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DUALITY FOR A NON-TOPOLOGICAL VERSION OF THE MASS TRANSPORTATION PROBLEM

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Summary. A duality theorem is proved for a non-topological version of the mass transportation problem with a given marginal difference. The theorem describes the cost functions for which the duality relation holds.

1. Introduction. The present paper is concerned with a non-topological version of the mass transportation problem. Before stating the problem, I will recall its topological version (Levin (1984, 1987, 1990a) and Levin and Milyutin (1979); for the case of compact spaces and continuous cost functions see also Levin (1974, 1975, 1977, 1978)).

Given a topological space X, a Radon measure ρ on it with $\rho(X) = 0$, and a cost function $c : X \times X \to \boldsymbol{R}^1 \cup \{+\infty\}$, the problem is to minimize the functional

$$c(\mu) := \int_{X\times X} c(x,y)\mu(d(x,y))$$

over all positive Radon measures μ having the given marginal difference ρ. In other words, the optimal value of

$$\mathcal{A}(c,\rho) := \inf\{c(\mu) : \mu \geq 0, \pi_1\mu - \pi_2\mu = \rho\} \tag{1}$$

is to be determined, where

$$(\pi_1\mu)(B) := \mu(B\times X), \quad (\pi_2\mu)(B) := \mu(X\times B)$$

for any Borel set B in X.

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Another (in general, non-equivalent) form of the mass transportation problem is the problem with fixed marginals. It consists in finding the optimal value

$$\mathcal{C}(c,\sigma_1,\sigma_2) := \inf\{c(\mu) : \mu \geq 0, \pi_1\mu = \sigma_1, \pi_2\mu = \sigma_2\} \tag{2}$$

for given positive Radon measures, σ_1 and σ_2, on X with $\sigma_1(X) = \sigma_2(X)$.

Extremal problems (1) and (2) are often referred to as two forms of the (general) Monge-Kantorovich problem. The classical Monge-Kantorovich problem, extending the old (1781) "déblais et remblais" problem of Monge, answers the case when X is a compact metric space and the cost function c is its metric. The classical version of problem (2) was first posed and examined by Kantorovich (1942), and the classical version of problem (1) was investigated later by Kantorovich and Rubinstein (1957, 1958) (see also Kantorovich and Akilov (1984)). These two forms of the classical Monge-Kantorovich problem are equivalent in the sense that $\mathcal{A}(c,\sigma_1-\sigma_2) = \mathcal{C}(c,\sigma_1,\sigma_2)$ and in problem (1) there exists an optimal measure μ satisfying $\pi_1\mu = \sigma_1, \pi_2\mu = \sigma_2$; see Kantorovich and Akilov (1984). However, in the general setting, these two forms of the Monge-Kantorovich problem are not equivalent, and the equality

$$\mathcal{A}(c,\sigma_1-\sigma_2) = \mathcal{C}(c,\sigma_1,\sigma_2)$$

for all $\sigma_1 \geq 0, \sigma_2 \geq 0$ with $\sigma_1 X = \sigma_2 X$ holds if and only if c satisfies the triangle inequality; see Levin (1990a, Theorem 9.2).

Problem (1) in the general case is much more difficult than the corresponding problem (2), so in the sequel I will concentrate on it, i.e., on mass transportation problems with a fixed marginal difference.

The main result for problem (1) is a duality theorem characterizing cost functions for which the duality relation $\mathcal{A}(c,\rho) = \mathcal{B}(c,\rho)$ holds for all $\rho \in V_0(X)$, where $\mathcal{B}(c,\rho)$ stands for the optimal value of the dual problem and $V_0(X)$ denotes the set of measures ρ satisfying $\rho(X) = 0$.

Before formulating the dual problem and the duality theorem, I will specify the classes from which the space X and the cost function c are chosen.

A space X is said to be in the class $\mathcal{L}_0$ if it is homeomorphic to a Baire subset of some compact space. Clearly, Polish spaces and σ-compact locally compact spaces belong to $\mathcal{L}_0$. Also it is easily seen that $X \times Y \in \mathcal{L}_0$ whenever $X \in \mathcal{L}_0$ and $Y \in \mathcal{L}_0$, hence the class $\mathcal{L}_0$ is wide enough.

A cost function c is said to belong to the class $\mathcal{C}$ if it is bounded below and if its Lebesgue sublevel sets $\{(x,y) : c(x,y) \leq \alpha\}$ $(\alpha \in \boldsymbol{R}^1)$ can be represented as results of the application of the A-operation to Baire sets in $X \times X$. Any Baire function that is bounded below belongs to $\mathcal{C}$.

A cost function c is said to belong to the class $\mathcal{C}_*$ if for all x, y with $x \neq y$ a representation

$$c(x,y) = \sup\{u(x) - u(y) : u \in Q\}$$

holds, where Q is a nonempty set in the space $C^b(X)$ of all bounded continuous real-valued functions on X. Any continuous metric on X belongs to $\mathcal{C}_*$. It follows from the results of Levin and Milyutin (1979) that if X is compact and $c(x,x) = 0$ for all $x \in X$, then $c \in \mathcal{C}_*$ if and only if c is lower semicontinuous and satisfies the triangle inequality

$$c(x,y) + c(y,z) \geq c(x,z) \quad \text{for all} \quad x,y,z \in X.$$

The dual problem to (1) consists in finding the value

$$\mathcal{B}(c,\rho) := \sup\left\{\int_X u(x)\rho(dx) : u \in Q(c)\right\},$$

where

$$Q(c) := \{u \in C^b(X) : u(x) - u(y) \leq c(x,y) \text{ for all } \quad x,y \in X\}$$

(by definition, $\mathcal{B}(c,\rho) = -\infty$ if $Q(c) = \emptyset$).

Given a cost function c, a reduced cost function c_*, connected with it, can be defined on $X \times X$ as follows:

$$c_*(x,y) := \min\Big\{c(x,y), \inf_n \inf\{c(x,z_1) + c(z_1,z_2) + \dots + c(z_n,y) : z_1, \dots, z_n \in X\}\Big\}. \tag{3}$$

If $c(x,x) = 0$ for all $x \in X$, this formula can be rewritten as follows:

$$c_*(x,y) = \lim_{n\to\infty} \inf\{c(x,z_1) + c(z_1,z_2) + \dots + c(z_n,y) : z_1, \dots, z_n \in X\}.$$

The duality theorem for the topological version of the mass transportation problem can be now formulated as follows:

Theorem 1.1 (Levin (1987, Theorem 2, 1990a, Theorem 9.4)). *Let $X \in \mathcal{L}_0$ and $c \in \mathcal{C}$. Then the following statements, (A) and (B), are equivalent:*

(A) The duality relation $\mathcal{A}(c,\rho) = \mathcal{B}(c,\rho)$ holds for all $\rho \in V_0(X)$;

(B) Two conditions, (B_1) and (B_2), are satisfied as follows:

(B_1) $\mathcal{A}(c,\rho) = \lim_{N\to\infty} \mathcal{A}(c \wedge N, \rho)$ for all $\rho \in V_0(X)$, where $(c \wedge N)(x,y) := \min(c(x,y), N)$;

(B_2) either $c_* \in C_*$ (and this is the case when $\mathcal{A}(c,\rho) = \mathcal{B}(c,\rho) = \mathcal{C}(c_*, \rho_+, \rho_-) > -\infty$ for all $\rho = \rho_+ - \rho_- \in V_0(X)$), or *$c_*$ is unbounded from below (and this is the case when $\mathcal{A}(c,\rho) = \mathcal{B}(c,\rho) = -\infty$ for all $\rho \in V_0(X)$).*

REMARKS: 1. The fulfillment of the condition (B_1) is obvious when c is bounded above. Also, (B_1) may be abandoned provided that c satisfies the triangle inequality (which implies $c = c_*$) and that there exists a bounded universally measurable function $v : X \to \boldsymbol{R}^1$ such that $v(x) - v(y) \leq c(x,y)$ for all $x, y \in X$. Moreover, under these assumptions on c, the equivalence (A) $\Leftrightarrow$ (B_2) holds even if X and c are chosen from broader classes than $\mathcal{L}_0$ and $\mathcal{C}$: X may be homeomorphic to a universally measurable subset in a compact space while c may be bounded below and universally measurable; see Theorem 9.2 in Levin (1990a).

2. In general, the assumption (B_1) cannot be omitted even if c is a nonnegative continuous function $X \times X \to \boldsymbol{R}^1$. For $X = \{0, 1, 2, \ldots\}$ an example is given in Levin (1990a, p. 147) of such a function c together with a measure $\rho \in V_0(X)$, for which $\mathcal{A}(c,\rho) = +\infty$ and $\mathcal{A}(c \wedge N, \rho) = \mathcal{B}(c,\rho) = 0$ for all $N > 0$.

The duality theorem 1.1 has applications to measure theory and to some aspects of utility theory in mathematical economics (see Levin (1981, 1983a, 1983b, 1985a, 1986, 1990a). Some stochastic applications of this theorem and of similar duality theorems for closely related extremal problems may be found in Rachev (1984) and Levin and Rachev (1989).

The goal of the present paper is to study the duality for an abstract (non-topological) version of the mass transportation problem. In this version, X is an arbitrary nonempty set, all measures under consideration are assumed to be finite linear combinations of Dirac measures, and the Banach space $C^b(X)$ in the statement of the dual problem is replaced by the linear space $\boldsymbol{R}^X$ of all functions $u : X \to \boldsymbol{R}^1$. The main result is a duality theorem asserting the validity of a duality relation similar to (A).

2. Duality Theorem for a Non-Topological Version of the Mass Transportation Problem. Let X be an arbitrary nonempty set and let $E(X)$ denote the linear space $\boldsymbol{R}^X$ of all real-valued functions on X. Equipped with the product topology, $E(X)$ is a Hausdorff locally convex linear topological space. The conjugate space $E(X)^*$ consists of functions $\sigma : X \to \boldsymbol{R}^1$ with finite supports $\operatorname{supp}\sigma = \{x : \sigma(x) \neq 0\}$. Any such σ will be treated as a (signed) finite measure on the σ-algebra 2^X. This measure is represented as a linear combination with coefficients $\sigma(x)$ of Dirac measures at points

$x \in \operatorname{supp}\sigma$. The pairing between $E(X)$ and $E(X)^*$ is given by

$$\langle \varphi, \sigma \rangle = \int_X \varphi(x)\sigma(dx) = \sum_{x \in \operatorname{supp}\sigma} \varphi(x)\sigma(x) \quad (\varphi \in E(X), \sigma \in E(X)^*).$$

Let $E(X)^*_+$ (resp. $E(X \times X)^*_+$) denote the convex cone of nonnegative measures in $E(X)^*$ (resp. in $E(X \times X)^*$) and let $E(X)^*_0$ stand for the linear subspace in $E(X)^*$ consisting of measures σ with $\sigma(X) = 0$.

Given a cost function $c : X \times X \to \boldsymbol{R}^1 \cup \{+\infty\}$ and a measure $\rho \in E(X)^*_0$, two linear extremal problems are considered. One of them, a version of the mass transportation problem with a given marginal difference, consists in finding the optimal value

$$\mathcal{A}_0(c, \rho) := \inf \left\{ \int_{X \times X} c(x,y)\mu(d(x,y)) : \mu \in E(X \times X)^*_+, \pi_1\mu - \pi_2\mu = \rho \right\}.$$

The other problem, dual to the first one, is to find the optimal value

$$\mathcal{B}_0(c, \rho) := \sup \left\{ \int_X u(x)\rho(dx) : u \in Q_0(c) \right\},$$

where

$$Q_0(c) := \{u \in E(X) : u(x) - u(y) \le c(x,y) \quad \text{for all} \quad x, y \in X\}.$$

Clearly $\mathcal{B}_0(c,\rho) \le \mathcal{A}_0(c,\rho)$ for all $\rho \in E(X)^*_0$.

Together with the original cost function c, the reduced cost function c_* defined by formula (3) will be considered.

THEOREM 2.1 (DUALITY THEOREM). *(i) If $c_*(x,y) < +\infty$ for all $x, y \in X$, then either $c_*(x,y) > -\infty$ for all $x, y \in X$ or $c_*(x,y) = -\infty$ for all $x, y \in X$. In the first case, $Q_0(c)$ is nonempty,*

$$c_*(x,y) = \sup_{u \in Q_0(c)} (u(x) - u(y)) \quad \text{for all } x, y \in X \text{with } x \ne y, \tag{4}$$

and

$$\mathcal{A}_0(c,\rho) = \mathcal{B}_0(c,\rho) > -\infty \quad \textit{whenever} \quad \rho \in E(X)^*_0. \tag{5}$$

In the second case, $Q_0(c)$ is empty, and

$$\mathcal{A}_0(c,\rho) = \mathcal{B}_0(c,\rho) = -\infty \quad \textit{whenever} \quad \rho \in E(X)^*_0.$$

(ii) If $Q_0(c) = \emptyset$ but $c_* \not\equiv -\infty$, then there exist x and y in X, $x \ne y$, such that $\mathcal{A}_0(c, \epsilon_x - \epsilon_y) = +\infty$ and $\mathcal{B}_0(c, \epsilon_x - \epsilon_y) = -\infty$. (Here and below ϵ_x denotes the Dirac measure at x.)

REMARKS: 1. An optimal measure for the mass transportation problem need not exist even if $Q_0(c)$ is nonempty. For the topological version of the problem, a very simple counterexample is given in Levin (1974) as follows: $X = [0,1], c(x,y) = (x-y)^2$, and $\rho = \epsilon_0 - \epsilon_1$. The same counterexample applies to the non-topological version as well.

2. The case (ii) actually occurs, which may be illustrated by the following example (cf. Levin and Milyutin (1979, Remark 2.2)):

$$c(x,y) = \begin{cases} +\infty & \text{if } x \neq y, \\ -1 & \text{if } x = y. \end{cases}$$

Clearly $Q_0(c) = \emptyset$ and

$$c_*(x,y) = \begin{cases} +\infty & \text{if } x \neq y, \\ -\infty & \text{if } x = y. \end{cases}$$

The following result supplements the duality theorem.

THEOREM 2.2 *Suppose $c_*(x,y) < +\infty$ for all $x, y \in X$. Then the following statements are equivalent:*

(a) $Q_0(c)$ is nonempty;

(b) $c_*(x,y) > -\infty$ for all $x, y \in X$;

(c) $c_*(x,x) \geq 0$ for all $x \in X$;

(d) $\sum_{i=1}^{n} c(x_{i-1}, x_i) \geq 0$ for each cycle $x_0, x_1, \ldots, x_n = x_0$.

In the case when $c(x,y) < +\infty$ for all $x, y \in X$ and $c(x,x) = 0$ for all $x \in X$, this theorem is proved in Levin (1990b, Lemma 2). In the general case the proof is practically the same. In Levin (1990b) applications of the theorem to the mass transportation problem with a smooth cost function and to cyclically monotone operators are given. Applications to dynamic optimization problems and to mathematical economics will be given elsewhere.

3. Proof of the Duality Theorem. Let c be any function $X \times X \to \boldsymbol{R}^1 \cup \{+\infty\}$.

LEMMA 3.1. $\mathcal{B}_0(c,\rho) = \mathcal{B}_0(c_*,\rho)$ for all $\rho \in E(X)_0^*$.

This follows from the obvious equality $Q_0(c) = Q_0(c_*)$.

Note that c_* may take the value $-\infty$, so if for a given μ the integral

$$c_*(\mu) = \int_{X \times X} c_*(x,y)\mu(d(x,y))$$

makes no sense — that is, if $c_{*+}(\mu) = c_{*-}(\mu) = +\infty$ —,then it will be assumed, that by definition, $c_*(\mu) = +\infty$. With this convention the value $\mathcal{A}_0(c_*, \rho)$ is determined for every $\rho \in E(X)^*_0$ and $\mathcal{A}_0(c, \rho) \geq \mathcal{A}_0(c_*, \rho)$.

LEMMA 3.2. *For any $\mu \in E(X \times X)^*_+$ and any integer n there exists a measure $\mu_n \in E(X \times X)^*_+$ such that $\pi_1\mu_n - \pi_2\mu_n = \pi_1\mu - \pi_2\mu$ and*

$$c(\mu_n) \leq \begin{cases} c_*(\mu) + (1/n)\mu(X \times X) & \text{if } c_*(\mu) > -\infty, \\ -d_n(\mu) & \text{if } c_*(\mu) = -\infty, \end{cases}$$

where $d_n(\mu) \to \infty$ as $n \to \infty$.

PROOF. We assume $c_*(\mu) < +\infty$ since otherwise the statement is trivial. Let

$$\mu = \sum_{k=1}^{m} a_k \epsilon_{(x_k, y_k)},$$

where $a_k > 0$, $k = 1, \ldots, m$; hence

$$c_*(\mu) = \sum_{k=1}^{m} a_k c_*(x_k, y_k).$$

Using the definition of c_*, choose points $z_{kn1}, \ldots, z_{knm(k,n)}$ in X such that

$$\sum_{i=0}^{m(k,n)} c(z_{kni}, z_{kni+1}) \leq b_{kn} := \begin{cases} c_*(x_k, y_k) + 1/n & \text{if } c_*(x_k, y_k) > -\infty, \\ -n & \text{if } c_*(x_k, y_k) = -\infty, \end{cases}$$

where $z_{kn0} = x_k$, $z_{knm(k,n)+1} = y_k$. Denote

$$M = \{(x, y) \in \operatorname{supp}\mu : c_*(x, y) > -\infty\}$$

and take into account that $c_*(\mu) = -\infty$ implies $\mu((X \times X) \setminus M) > 0$. It is easily seen that the measure

$$\mu_n := \sum_{k=1}^{m} \sum_{i=0}^{m(k,n)} a_k \epsilon_{(z_{kni}, z_{kni+1})}$$

has all the required properties with

$$d_n(\mu) = n\mu((X \times X) \setminus M) - \int_M \left(c_*(x, y) + \frac{1}{n}\right) \mu(d(x, y)).$$

LEMMA 3.3. $\mathcal{A}_0(c, \rho) = \mathcal{A}_0(c_*, \rho)$ for every $\rho \in E(X)^*_0$.

This is a simple consequence of Lemma 3.2.

Let us extend $\mathcal{A}_0(c,\cdot)$ to the whole space $E(X)^*$ by setting

$$\mathcal{A}_0(c,\rho) = +\infty \quad \text{if } \rho \notin E(X)_0^*.$$

Observe that the extended functional is sublinear (we assume by definition $+\infty + (-\infty) = +\infty$). The following lemma describes its subdifferential

$$\begin{aligned}\partial\mathcal{A}_0(c,\cdot)(0) :&= \{u \in E(X) : \langle u,\rho\rangle \le \mathcal{A}_0(c,\rho) \text{ for all } \rho \in E(X)^*\} \\ &= \{u \in E(X) : \langle u,\rho\rangle \le \mathcal{A}_0(c,\rho) \text{ for all } \rho \in E(X)_0^*\}.\end{aligned}$$

LEMMA 3.4. $\partial\mathcal{A}_0(c,\cdot)(0) = Q_0(c)$.

PROOF. The inclusion $Q_0(c) \subseteq \partial\mathcal{A}_0(c,\cdot)(0)$ is obvious. On the other hand, if $u \in \partial\mathcal{A}_0(c,\cdot)(0)$, then for any $x, y \in X$

$$u(x) - u(y) = \langle u, \epsilon_x - \epsilon_y\rangle \le \mathcal{A}_0(c, \epsilon_x - \epsilon_y) \le c(\epsilon_{(x,y)}) = c(x,y),$$

that is $u \in Q_0(c)$.

LEMMA 3.5. *Assume $c_*(x,y) > -\infty$ for all $x, y \in X$. Then for any $\rho \in E(X)_0^*$ there exists a measure $\mu_0 \in E(X\times X)_+^*$ such that $\pi_1\mu_0 = \rho_+, \pi_2\mu_0 = \rho_-$, and*

$$\mathcal{A}_0(c_*,\rho) = c_*(\mu_0) = \min\{c_*(\mu) : \quad \mu \in E(X\times X)_+^*, \pi_1\mu = \rho_+, \pi_2\mu = \rho_-\},$$

where $\rho = \rho_+ - \rho_-$ is the Jordan decomposition of ρ. In particular,

$$\mathcal{A}_0(c_*, \epsilon_x - \epsilon_y) = \begin{cases} c_*(x,y) & \text{if } x \ne y \quad (\mu_0 = \epsilon_{(x,y)}), \\ 0 & \text{if } x = y \quad (\mu_0 = 0). \end{cases}$$

PROOF (cf. proof of Lemma 5 in Levin (1978)). The result will follow if, for each $\mu \in E(X\times X)_+^*$ with $\pi_1\mu - \pi_2\mu = \rho$, a measure $\mu' \in E(X\times X)_+^*$ can be found such that $\pi_1\mu' = \rho_+, \pi_2\mu' = \rho_-$, and $c_*(\mu') \le c_*(\mu)$. (Indeed, in such a case the problem consists in finding $\inf\{c_*(\mu') : \mu' \in E(X\times X)_+^*, \pi_1\mu' = \rho_+, \pi_2\mu' = \rho_-\}$, and since this is a finite-dimensional linear program with a compact constraint set, the infimum is attained.)

A triple of points in X, $\{x_0, y_0, z_0\}$, is said to be a transshipment with respect to μ if $z_0 \ne x_0, z_0 \ne y_0$ and $\mu(x_0,z_0) > 0, \mu(z_0,y_0) > 0$. If μ has such a transshipment, then form a new measure μ_1 by setting

$$\mu_1(x,y) = \begin{cases} \mu(x_0,z_0) - a & \text{if } x = x_0, y = z_0, \\ \mu(z_0,y_0) - a & \text{if } x = z_0, y = y_0, \\ \mu(x_0,y_0) + a & \text{if } x = x_0, y = y_0, \\ \mu(x,y) & \text{in all other cases,} \end{cases}$$

where $a := \min(\mu(x_0, z_0), \mu(z_0, y_0))$. It is easily seen that $\mu_1 \in E(X \times X)^*_+$, $\pi_1\mu_1 - \pi_2\mu_1 = \pi_1\mu - \pi_2\mu = \rho$, and μ_1 has at least one transshipment less than μ. Further, since c_* satisfies the triangle inequality, we have

$$c_*(\mu_1) = c_*(\mu) - ac_*(x_0, z_0) - ac_*(z_0, y_0) + ac_*(x_0, y_0) \le c_*(\mu).$$

After repeating such a procedure several times, we get a measure with no transshipments, $\mu_n \in E(X \times X)^*_+$, such that $\pi_1\mu_n - \pi_2\mu_n = \rho$ and $c_*(\mu_n) \le c_*(\mu_{n-1}) \le \ldots \le c_*(\mu_1) \le c_*(\mu)$.

Now define the measure $\mu' \in E(X \times X)^*_+$, via

$$\mu'(x, y) := \begin{cases} \mu_n(x, y) & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

and observe that

$$\operatorname{supp}\pi_1\mu' \cap \operatorname{supp}\pi_2\mu' = \emptyset \tag{6}$$

and

$$c_*(\mu') = c_*(\mu_n) - \sum_{(x,x)\in\operatorname{supp}\mu_n} c_*(x, x)\mu_n(x, x) \le c_*(\mu_n)$$

(since $c_* > -\infty$, one has $c_*(x, x) \ge 0$ for all $x \in X$). Clearly $\pi_1\mu' - \pi_2\mu' = \pi_1\mu_n - \pi_2\mu_n = \rho$ and, in view of (6), this means $\pi_1\mu' = \rho_+$, $\pi_2\mu' = \rho_-$.

Proof of Theorem 2.1. (*i*) It follows easily from the triangle inequality for c_* that either $c_*(x, y) > -\infty$ for all $x, y \in X$ or $c_*(x, y) = -\infty$ for all $x, y \in X$.

If $c_*(x, y) > -\infty$ for all $x, y \in X$, then c_* takes only finite values. In this case the set $Q_0(c)$ is nonempty because it contains the functions u_*, $u_*(x) = c_*(x, x_*)$ for $x \neq x_*$ and $u_*(x_*) = 0$, where x_* is any fixed point in X.

Let us verify (4). Assume the contrary. Then

$$c_*(x_0, y_0) > \sup_{u \in Q_0(c)} (u(x_0) - u(y_0)) = \mathcal{B}_0(c, \epsilon_{x_0} - \epsilon_{y_0})$$

for some $x_0, y_0 \in X$ with $x_0 \neq y_0$. The function

$$u_0(x) = \begin{cases} c_*(x, y_0) & \text{if } x \neq y_0, \\ 0 & \text{if } x = y_0, \end{cases}$$

belongs to $Q_0(c)$, hence

$$\mathcal{B}_0(c, \epsilon_{x_0} - \epsilon_{y_0}) \ge u_0(x_0) - u_0(y_0) = c_*(x_0, y_0),$$

a contradiction.

Let us verify (5). Note that the space $E(X)^*$ is topologized in a natural way as the topological direct sum of real lines $R_{(x)}, x \in X$. As is well known, this topology — denote it by t — is the finest locally convex topology on $E(X)^*$, and $(E(X)^*, t)^* = E(X)$. Since

$$E(X)_0^* = \{\rho \in E(X)^* : \langle \mathbf{1}, \rho \rangle = 0\},$$

where $\mathbf{1}(x) = 1$ for all $x \in X$, $E(X)_0^*$ is a closed hypersubspace in $E(X)^*$ and the induced topology $t \mid E(X)_0^*$ is obviously the finest locally convex topology on $E(X)_0^*$. Since $c_*(x, y) < +\infty$ for all $x, y \in X$, we have

$$-\infty < \mathcal{B}_0(c_*, \rho) \leq \mathcal{A}_0(c_*, \rho) < +\infty$$

whenever $\rho \in E(X)_0^*$. The restriction of $\mathcal{A}_0(c, \cdot) = \mathcal{A}_0(c_*, \cdot)$ on $E(X)_0^*$ is thus a proper sublinear functional $E(X)_0^* \to \boldsymbol{R}^1$, hence the functional is continuous with respect to the topology $t \mid E(X)_0^*$. Now, because $E(X)_0^*$ is closed in $E(X)^*$, $\mathcal{A}_0(c, \cdot)$ is lower semicontinuous as a functional on the whole space $(E(X)^*, t)$. As is known from convex analysis (see, e.g., Levin (1985b, Theorem 0.3 and its Corollary 3)), in such a case the equality

$$\mathcal{A}_0(c, \rho) = \sup\{\langle u, \rho \rangle : u \in \partial \mathcal{A}_0(c, \cdot)(0)\}$$

holds for all $\rho \in E(X)_0^*$ which, in view of Lemma 3.4, may be rewritten as

$$\mathcal{A}_0(c, \rho) = \mathcal{B}_0(c, \rho) \quad \text{for every} \quad \rho \in E(X)_0^*.$$

If now $c_*(x, y) = -\infty$ for all $x, y \in X$, then, by Lemma 3.3, $\mathcal{A}_0(c, \rho) = -\infty$ for all $\rho \in E(X)_0^*$. Also $\mathcal{B}_0(c, \rho) = -\infty$ for all $\rho \in E(X)_0^*$ because $Q_0(c) = \emptyset$.

The statement (i) is now completely proved.

(ii) If $Q_0(c) = \emptyset$ but $c_* \not\equiv -\infty$, then $c_*(x_0, y_0) = +\infty$ for some $x_0, y_0 \in X$ with $x_0 \neq y_0$. Because $Q_0(c) = \emptyset$, we have $\mathcal{B}_0(c, \epsilon_{x_0} - \epsilon_{y_0}) = -\infty$. On the other hand, applying Lemma 3.3 yields

$$\mathcal{A}_0(c, \epsilon_{x_0} - \epsilon_{y_0}) = \mathcal{A}_0(c_*, \epsilon_{x_0} - \epsilon_{y_0}).$$

Let μ be an arbitrary measure in $E(X \times X)_+^*$ with $\pi_1\mu - \pi_2\mu = \epsilon_{x_0} - \epsilon_{y_0}$. Arguing as in the proof of Lemma 3.5, we obtain a measure with no transshipments, $\mu_* \in E(X \times X)_+^*$, such that $\pi_1\mu_* - \pi_2\mu_* = \epsilon_{x_0} - \epsilon_{y_0}$ and $c_*(\mu_*) \leq c_*(\mu)$. (Notice that the supposition of Lemma 3.5 that $c_* > -\infty$ is not used in this argument.) It follows that

$$\mu_* = \epsilon_{(x_0, y_0)} + \sum_{(x,x) \in \operatorname{supp} \mu_*} \alpha(x) \epsilon_{(x,x)},$$

where $\alpha(x) \geq 0$. Since $c_*(x_0, y_0) = +\infty$, we have $c_{*+}(\mu_*) = +\infty$ hence $c_*(\mu_*) = +\infty$, and in view of the arbitrariness of μ, $\mathcal{A}_0(c_*, \epsilon_{x_0} - \epsilon_{y_0}) = +\infty$.

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