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PRODUCTS OF VECTOR MEASURES

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Theorems are given regarding the existence of products of finitely and infinitely many Banach space valued measures. A sequence of mesures is constructed for which all finite product measures exist, but the infinite dimensional product does not.

0. Introduction. Theorems regarding products ("amalgamations") of operator valued measures have been known for some time (see pp. 86–107 in Berberian (1966)). These measures are required to be countably additive with respect to the strong operator topology and thus are not Banach space valued measures in the generally accepted sense; also, the two factor measures μ_1 and μ_2 are usually assumed to commute: $\mu_1(E)\mu_2(F) = \mu_2(F)\mu_1(E)$: such a requirement is imposed so that the product of self-adjoint valued measures will have self-adjoint values. The articles Duchon (1969), März and Shortt (1994) and Ohba (1977) represent attempts to develop a theory of product measure for not necessarily commuting, Banach space valued measures; we continue this line of enquiry.

An example of Dudley (1989) has shown that amalgamations of spectral measures cannot always be formed without some regularity assumption for one of the factor measures (as in Theorem 33 of Berberian (1966)). The same idea holds for general Banach space valued measures, where the notion of a perfect vector measure is useful. Section 1 lays out some basic results for perfect measures; for measures taking values in the positive cone of a Banach lattice, the theory runs parallel to the classical one developed by Gnedenko and Kolmogoroff (cp. Ramachandran (1979)): Lemma 1.2 effects this similarity. The fundamental result that enables our analysis is one proved in Shortt (1994) and stated here as Lemma 1.5.

After exhibiting an example to which known existence results do not apply, we prove a new existence theorem for products of Banach-space valued measures (Theorem 2.1). The hypotheses required are weaker, e.g., than those used in Duchon (1969) or Ohba (1977). A corresponding result for products of infinitely many measures forms Theorem 2.2. Finally, Example 2.3 shows

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a sequence of measures for which no product measure exists; yet, product measures exist for every finite subset of measures drawn from this sequence.

1. Perfect Vector Measures. Let $\mathcal{F}$ be a field (i.e. algebra) of subsets of a set X and let $(B, \|\cdot\|)$ be a Banach space. (We allow either real or complex scalars.) A function $\mu : \mathcal{F} \to B$ is a *B-valued charge* if it is finitely additive: $\mu(F_1 \cup F_2) = \mu(F_1) + \mu(F_2)$ for disjoint F_1 and F_2 in $\mathcal{F}$. A function $\mu : \mathcal{F} \to B$ is a *B-valued measure* if it is countably additive: $\mu(F_1 \cup F_2 \cup \cdots) = \mu(F_1) + \mu(F_2) + \cdots$ as a norm-convergent series in B whenever (F_n) is a disjoint sequence in $\mathcal{F}$ whose union also belongs to $\mathcal{F}$. Basic references on the subject of vector measures include the classic treatise of Dunford and Schwartz (1958) and the monograph of Diestel and Uhl (1977). The notation and terminology of this article conform by and large to that of these standard sources.

As is customary, we let $\|\mu\| : \mathcal{F} \to R$ denote the semi-variation of a B-valued charge μ

$$\|\mu\|(F) = \sup\{|\varphi \circ \mu|(F) : \varphi \in B^*, |\varphi| \leq 1\},$$

where $|\varphi \circ \mu|$ is the total variation of the real measure $\varphi \circ \mu$. Recall that every B-valued measure on a σ-field $\mathcal{F}$ is bounded, i.e. is of bounded semi-variation. (This follows from Corollary 19 on p. 9 of Diestel and Uhl (1977).)

LEMMA 1.1. *Let $\mu : \mathcal{F} \to B$ be a bounded B-valued measure on a field $\mathcal{F}$. Then μ extends uniquely to a B-valued measure $\bar{\mu} : \sigma(\mathcal{F}) \to B$ defined on the σ-field $\sigma(\mathcal{F})$ generated by $\mathcal{F}$.*

INDICATION: See Theorem 2, p. 27 in Diestel and Ulh (1977).

It is worth noting (Dudley and Pakula (1972)) that the preceding Lemma becomes false if the hypothesis of boundedness is omitted.

We note that if $\mu : \mathcal{F} \to B$ is a B-valued charge, then $\|\mu(F)\| \leq \|\mu\|(F)$ for each $F \in \mathcal{F}$. When the Banach space B is equipped with additional structure, then more can be said.

Recall that a *Banach lattice* is a partially ordered Banach space $(B, \|\cdot\|, \leq)$ such that

i) $(B, \leq)$ is a vector lattice, and

ii) $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$ for $x, y \in B$.

Here, $|x| = (x \vee 0) - (x \wedge 0)$.

The reader is referred to the text of Kelley, Namioka, et al. (1976) and Schaefer (1974) for readable accounts of the elementary theory of Banach lattices.

LEMMA 1.2. *Let $(B, \leq)$ be a Banach lattice with positive cone $B^+ = \{x \in B : x \geq 0\}$. Let $\mu : \mathcal{F} \to B^+$ be a charge. For each $F \in \mathcal{F}$, we have* $\|\mu\|(F) = \|\mu(F)\|$.

INDICATION: This is Lemma 1.1 of Shortt (1994).

A class $\mathcal{K}$ of subsets of a set X is *compact* if it has the following property: given any sequence (K_n) drawn from $\mathcal{K}$ such that $K_1 \cap K_2 \cap \cdots \cap K_n \neq \emptyset$ for each n, the intersection $K_1 \cap K_2 \cap \cdots$ is non-empty. Let $\mu : \mathcal{F} \to B$ be a B-valued charge on a field $\mathcal{F}$. We say that μ is a *compact charge* if there is a compact class $\mathcal{K}$ of subsets of X such that for every $F \in \mathcal{F}$ and $\epsilon > 0$, there are sets $F' \in \mathcal{F}$ and $K \in \mathcal{K}$ with $F' \subseteq K \subseteq F$ and $\|\mu\|(F - F') < \epsilon$. In this case, we say that the class $\mathcal{K}$ *μ-approximates* $\mathcal{F}$. If $\mathcal{F}$ is a σ-field, we say that a charge $\mu : \mathcal{F} \to B$ is *perfect* if the restriction of μ to every countably generated sub-σ-field of $\mathcal{F}$ is compact. Clearly, every compact charge on a σ-field is perfect.

Every B-valued measure μ on a σ-field $\mathcal{F}$ has a control measure, i.e. a finite, positive, real measure m on $\mathcal{F}$ such that $\|\mu\|(F) \to 0$ if and only if $m(F) \to 0$. (See Dunford and Schwartz (1958) IV.10.5 (p. 321) .) One easily deduces

LEMMA 1.3. *Let $\mu : \mathcal{F} \to B$ be a B-valued measure on a σ-field $\mathcal{F}$ with control measure m. Then μ is compact [resp. perfect] if and only if m is compact [resp. perfect].*

Let X be a topological space with Baire σ-field $\mathcal{B}(X)$. A charge $\mu : \mathcal{B}(X) \to B$ is *tight* if, for each $\epsilon > 0$ and $F \in \mathcal{B}(X)$, there is a compact set $K \subseteq F$ with $\|\mu\|(F - K) < \epsilon$. Clearly, every tight measure is compact and hence perfect. It is well-known that for certain spaces X, every finite, positive, real measure m on $\mathcal{B}(X)$ is tight. The class of such spaces includes the compact Hausdorff spaces and those separable metric spaces that are Borel subsets of their completions ("absolute Borel" spaces). See, e.g., Theorem 7.1.4 (Ulam's Theorem) and Theorem 7.1.5 in Dudley (1989). The use of control measures allows the easy transfer of these results to vector measures:

LEMMA 1.4 *Let X be either a compact Hausdorff space or an absolute Borel metrisable space with Baire σ-field $\mathcal{B}(X)$. Then every B-valued measure $\mu : \mathcal{B}(X) \to B$ is tight (hence compact, hence perfect).*

The following is a recently proved generalisation of a result of Marczewski and Ryll-Nardzewski (1953). Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be a σ-fields of subsets of sets X_1 and X_2, respectively. Then $\mathcal{F}_1 \times \mathcal{F}_2$ is the field on $X_1 \times X_2$ generated by all rectangles $F_1 \times F_2$ for $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$. Further, $\mathcal{F}_1 \otimes \mathcal{F}_2$ is the σ-field generated by $\mathcal{F}_1 \times \mathcal{F}_2$.

LEMMA 1.5. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be σ-fields of subsets of sets X_1 and X_2, respectively, and let $\rho : \mathcal{F}_1 \times \mathcal{F}_2 \to B^+$ be a charge taking values in the positive cone of a Banach lattice $(B, \leq)$. Define charges $\mu_1 : \mathcal{F}_1 \to B^+$ and $\mu_2 : \mathcal{F}_2 \to B^+$ by the rule*

$$\mu_1(F_1) = \rho(F_1 \times X_2) \qquad \mu_2(F_2) = \rho(X_1 \times F_2).$$

If μ_1 is a measure, and μ_2 is a perfect measure, then ρ is a measure on $\mathcal{F}_1 \times \mathcal{F}_2$.

INDICATION: This is Theorem 3.1 in Shortt (1994).

COROLLARY 1.6. *In the situation above, ρ extends uniquely to a measure $\bar{\rho} : \mathcal{F}_1 \otimes \mathcal{F}_2 \to B^+$.*

PROOF. Lemma 1.2 shows that $\|\rho\| = \|\rho(X_1 \times X_2)\|$, so that ρ is bounded. Lemma 1.1 gives the extension $\bar{\rho} : \mathcal{F}_1 \otimes \mathcal{F}_2 \to B$. We then consider the collection $\mathcal{C} = \{B \in \mathcal{F}_1 \otimes \mathcal{F}_2 : \bar{\rho}(B) \geq 0\}$. If (x_n) is a summable sequence in B^+ then $\Sigma x_n \geq 0$. Using this fact, it is easy to prove that $\mathcal{C}$ is a σ-field containing $\mathcal{F}_1 \times \mathcal{F}_2$. Thus $\mathcal{C} = \mathcal{F}_1 \otimes \mathcal{F}_2$.

2. Finite and Infinite Products. Let B_1, B_2 and B_3 be Banach spaces and let $f : B_1 \times B_2 \to B_3$ be a separately continuous bilinear form. Important examples include the case where $B = B_1 = B_2 = B_3$ is a Banach algebra, and $f(x, y) = xy$ is the product in B, and the case where $H = B_1 = B_2$, is a Hilbert space, $B_3 = \mathbb{R}$, and $f(x, y) = \langle x, y \rangle$.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be σ-fields of subsets of sets X_1 and X_2, respectively. Let $\mathcal{F}_1 \times \mathcal{F}_2$ and $\mathcal{F}_1 \otimes \mathcal{F}_2$ be, as before, the field and σ-field on $X_1 \times X_2$ generated by the collection of all rectangles $F_1 \times F_2$ for $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$. Let $\mu_1 : \mathcal{F}_1 \to B_1$ and $\mu_2 : \mathcal{F}_2 \to B_2$ be B_i-valued measures and let $\mu = \mu_1 \times \mu_2$ be the unique B_3-valued charge on $\mathcal{F}_1 \times \mathcal{F}_2$ such that $\mu(F_1 \times F_2) = f(\mu_1(F_1), \mu_2(F_2))$ for $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$. We address the question of whether $\mu_1 \times \mu_2$ is countably additive and whether $\mu_1 \times \mu_2$ can be extended to a measure on $\mathcal{F}_1 \otimes \mathcal{F}_2$.

It is known that the answer to the first question can be in the negative. For the special case where $f(x, y) = \langle x, y \rangle$ (inner product in Hilbert space), this is shown by an example of Dudley and Pakula (1972). In another special case, an example of Dudley (1973) applies. Since this example was originally given in a somewhat different context, where σ-additivity for the norm was neither assumed for the μ_i nor required for $\mu_1 \times \mu_2$, we present the idea afresh.

Let $1 < p < \infty$ and suppose that $\frac{1}{p} + \frac{1}{q} = 1$. Define $B_1 = L^p[0,1], B_2 = L^q[0,1]$, and $B_3 = L^1[0,1]$, setting $f(u, v) = uv$ (point-wise product). Let λ be Lebesgue measure on $[0,1]$ and let $S \subseteq [0,1]$ be such that $\lambda^*(S) = 1$ and $\lambda_*(S) = 0$. Put $X_1 = S$ and $X_2 = [0,1] - S$, with $\mathcal{F}_i = \{F \cap X_i : F$

Borel$\}, i = 1, 2$. Define $\mu_1(F \cap X_1) = I_F$ and $\mu_2(F \cap X_2) = I_F$: these are B_1- and B_2-valued measures. The argument in Dudley (1973) shows that $\mu_1 \times \mu_2$ is not countably additive on $\mathcal{F}_1 \times \mathcal{F}_2$, so extension to a measure on $\mathcal{F}_1 \otimes \mathcal{F}_2$ is impossible.

In this section, conditions are given under which product measure $\mu_1 \times \mu_2$ is countably additive and admits of a countably additive extension $\mu_1 \otimes \mu_2$. Subsequently, an existence result for infinite products is developed. An example is given where finite products exist, but infinite ones do not.

Let $\mu_1 : \mathcal{F}_1 \to B_1$ and $\mu_2 : \mathcal{F}_2 \to B_2$ be B-valued measures on σ-fields $\mathcal{F}_1$ and $\mathcal{F}_2$ and let $f : B_1 \times B_2 \to B_3$ be a (separately) continuous bilinear form, where $(B_3, \leq)$ is a Banach lattice. Let $\rho = \mu_1 \times \mu_2$ be the product charge defined above. Supose that $\rho(F) \geq 0$ for every $F \in \mathcal{F}_1 \times \mathcal{F}_2$. Define charges $\rho_1 : \mathcal{F}_1 \to B_3^+$ and $\rho_2 : \mathcal{F}_2 \to B_3^+$ by $\rho_1(F_1) = \rho(F_1 \times X_2) = f(\mu_1(F_1), \mu_1(X_2))$ and $\rho_2(F_2) = \rho(X_1 \times F_2) = f(\mu_1(X_1), \mu_2(F_2))$. Then continuity of f ensures that ρ_1 and ρ_2 are B_3-valued measures.

THEOREM 2.1. *In the situation just described, if the measure ρ_2 is perfect, then $\mu_1 \times \mu_2$ is countably additive on $\mathcal{F}_1 \times \mathcal{F}_2$ and extends to a unique measure $(\mu_1 \otimes \mu_2) : \mathcal{F}_1 \otimes \mathcal{F}_2 \to B_3^+$.*

PROOF. The theorem follows from Lemma 1.5 and Corollary 1.6.

With further work, it can be shown that if the measures μ_1 and μ_2 are perfect, then so is the product measure $\mu_1 \otimes \mu_2$. *Mutatis mutandis*, the argument parallels the proof of the classical result of Marczewski (1953); for another exposition of the classical, real-valued case, see Theorem 3.1.1 (Volume I, p. 35) of Dunford and Schwartz (1958).

We note two special cases of the theorem: the first with $B_1 = B_2$ a Hilbert space and $B_3 = \mathbb{R}$ and $f(x, y) = \langle x, y \rangle$; the second, with $B_1 = B_2 = B_3$ a Banach algebra with $f(x, y) = xy$. The necessary continuity for f follows from the continuity of the inner product in the first case, from $||xy|| \leq ||x|| \cdot ||y||$ in the second.

In the paper of Dudley and Pakula (1972) addressing the case where $f(x, y) = \langle x, y \rangle$, it is noted that a satisfactory existence theorem for the inner product of Hilbert spaced valued measures can be derived from work of Bartle (1956) and found in Duchon (1969): this result assumes that one of the measures μ_i has finite total variation. The characteristics of the counter-example presented in Dudley and Pakula (1972) led the authors to suggest "that it may be difficult to find any reasonably broad conditions under which $\mu_1 \times \mu_2$ would be countably additive while $|\mu_1| = |\mu_2| = \infty$." In view of this assertion, it is noteworthy that Theorem 2.1 makes no assumptions about finite variation for the μ_i. Indeed, returning to a version of the example outlined above, let $B_1 = L^p[0,1], B_2 = L^q[0,1]$ and $B_3 = L^1[0,1]$ and $f(u,v) = uv$. Taking $X_1 = X_2 = [0,1]$ and $\mathcal{F}_1 = \mathcal{F}_2$ the Borel σ-field on $[0,1]$, put $\mu_1(F) = I_F$ and $\mu_2(F) = I_F$. Then μ_1 and μ_2 are perfect (Lemma 1.4), so that $\mu_1 \times \mu_2$ is countably additive and extends to a measure $(\mu_1 \otimes \mu_2) : \mathcal{F}_1 \times \mathcal{F}_2 \to L^1[0,1]$ (Theorem 2.1). In this example, we have $|\mu_1|(F) = |\mu_2|(F) = \infty$ whenever $1 < p < \infty$ and F is not a null-set: see Example 16, p. 7, in Diestel and Uhl (1977).

S. Obha (1977) proved an existence theorem for products of vector measures under special assumptions ("D-continuity") on one of the factors; these conditions do not apply in the foregoing example.

We now return our attention to the existence of products of an infinite sequence of vector measures.

Let $B_1, B_2 \ldots$ be Banach spaces and let $(B_\infty, \leq)$ be a Banach lattice. For each $n \geq 1$, suppose that $f_n : B_1 \times \cdots \times B_n \to B_\infty$ is a separately continuous multi-linear form. We assume that the f_n are connected in the following way: there are elements $e_i \in B_i (i \geq 1)$ such that

$$f_{n+1}(x_1, \ldots, x_n, e_{n+1}) = f_n(x_1, \ldots, x_n)$$

for all $x_1 \in B_i$. (We have in mind the example where $B_\infty = B_1 = B_2 = \cdots$ is a Banach algebra with unit $e = e_1 = e_2 \cdots$, and $f_n(x_1, \cdots x_n) = x_1 \cdots x_n$.)

Let $\mathcal{F}_1, \mathcal{F}_2 \ldots$ be σ-fields of subsets of sets $X_1, X_2 \ldots$ and let $\mu_i : \mathcal{F}_i \to B_i (i \geq 1)$ be B_i-valued measures with $\mu_i(X_i) = e_i$ for $i \geq 1$. Put $X = X_1 \times X_2 \times \cdots$ and let $\mathcal{S}_n$ be the σ-field on X generated by rectangles $F_1 \times \cdots \times F_n \times X_{n+1} \times X_{n+2} \times \cdots$ for $F_i \in \mathcal{F}_i$. Let $\mathcal{S} = \cup \mathcal{S}_n$ and define $\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \cdots = \sigma(\mathcal{S})$. We seek a B_∞-valued measure ρ on $\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \cdots$ such that $\rho(F_1 \times \cdots \times F_n \times X_{n+1} \times \cdots) = f_n(\mu_1(F_1), \ldots, \mu_n(F_n))$ for all $F_i \in \mathcal{F}_i$ and $n \geq 1$.

THEOREM 2.2. *In the situation just described, suppose that the measures μ_i are perfect and that the product charges $\mu_1 \times \cdots \times \mu_n$ defined on $\mathcal{F}_1 \times \cdots \times \mathcal{F}_n$ by the rule*

$$(\mu_1 \times \cdots \times \mu_n)(F_1 \times \cdots \times F_n) = f_n(\mu_1(F_1), \ldots, \mu_n(F_n))$$

take values in B^+_∞. Then there is a unique B_∞-valued measure $(\mu_1 \otimes \mu_2 \otimes \cdots) : \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \cdots \to B^+_\infty$ such that

$$(\mu_1 \otimes \mu_2 \otimes \cdots)(F_1 \times \cdots \times F_n \times X_{n+1} \times \cdots) = f_n(\mu_1(F_1), \ldots, \mu_n(F_n))$$

for all $F_i \in \mathcal{F}_i$ and all n. The measure $\mu_1 \otimes \mu_2 \otimes \cdots$ is perfect.

OUTLINE OF PROOF. Repeated application of Theorem 2.1 yields product measures $\mu_1 \otimes \cdots \otimes \mu_n$ on $\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n$ taking values in B_∞^+. We define a charge $\rho : \mathcal{S} \to B_\infty^+$ by putting

$$\rho(F \times X_{n+1} \times X_{n+2} \times \cdots) = (\mu_1 \otimes \cdots \otimes \mu_n)(F)$$

for $F \in \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n$. It must be proved that ρ is countably additive; this, combined with the fact that $\|\rho\|(X) = \|\rho(X)\| < \infty$ (Lemma 1.2) and Lemma 1.1 yields the desired extension on ρ. The argument is much the same as in the classical setting: the proof of Marczewski and Ryll-Nardzewski (1953) applies *mutatis mutandis* to show that ρ is a compact charge on $\mathcal{S}$, hence countably additive. (For another exposition, see Ramachandran (1979) Theorem 3.1.1.)

We conclude with an example of a sequence of measures for which all finite product measures exist, but infinite product measure does not.

EXAMPLE 2.3. Let λ be Lebesgue measure on $[0,1]$ and let $[0,1] = P_1 \cup P_2 \cup \cdots$ be a partition of the interval into sets P_i such that $\lambda_*(P_i) = 0$. (See [7], p. 81.) For $n \geq 1$, define $X_n = [0,1] - (P_1 \cup \cdots \cup P_n)$ and let $\mathcal{F}_n$ be the σ-field on X_n given by $\mathcal{F}_n = \{F \cap X_n : F \text{ Borel}\}$. Let (p_n) be a sequence of positive real numbers with $\Sigma 1/p_n = 1$. Put $B_\infty = L^1[0,1]$ and $B_n = L^{p_n}[0,1]$ for $n = 1, 2, \ldots$. Let e_n be the constant function 1 for each n. Then the product mappings $f_n : B_1 \times \cdots \times B_n \to B_\infty$ we define by $f_n(u_1, \ldots, u_n) = u_1 \cdots u_n$ (point-wise product). Let $\mu_n : \mathcal{F}_n \to B_n^+$ be the measure given by $\mu_n(F \cap X_n) = I_F$ (indicator function).

We assert that for each $n \geq 1$, there exists a measure

$$(\mu_1 \otimes \cdots \otimes \mu_n) : \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n \to B_\infty^+$$

such that $(\mu_1 \otimes \cdots \otimes \mu_n)(F_1 \times \cdots \times F_n) = f_n(\mu_1(F_1), \ldots, \mu_n(F_n)) = \mu_1(F_1) \cdots \mu_n(F_n) = I_{F_1 \cap \cdots \cap F_n}$ for all $F_i \in \mathcal{F}_i$. However, these is no measure $(\mu_1 \otimes \mu_2 \otimes \cdots) : \mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \cdots \to B_\infty^+$ such that $(\mu_1 \otimes \mu_2 \otimes \cdots)(F_1 \times \cdots \times F_n \times X_{n+1} \times \cdots) = I_{F_1 \cap \cdots \cap F_n}$ for all n and $F_i \in \mathcal{F}_i$.

For each $n \geq 1$, define the diagonal $\Delta_n = \{(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n : x_1 = x_2 = \cdots = x_n\}$. Then $\mu_1 \otimes \cdots \otimes \mu_n$ is defined so that

$$(\mu_1 \otimes \cdots \otimes \mu_n)\{(x, \ldots, x) : x \in F \cap X_n\} = I_F$$

$$(\mu_1 \otimes \cdots \otimes \mu_n)(X_1 \times \cdots \times X_n - \Delta_n) = 0$$

for all Borel $F \subseteq [0,1]$.

We now derive a contradiction from the supposed existence of product measure $\mu_1 \otimes \mu_2 \otimes \cdots$ on $\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \cdots$. Consider the diagonal $\Delta = \{(x,x) : x \in [0,1]\}$ in $[0,1]^2$ and write $[0,1]^2 - \Delta = \cup_n(E_n \times F_n)$, where the (E_n) and

F_n) are sequences of Borel sets with $E_n \cap F_n = \emptyset$; they may be taken to be intervals. For $1 \leq i < j$, define $E(n,i,j) = C_1 \times C_2 \times \cdots$, where $C_k = [0,1]$ for $k \notin \{i,j\}, C_i = E_n, C_j = F_n$. Let Δ_ω be the diagonal $\Delta_\omega = \{x : x(1) = x(2) = \cdots\}$ in $[0,1]^\omega$ and note that $[0,1]^\omega = \Delta_\omega \cup \bigcup E(n,i,j)$, where the union is taken over all $i < j$ and $n \geq 1$. Then $X = \bigcup X \cap E(n,i,j)$, and we see that $(\mu_1 \otimes \mu_2 \otimes \cdots)(X) = I_{[0,1]}$ and, at the same time,

$$(\mu_1 \otimes \mu_2 \otimes \cdots)(X \cap E(n,i,j)) = (\mu_i \otimes \mu_j)((E_n \cap X_i) \times (F_n \cap X_j)) = I_{E_n} I_{F_n} = 0,$$

a contradiction.

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