dh) \\ &\leq \sup_{t \in T'} \int \inf_{s \in T'} \int u(h'_s)\, \sigma[p_t(h)](dh')\, \sigma(dh) \\ &\leq \sup_{t \in T'} \int (\Gamma^1 L)(h_t)\, \sigma(dh). \\ &\leq Q(x_0), \end{aligned}$$

where the first equality is by virtue of the fact that $w_n \uparrow u_*$, the third equality is by (5.4), and the second inequality is by virtue of the fact that $(\Gamma^1 L)(h_t)$ is the value of the game $G'(h_t)$. Since $\sigma \in \Sigma(x_0)$ was arbitrary, we have:

$$W(x_0) \leq Q(x_0).$$

This completes the proof. □

The right side of (5.2) is the optimal reward function of a leavable gambling problem, in the sense of Dubins and Savage [3], with utility function $\Gamma^1 L$. Now Dubins and Savage have given a scheme for calculating the optimal reward function of a leavable problem. We will now apply this scheme to the deficient function $\Gamma^1 L$ to calculate W. Set

$$W_0 = \Gamma^1 L$$

and

$$W_{n+1} = \Gamma^1 W_n, \quad n \geq 0.$$

Then

$$W = \sup_n W_n.$$

The next result is an immediate consequence of the fundamental theorem of gambling [3, Corollary 2.14.1].

Corollary 5.5 *The function W is the least bounded, excessive function v such that $v \geq \Gamma^1 L$. Hence, W is an invariant function.*

We will now characterize W directly in terms of the utility function u. This result is proved for countable S in [12, 4.10.9(i)].

Theorem 5.6 *The function W is the least bounded, excessive function v such that $\int v_* d\sigma \geq \int u_* d\sigma$ for all $\sigma \in \Sigma(x)$ and $x \in X$.*

Proof. Fix $x \in X$ and $\sigma \in \Sigma(x)$. Let t be a Borel measurable stop rule in $\mathcal{T}'$. Now

$$\begin{aligned}\int u_* d\sigma &= \int\int u_* d\sigma[p_t]\, d\sigma \\ &\leq \int W(h_t)\, d\sigma.\end{aligned}$$

Hence,

$$\begin{aligned}\int u_* d\sigma &\leq \liminf_t \int W(h_t)\, d\sigma \\ &= \int W_* d\sigma,\end{aligned}$$

where the liminf is taken over the directed set $\mathcal{T}'$ of stop rules and the equality is by virtue of the Fatou equation ([16]).

Suppose next that v is a bounded excessive function such that $\int v_* d\sigma \geq \int u_* d\sigma$ for every $\sigma \in \Sigma(x)$. Since v is excessive, it is easy to verify that

$v(x), v(h_1), v(h_2), \ldots$ is a bounded supermartingale under $\sigma \in \Sigma(x)$. So, by the Optional Sampling Theorem,

$$\int v(h_t)\, d\sigma \leq v(x)$$

for any $t \in \mathcal{T}'$ and $\sigma \in \Sigma(x)$. Hence,

$$\liminf_t \int v(h_t)\, d\sigma \leq v(x),$$

so that by the Fatou equation, for any $\sigma \in \Sigma(x)$,

$$\int v_* d\sigma \leq v(x).$$

It follows that

$$\int u_* d\sigma \leq v(x).$$

Taking the sup over $\sigma \in \Sigma(x)$, we get

$$W(x) \leq v(x).$$

This completes the proof. □

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