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Copulae of Capacities on Product Spaces

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We consider a nonadditive probability measure (capacity) on a product space and we define the distribution function associated to it. We show that, under suitable conditions on the capacity, its distribution function has many of the properties of the distribution functions of finitely additive probability measures. In particular, if the capacity is convex, then there exists a function that links the multivariate distribution function to its marginals. This function enjoys many of the properties of a copula.

1. Introduction. Many areas of applications require the use of set functions that, like measures, are monotone with respect to set inclusion, but, unlike measures, are not additive, not even finitely additive. For instance, in cooperative game theory, the characteristic function, which is defined on the power set of the set of players, is monotone, but not additive. This corresponds to the intuitive idea that the bigger a coalition, the stronger it is, but its strength in general does not coincide with sum of the strengths of its components (see e.g. Aumann and Shapley (1974)).

The theories of inference proposed by Dempster (1967, 1968) and Shafer (1976) are based on belief functions, which are again nonadditive set functions, a particular case of which is given by the usual probability measures. Their approach allows one to employ an updating mechanism which is much more flexible than the usual Bayesian one. A similar generalization has been proposed, with different motivations, in decision theory, by Schmeidler (1986, 1989) and Gilboa (1987). They have relaxed the axioms of Anscombe and Aumann and Savage, respectively and have obtained a paradigm for choice under uncertainty that is similar to the usual maximization of expected utility, except that the integration is performed with respect to nonadditive probabilities

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and therefore the Choquet integral has to be used, instead of a Riemann or Lebesgue integral. The work by Schmeidler and Gilboa has been expanded and employed by several authors, and we refer to Gilboa and Schmeidler (1992a) for a bibliography.

Robust statistics has used nonadditive probabilities both to represent uncertainty about the statistical model (Huber and Strassen (1973)), and, in a Bayesian framework, to represent uncertainty about the prior distribution of a parameter (Wasserman and Kadane (1990, 1992)).

Choquet's theory of capacities (Choquet (1953–54)) is the mathematical core of all this research, even if we should notice that what he calls a capacity is not what afterwards the applied literature has often called a capacity. We will follow the use of the decision theoretical literature and employ the term capacity to indicate a nonadditive probability (in a sense that will be made clear below). A nice introduction to the topic of nonadditive measures and integrals can be found in Denneberg (1994).

The main objective of this note is to study some properties of capacities on a finite dimensional space (which, to keep things simple, will always be $\Re^d$). In particular, after defining the distribution function of a capacity, we will examine the possibility of extending the concept of copula to this more general situation. This concept was introduced by Sklar (1959) with reference to distribution functions of probability measures (see Chapter 6 of Schweizer and Sklar (1983) for a description of its properties). A copula links a multivariate distribution function to its marginals. A generalization of this idea to measures on Polish product spaces was developed in Scarsini (1989). Note that, since capacities are not measures, we cannot apply the methods of the latter paper in the present context. What we now do instead is to use the assumption of the convexity of the capacity to establish the existence of a function that links the multivariate distribution function to its marginals. By analogy with the additive case, this function will be called a generalized copula of the capacity. If furthermore the capacity is d-monotone, then the generalized copula of the capacity has all the usual properties of a copula. The use of this result is somehow limited by the fact that the distribution function of a capacity does not characterize it, unlike what happens in the σ-additive case. Nevertheless there are instances where it is not necessary to know the value of a capacity on the whole Borel class, but only on a suitable subclass. For instance, in order to establish stochastic ordering results, only the value of the capacity on (a subclass of) the classes of lower or upper sets is needed (see e.g. Scarsini (1992), Dyckerhoff and Mosler (1993)).

Some conventions: Given a set A, we will indicate its complement as A^c. We will use the terms increasing and decreasing in the weak sense. The

Euclidean space $\Re^d$ will be equipped with the componentwise ordering, unless otherwise specified.

2. Main Results. Let Ω be a set and let Σ be a class of subsets of Ω. A function $\nu : \Sigma \to [0,1]$ is called a capacity if it satisfies the following three properties:

(1) it is grounded, i.e. $\nu(\emptyset) = 0$,

(2) it is monotone, i.e. $A, B \in \Sigma$, $A \subseteq B \Longrightarrow \nu(A) \leq \nu(B)$,

(3) it is normalized, i.e. $\nu(\Omega) = 1$.

The original definition of capacity, due to Choquet (1953–54) is actually different from this one, in that it does not assume normalization, but it requires some topological assumptions about the space on which the capacity is defined. For instance (Dellacherie (1971)), if K is a metrizable compact space, $\phi : 2^K \to \Re_+$ is a Choquet capacity if

(α) $\phi(\emptyset) = 0$,

(β) $A \subset B \Longrightarrow \phi(A) \leq \phi(B)$,

(γ) if $\{A_n\}$ is an increasing sequence of subsets of K, then $\phi(\cup_n A_n) = \sup_n \phi(A_n)$,

(δ) if $\{B_n\}$ is a decreasing sequence of compact subsets of K, then $\phi(\cap_n A_n) = \inf_n(B_n)$.

The reason for using our definition is that it is quite common in mathematical economics (see Gilboa and Schmeidler (1992a, 1992b) and Denneberg (1994) for a list of references), and that the continuity conditions (γ) and (δ) are not needed for most of our results.

The following definitions can be found for instance in Gilboa and Schmeidler (1992a, 1992b). Given a capacity ν, the set function $\tilde{\nu}$ defined by $\tilde{\nu}(A) = 1 - \nu(A^c)$ is called the dual of ν. It is easy to prove that $\tilde{\nu}$ is a capacity, as well. A capacity is called

(i) *convex* if $\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$ for all $A, B \in \Sigma$,

(ii) *k-monotone* if for every $n \in \{1, \ldots, k\}$ and $A_1, \ldots, A_n \in \Sigma$

$$\nu\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{\{I:\emptyset \neq I \subseteq \{1,\ldots,n\}\}} (-1)^{|I|+1} \nu\left(\bigcap_{i \in I} A_i\right), \qquad (1)$$

where $|I|$ is the cardinality of I (for instance, when $k = 2$, (1) becomes $\nu(A \cup B) \geq \nu(A) + \nu(B) - \nu(A \cap B)$, therefore 2-monotonicity is equivalent to convexity),

(iii) *totally monotone* if it is k-monotone for every integer k,

(iv) *continuous from below* if $A_n \in \Sigma$, $A_n \subset A_{n+1}$ for all integer n implies $\lim_{n\to\infty} \nu(A_n) = \nu(\cup_{n=1}^{\infty} A_n)$.

A capacity is called *concave* (resp. *k-alternating*, resp. *totally alternating*, resp. *continuous from above*) if its dual is convex (resp. k-monotone, resp. totally monotone, resp. continuous from below).

EXAMPLE 1. Given a class $\mathcal{P}$ of probability measures on a measurable space $(\Omega, \mathcal{S})$, let μ_1, μ_2 be defined as follows

$$\mu_1(A) = \sup_{P\in\mathcal{P}} P(A), \qquad \mu_2(A) = \inf_{P\in\mathcal{P}} P(A).$$

Then μ_1, μ_2 are capacities.

EXAMPLE 2. Let P be a probability measure and let $\gamma : [0,1] \to [0,1]$ be an increasing one-to-one function. Then $\mu_3 = \gamma \circ P$ is a capacity. If γ is convex (concave) in the usual sense, then μ_3 is convex (concave) in the sense of (iii).

EXAMPLE 3. For any Borel set $A \in \Re^d$ let $\dim(A)$ be its topological dimension. If

$$\mu_4(A) = \frac{1 + \dim(A)}{1 + d},$$

then μ_4 is a capacity.

EXAMPLE 4. Let Θ be a finite space and let $m : 2^{\Theta} \to [0,1]$ satisfy

- $m(\emptyset) = 0$ and
- $\sum_{A\subset\Theta} m(A) = 1$.

Define μ_5 as follows

$$\mu_5(A) = \sum_{B\subset A} m(B).$$

Then μ_5 is a totally monotone capacity (Shafer (1976)).

Define $\overline{\Re} = \Re \cup \{-\infty, \infty\}$. From now on, ν will be a capacity on $(\overline{\Re}^d, \mathrm{Bor}(\overline{\Re}^d))$ (where $\mathrm{Bor}(\Re^d)$ is the class of Borel sets in $\Re^d$), and ν_i will be its i-th projection: for $A \in \mathrm{Bor}(\overline{\Re})$,

$$\nu_i(A) = \nu(\overline{\Re} \times \cdots \times \overline{\Re} \times \underset{i}{A} \times \overline{\Re} \cdots \times \overline{\Re}).$$

DEFINITION 5. The *distribution function associated to* ν is the function $F_\nu : \overline{\Re}^d \to \Re$ given by

$$F_\nu(x_1, \ldots, x_d) = \nu([-\infty, x_1] \times \cdots \times [-\infty, x_d]).$$

Correspondingly the distribution function associated to ν_i is the function F_{ν_i} : $\overline{\Re} \to \Re$ given by

$$F_{\nu_i}(x) = \nu_i([-\infty, x]).$$

It is easy to see that F_ν is increasing, since ν is monotone. In general F_ν is not right continuous and, of course, does not characterize ν on the whole Borel σ-field, since even the distribution function of a finitely additive probability measure in general does not have these properties. Other properties of F_ν can be obtained under particular assumptions, as the following lemma will show.

Let $\triangle_{s_i=x_i}^{y_i} f(s_1 \ldots, s_i, \ldots, s_d) = f(s_1, \ldots, y_i, \ldots, s_d) - f(s_1, \ldots, x_i, \ldots, s_d)$. A function $F : \overline{\Re}^d \to \Re$ is called n-increasing ($n \leq d$) if

$$\mathop{\triangle}_{s_{i_1}=x_{i_1}}^{y_{i_1}} \cdots \mathop{\triangle}_{s_{i_n}=x_{i_n}}^{y_{i_n}} f(\ldots, s_{i_1}, \ldots, s_{i_n}, \ldots) \geq 0$$

for every possible choice of the indices $i_1, \ldots, i_n$, every $x_{i_1} \leq y_{i_1}, \ldots, x_{i_n} \leq y_{i_n}$ and every possible value of the other coordinates.

Any d-variate distribution function associated with a finitely additive probability measure is n-increasing for all $n \leq d$. This is true also for d-monotone capacities. In the case of probability measures the result stems from the fact that the probability of any d-dimensional rectangle in $\overline{\Re}^d$ is non-negative and this probability can be expressed as the multiple finite difference of the distribution function. The same procedure cannot be applied here, due to lack of additivity, but the definition of d-monotonicity and a suitable choice of the sets to which this property is applied, give the result.

Lemma 6. *If ν is d-monotone, then F_ν is n-increasing for every $n \leq d$.*

Proof. Without any loss of generality, we will prove that F_ν is n-increasing in its first n variables. Let $n \leq d$, $x_i \leq y_i$, for $i \in \{1, \ldots, n\}$, and let

$$A_i = \mathop{\times}_{j=1}^{d} [-\infty, z_j],$$

where

$$z_j = \begin{cases} x_j, & \text{if } i = j \\ y_j, & \text{otherwise.} \end{cases}$$

Let $E_1, \ldots, E_n$ be disjoint sets such that

$$\bigcup_{j=1}^{n} E_j = \mathop{\times}_{j=1}^{n} (x_j, y_j] \mathop{\times}_{h=n+1}^{d} [-\infty, y_h].$$

Define $D_i = A_i \cup E_i$, for $i = 1, \ldots, n$. Therefore $D_i \cap D_j = A_i \cap A_j$.

If ν is d-monotone, then

$$\begin{aligned}\nu\left(\bigcup_{j=1}^{n} D_j\right) &\geq \sum_{\{I:\emptyset\neq I\subseteq\{1,\ldots,n\}\}} (-1)^{|I|+1}\nu\left(\bigcap_{i\in I} D_j\right)\\ &= \sum_{\{I:\emptyset\neq I\subseteq\{1,\ldots,n\}\}} (-1)^{|I|+1}\nu\left(\bigcap_{i\in I} A_j\right).\end{aligned} \tag{2}$$

Since

$$\nu\left(\bigcap_{i\in I} A_j\right) = F_\nu(w_1,\ldots,w_d),$$

where

$$w_i = \begin{cases} x_i, & \text{if } i\in I\\ y_i, & \text{if } i\in I^c,\end{cases}$$

and

$$\nu\left(\bigcup_{j=1}^{n} D_j\right) = F_\nu(y_1,\ldots,y_d),$$

then (2) becomes

$$\sum_{\substack{w_i\in\{x_i,y_i\}\\ i\in\{1,\ldots,n\}}} F_\nu(w_1,\ldots,w_n,y_{n+1},\ldots,y_d) \geq 0,$$

namely F_ν is n-increasing. ∎

In particular for the case $d = 2$, the distribution function of any convex capacity is increasing and 2-increasing.

COROLLARY 7. *If ν is a d-monotone capacity on $(\overline{\Re}^d, \mathrm{Bor}(\overline{\Re}^d))$, then there exists a finitely additive probability measure μ on $(\overline{\Re}^d, \mathrm{Bor}(\overline{\Re}^d))$, such that $F_\mu = F_\nu$.*

PROOF. By Lemma 6, if ν is d-monotone, then its distribution function F_ν is n-increasing for every $n \leq d$, therefore it coincides with the distribution function of a finitely additive probability measure. ∎

As in the additive case, the multivariate distribution function of a convex capacity satisfies the Fréchet bounds (Fréchet (1951)).

THEOREM 8. *If ν is convex, then*

$$\max\left(\sum_{i-1}^{d} F_{\nu_i}(x_i) - d + 1, 0\right) \leq F_\nu(x_1,\ldots,x_d) \leq \min(F_{\nu_1}(x_1),\ldots,F_{\nu_d}(x_d)).$$

PROOF. The second inequality is immediate, by monotonicity of ν.

We now prove the first inequality. Let

$$A_i = (\overline{\Re} \times \cdots \times \overline{\Re} \times [-\infty, x_i] \times \overline{\Re} \times \cdots \times \overline{\Re}).$$

By convexity,

$$\begin{aligned}
\nu(A_1 \cap \ldots \cap A_d) &\geq \nu(A_1) + \nu(A_2 \cap \ldots \cap A_d) - \nu(A_1 \cup (A_2 \cap \ldots \cap A_d)) \\
&\geq \nu(A_1) + \nu(A_2) + \nu(A_3 \cap \ldots \cap A_d) - \nu(A_2 \cup (A_3 \cap \ldots \cap A_d)) - 1 \\
&\;\;\vdots \\
&\geq \nu(A_1) + \cdots + \nu(A_d) - d + 1,
\end{aligned}$$

which is exactly the result. ∎

The following theorem will prove that the distribution function of any convex capacity depends on its arguments only through its marginals (as in the additive case).

THEOREM 9. *Let ν be a convex capacity on $(\overline{\Re}^d, \text{Bor}(\overline{\Re}^d))$. Then there exists a function $C_\nu : [0,1]^d \to [0,1]$, called a generalized copula of ν, such that*

a. $F_\nu(x_1, \ldots, x_d) = C_\nu(F_{\nu_1}(x_1), \ldots, F_{\nu_d}(x_d))$,

b. $C_\nu(s_1, \ldots, s_d) = 0$ *if* $s_i = 0$ *for some* $i \in \{1, \ldots, d\}$,

c. $C_\nu(1, \ldots, 1, s_i, 1, \ldots, 1) = s_i$,

d. C_ν *is increasing.*

PROOF. a. We need to prove that, for every i, and for every $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$, if $F_{\nu_i}(z) = F_{\nu_i}(y)$, then

$$F_\nu(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_d) = F_\nu(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_d).$$

Assume, without any loss of generality, that $z \leq y$, and call

$$A = (\overline{\Re} \times \cdots \times \overline{\Re} \times [-\infty, z] \times \overline{\Re} \times \cdots \times \overline{\Re}),$$

$$B = ([-\infty, x_1] \times \cdots \times [-\infty, x_{i-1}] \times [-\infty, y] \times [-\infty, x_{i+1}] \times \cdots \times [-\infty, x_d]),$$

$$D = (\overline{\Re} \times \cdots \times \overline{\Re} \times [-\infty, y] \times \overline{\Re} \cdots \times \overline{\Re}).$$

Therefore

$$A \cap B = ([-\infty, x_1] \times \cdots \times [-\infty, x_{i-1}] \times [-\infty, z] \times [-\infty, x_{i+1}] \times \cdots \times [-\infty, x_d]).$$

Since $A \subseteq A \cup B \subseteq D$ and $\nu(A) = \nu(D)$ by hypothesis, we have $\nu(A) = \nu(A \cup B) = \nu(D)$. The convexity of ν implies

$$\nu(A \cup B) + \nu(A \cap B) \geq \nu(A) + \nu(B)$$

and therefore $\nu(A \cap B) = \nu(B)$, namely

$$F_\nu(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_d) = F_\nu(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_d).$$

Therefore C_ν is uniquely defined on $\times_{i=1}^d \text{Ran}(F_{\nu_i})$. It can be (nonuniquely) extended to $[0,1]^d$ by using, for instance, the procedure indicated in Deheuvels (1978), Schweizer and Sklar (1983, p. 84) (see also Scarsini (1984)).

b. By monotonicity of ν, if $\nu_i([-\infty, z]) = 0$, then $\nu([-\infty, x_1] \times \cdots \times [-\infty, z] \times \cdots \times [-\infty, x_d]) = 0$. The definition of F_ν, F_{ν_i} gives the result.

c. Note that $1 \in \text{Ran}(F_{\nu_j})$, for any $j \in \{1, \ldots, d\}$. If $s_i \in \text{Ran}(F_{\nu_i})$, let $F_{\nu_i}(x_i) = s_i$. Then

$$\begin{aligned} C_\nu(1, \ldots, 1, s_i, 1, \ldots, 1) &= C_\nu(F_{\nu_1}(\infty), \ldots, F_{\nu_{i-1}}(\infty), F_{\nu_i}(x_i), \\ &\qquad F_{\nu_{i+1}}(\infty), \ldots, F_{\nu_d}(\infty)) \\ &= \nu(\overline{\Re} \times \cdots \times \overline{\Re} \times [-\infty, x_i] \times \overline{\Re} \times \cdots \times \overline{\Re}) \\ &= \nu_i([-\infty, x_i]) \\ &= F_{\nu_i}(x_i) \\ &= s_i. \end{aligned}$$

The above mentioned extension procedure of Deheuvels and Schweizer and Sklar insures that the result holds also when $s_i \notin \text{Ran}(F_{\nu_i})$.

d. This is an immediate consequence of the monotonicity of ν. ∎

If ν is a probability measure, then the generalized copula C_ν is the usual copula, as defined by Sklar (1959). As Theorem 9 shows, if ν is convex, then C_ν has some of the properties of the usual copula. For it to be n-increasing (for every $n \leq d$), though, convexity is not sufficient and d-monotonicity is required. Then the result is an immediate corollary of Lemma 6.

THEOREM 10. *If ν is d-monotone, then C_ν is n-increasing for every $n \leq d$.*

PROOF. By Lemma 6 and Corollary 7 there exists a finitely additive probability measure whose distribution function coincides with F_ν. Therefore its copula will be n-increasing for $n \leq d$. ∎

As a consequence of Theorem 8 we obtain that, if ν is convex, then the Fréchet bounds hold for C_ν, too.

COROLLARY 11. *Let ν be convex, then C_ν satisfies the following inequalities*

$$C_-(s_1,\dots,s_d) \le C_\nu(s_1,\dots,s_d) \le C_+(s_1,\dots,s_d),$$

where

$$C_-(s_1,\dots,s_d) = \left(\sum_{i=1}^{d} s_i - d + 1\right)^+,$$

$$C_+(s_1,\dots,s_d) = \min(s_1,\dots,s_d).$$

In the additive case, if the copula of a distribution function F_μ is C_+, then μ concentrates its probability on the set $S := \{(x_1,\dots,x_d) : F_{\mu_1}(x_1) = \cdots = F_{\mu_d}(x_d)\}$, and therefore any set whose intersection with S is empty has μ-probability zero. A weaker version of this result holds for convex continuous capacities.

THEOREM 12. *If ν is convex and continuous from below and $C_\nu = C_+$, then for every pair i,j*

$$\nu\{(x_1,\dots,x_d) : F_{\nu_i}(x_i) > F_{\nu_j}(x_j)\} = 0.$$

PROOF. We will start by proving that $\nu(\overline{\Re}\times\cdots\times(x_i,\infty]\times\overline{\Re}\times\cdots\times\overline{\Re}\times[-\infty,x_j)\times\overline{\Re}\times\cdots\times\overline{\Re}) = 0$, if $F_{\nu_i}(x_i) > F_{\nu_j}(x_j)$. Call

$$A(x_i,x_j) = \overline{\Re}\times\cdots\times\overline{\Re}\times[-\infty,x_i)\times\overline{\Re}\times\cdots\times\overline{\Re}\times[-\infty,x_j)\times\overline{\Re}\times\cdots\times\overline{\Re},$$

$$B(x_i,x_j) = \overline{\Re}\times\cdots\times\overline{\Re}\times(x_i,\infty]\times\overline{\Re}\times\cdots\times\overline{\Re}\times[-\infty,x_j)\times\overline{\Re}\times\cdots\times\overline{\Re}.$$

By convexity

$$\nu(A(x_i,x_j)\cup B(x_i,x_j)) + \nu(A(x_i,x_j)\cap B(x_i,x_j)) \ge \nu(A(x_i,x_j)) + \nu(B(x_i,x_j)).$$

From $C_\nu = C_+$, we obtain $\nu(A(x_i,x_j)\cup B(x_i,x_j)) = \nu(A(x_i,x_j))$, and since $A(x_i,x_j)\cap B(x_i,x_j) = \emptyset$, we have $\nu(A(x_i,x_j)\cap B(x_i,x_j)) = 0$. Therefore $\nu(B(x_i,x_j)) = 0$.

Now let y_i, y_j be such that $y_i \le x_i$, $y_j \le x_j$ and $F_{\nu_i}(y_i) > F_{\nu_j}(y_j)$. From what we have just proved $\nu(B(y_i,y_j)) = 0$. We will prove that $\nu(B(x_i,x_j)\cup B(y_i,y_j)) = 0$.

Call $D = B(x_i,x_j)\cup B(y_i,y_j)$. Again, by convexity $\nu(A(x_i,x_j)\cup D) + \nu(A(x_i,x_j)\cap D) \ge \nu(A(x_i,x_j)) + \nu(D)$ and by the fact that $C_\nu = C_+$, we have $\nu(A(x_i,x_j)\cup D) = \nu(A(x_i,x_j))$. Since $A(x_i,x_j)\cap D \subseteq B(y_i,y_j)$, we have $0 = \nu(A(x_i,x_j)\cap D) \ge \nu(D)$, hence $\nu(D) = 0$.

By iterating the procedure, we can prove that, for every k, $\nu(\bigcup_{n=1}^{k} B(z_i^n, z_j^n)) = 0$, whenever $F_{\nu_i}(z_i^n) > F_{\nu_j}(z_j^n)$, $n \in \{1,\dots,k\}$.

Since

$$\{(x_1, \ldots, x_d) : F_{\nu_i}(x_i) > F_{\nu_j}(x_j)\} = \bigcup_n B(z_i^n, z_j^n)$$

for some suitable sequence (z_i^n, z_j^n) with $F_{\nu_i}(z_i^n) > F_{\nu_j}(z_j^n)$, for all n, by using continuity from below of ν, we obtain the result. ∎

3. Concluding Remarks. In this note we have examined the behavior of nonadditive probabilities (capacities) on $\overline{\Re}^d$. In complete analogy to the usual additive case, we have defined the distribution function of a capacity and we have examined some of its properties. Most of these properties rely on the assumption of convexity (=2-monotonicity) of the capacity. Some other properties require higher degrees of monotonicity. For instance to a d-monotonic capacity, there corresponds a d-increasing distribution function. Furthermore, it is enough to assume convexity of the capacity to establish the existence of a function that relates the joint distribution function to its marginals, but d-monotonicity is required for this function to have all the analytic properties of a copula.

Due to the lack of additivity, the distribution function does not characterize a capacity (not even in the finite case). Nevertheless it can provide useful information when all that is needed is the probability of lower intervals (as in the case of some stochastic orderings or some dependence concepts).

We have used a "naive" approach, and have not resorted to any of the existing representations of capacities as additive measures on different (in general larger) spaces. Some of these representations can be found in Shafer (1976), Weber (1984), Murofushi and Sugeno (1989), Gilboa and Schmeidler (1992a, 1992b). In our future research we plan to make use of some of these representations to obtain new results and generalize known properties from the additive to the nonadditive framework.

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