Distributions with Fixed Marginals and Related Topics
IMS Lecture Notes – Monograph Series Vol. 28, 1996

Proximity of Probability Measures with Common Marginals in a Finite Number of Directions

By L. B. Klebanov* and S. T. Rachev†
Institute of Mathematical Geology, St. Petersburg
University of California, Santa Barbara

We provide estimates of the closeness between probability measures defined on $\mathbb{R}^n$ which have the same marginals in a finite number of arbitrary directions. Our estimates show that the probability laws get closer in the λ-metric which metrizes the weak topology when the number of coinciding marginals increases. Our results offer a solution to the computer tomography paradox stated in Gutmann, Kemperman, Reeds, and Shepp (1991).

1. Introduction and Statement of the Problem. Let Q_1 and Q_2 be a pair of probabilities, i.e. probability measures defined on the Borel σ-field of $\mathbb{R}$. Lorentz (1949) gave criteria for the existence of a probability density function $g(\cdot)$ on $\mathbb{R}^2$ taking only two values, 0 or 1, and having Q_1 and Q_2 as marginals. Kellerer (1961) generalized this result, obtaining the necessary and sufficient conditions for the existence of a density $f(\cdot)$ on $\mathbb{R}^2$ which satisfies the inequalities $0 \leq f(\cdot) \leq 1$, and has Q_1 and Q_2 as marginals (see also Strassen (1965) and Jacobs (1978)). Fishburn et al. (1990) were able to show that Kellerer's and Lorentz's conditions are equivalent, i.e. for any density $f(\cdot), 0 \leq f \leq 1$, on $\mathbb{R}^2$ there exists a density $g(\cdot)$ taking the values 0 and 1 only, which has the same marginals. In general, similar results hold for probability densities on $\mathbb{R}^m, \quad m \geq 2$, when the $(m-1)$-dimensional marginals are prescribed. A considerably stronger result was established by Gutmann et al. (1991). This is that, for any probability density $f(\cdot), 0 \leq f \leq 1$, on $\mathbb{R}^m$ and for any finite number of directions, there exists a probability density $g(\cdot)$ taking the values 0, 1 only, which has the same marginals as $f(\cdot)$ in the chosen directions. It follows that densities having the same marginals in a finite number of arbitrary directions may differ considerably in the uniform

* Research partly supported by the Russian Foundation of Fundamental Investigation Grant 93-011-1959.

† Research partly supported by grants from NSF DMS-9103451 and NATO CRG-900798.

AMS 1991 Subject Classification: Primary 60A10; Secondary 60E10

Key words and phrases: Probability measures with fixed marginals, computer tomography.

metric.

Let us note that the problem of the existence of probability measures with fixed marginals is important for the theory of probability metrics. This is especially true in studying the structure of minimal metrics (see Rachev (1991)). Most of our results will make use of relationships between different probability metrics, analyzed in the monograph by Kakosjan, Klebanov and Rachev (1988), referred to below as KKR (1988).

The purpose of this work is to show that, under moment-type conditions, measures having a "large" number of coinciding marginals are close to each other in the *weak* metrics. Our method is based on techniques used in the classical moment problem. The key idea in showing that measures with large numbers of common marginals are close to each other in the weak metrics is best understood by comparing three results. The first is the Theorem of Guttman et al. (1991) mentioned above. The second (see Karlin and Studden (1966), p. 265) states that if a finite number of moments $\mu_1, \cdots, \mu_n$ of a function $f, 0 \leq f \leq 1$, are given, then there exists a function g which takes the values 0 or 1 only, and possesses the moments $\mu_1, \cdots, \mu_n$. It is clear that these two results are similar; however, the condition of equality of the marginals is more complex than the condition of coincidence of the moments. Finally, the third result (see KKR (1988), p. 170-197) gives estimates of the closeness in the λ-metric on $\mathbb{R}^1$ for measures having common moments $\mu_1, \cdots, \mu_n (n < \infty)$. These estimates are expressed in terms of the truncated Carleman's series $\beta_m = \sum_{j=1}^{2m} \mu_{2j}^{-1/(2j)}, (2m < n)$. The result shows that the closeness in the λ-metric is of order $\beta_m^{-1/4}$. Of course, since the condition of common marginals seems to be more restrictive than the condition of equal moments, one should be able to construct a similar estimate expressed in terms of the common marginals only. Furthermore, the technique required for such a construction should be similar to that used in this paper.

Following this plan we derive estimates for closeness of measures in $\mathbb{R}^2$ having coinciding marginals in n directions. We first consider the case when one of the measures has compact support: in this case, the λ-closeness of measures has order $1/n$. Further, the compactness assumption will be relaxed by Carleman's assumption in the problem of moments. Here the λ-closeness of measures is of order $\beta_{n/2}^{-1/4}$. We also display estimates of the closeness of measures with ε-coinciding marginals. These estimates differ from the corresponding ones with equal marginals by an additional term of order $1/ln\frac{1}{\varepsilon}$. We conclude the paper by applying our results to the problems of computer tomography. In particular, we offer a solution of the paradox in computer tomography stated in Gutmann, et al. (1991), and compare this solution with analogous results in Khalfin and Klebanov (1990).

2. The Case of Probability Measures on $\mathbb{R}^2$. To avoid drowning the basic ideas in too much detail, let us consider only the 2-dimensional case in full.

Let $\theta_1, \cdots, \theta_n$ be n unit vectors on the plane and P_1, P_2 be two probabilities on $\mathbb{R}^2$, having the same marginals in the directions $\theta_1, \cdots, \theta_n$. To estimate the distance between P_1 and P_2, different weak metrics can be used; however, it seems, that the λ-metric is the most convenient for our purposes. This metric is defined as follows (see, e.g., Zolotarev(1986))

$$\lambda(P_1, P_2) = \min_{T>0} \quad \max\left\{ \max_{\|t\|\le T} \left| \int_{\mathbb{R}^2} e^{i(t,x)}(P_1 - P_2)(dx) \right|, \frac{1}{T} \right\}, \tag{1}$$

where $(\cdot,\cdot)$ is the inner product and $\|\cdot\|$ is the Euclidean norm. Clearly, λ metrizes weak convergence.

Our first result concerns the important case where one of the probability measures considered has compact support.

THEOREM 1. *Let $\theta_1, \cdots \theta_n$ be $n \ge 2$ unit vectors in $\mathbb{R}^2$, no two of which are collinear. Let the support of the probability P_1 be a subset of the unit disc, and let the probability P_2 have the same marginals as P_1 in the directions $\theta_1, \cdots \theta_n$. Set*

$$s = 2\left[\frac{n-1}{2}\right], \tag{2}$$

Then

$$\lambda(P_1, P_2) \le \left(\frac{2}{s!}\right)^{\frac{1}{s+1}}. \tag{3}$$

Here $[r]$ denotes the integer part of the number r. We can replace the right-hand side of (3) by C/s, where C is a constant; note that, as $s \to \infty$, $\left(\frac{2}{s!}\right)^{\frac{1}{s+1}} \sim e/s$.

PROOF. The metric (1) is invariant under rotations of the coordinate system, so that we can assume the following conditions to hold:

(a) the directions $\theta_j (j = 1, \cdots, n)$ are not parallel to the axes.

(b) there exists at least one pair of directions, say, θ_{j_1}, and θ_{j_2}, such that $\theta_{j_1} = (a, b), \theta_{j_2} = (a, -b)$, where $a \ne 0, b \ne 0$, i.e. the vectors θ_{j_1} and θ_{j_2} are symmetric about the horizontal axis.

Since P_1 has all moments and the marginals of P_1 and P_2 coincide, for any k we have

$$\int_{\mathbb{R}^2} (x, \theta_j)^k P_1(dx) = \int_{\mathbb{R}^2} (x, \theta_j)^k P_2(dx), \tag{4}$$

$j = 1, \cdots, n.$

We first show that P_2 has moments of any order. Consider (4) with $j = j_1, j = j_2$ and $x = (x_1, x_2)$; then

$$\int_{\mathbb{R}^2} \Big(x_1 a \pm x_2 b\Big)^k P_1\Big(dx_1, dx_2\Big) = \int_{\mathbb{R}^2} \Big(x_1 a \pm x_2 b\Big)^k P_2\Big(dx_1, dx_2\Big),$$

where the signs on the left-hand and right-hand side should coincide. It follows that,

$$\begin{aligned}\int_{\mathbb{R}^2}&[(x_1a+x_2b)^k+(x_1a-x_2b)^k]P_1(dx_1,dx_2)\\&=\int_{\mathbb{R}^2}[(x_1a+x_2b)^k+(x_1a-x_2b)^k]P_2(dx_1,dx_2),\end{aligned}\tag{5}$$

and all integrals are finite. Note that when k is even, the terms of the integrand all have positive coefficients and even powers. Therefore

$$\begin{aligned}(ax_1+bx_2)^k+(ax_1-bx_2)^k&=2\sum_{r=0}^{k/2}\binom{k}{2r}a^{2r}b^{2(\frac{k}{2}-r)}x_1^{2r}x_2^{2(\frac{k}{2}-r)}\\&\geq a^kx_1^k+b^kx_2^k.\end{aligned}$$

From this inequality and (5), we readily obtain the existence of the moments of P_2 of even order for each distinct coordinate, and therefore the moments of all orders.

Next we show that all moments of P_1 and P_2 of order $\leq n-1$ agree. Let

$$\mu_{rt}(P_l)=\int_{\mathbb{R}^2}x_1^rx_2^tP_l(dx),\quad l=1,2.$$

Then setting $\theta_j=(u_j,v_j)$ in (4) yields

$$\sum_{l=0}^{k}\binom{k}{l}u_j^lv_j^{k-l}\Big[\mu_{l,k-l}(P_1)-\mu_{l,k-l}(P_2)\Big]=0,$$

$j=1,\cdots,n;k\geq 0$. Now setting $z_j=v_j/u_j$ in the last equation leads to

$$\sum_{l=0}^{k}\binom{k}{l}z_j^{k-l}\Big[\mu_{l,k-l}(P_1)-\mu_{l,k-l}(P_2)\Big]=0,\tag{6}$$

$j=1,\cdots n$. Since no two of the directions $\theta_1,\cdots,\theta_n$ are collinear, the points $z_1,\cdots,z_2$ are distinct. Hence from (6) we find that the following polynomial of degree k of the variable z

$$\sum_{l=0}^{k}\binom{k}{l}z^{k-l}\Big[\mu_{l,k-l}(P_1)-\mu_{l,k-l}(P_2)\Big]\tag{7}$$

has n distinct roots $z_1,\cdots,z_n$. If $n\geq k+1$ then this is possible only if all coefficients of (7) are equal to zero, i.e.,

$$\mu_{l,k-l}(P_1)=\mu_{l,k-l}(P_2),\tag{8}$$

$l = 0, \cdots, k; k = 0, \cdots, n-1$. Then from (8) we get that for any unit vector t,

$$\int_{\mathbb{R}^2} (t,x)^k P_1(dx) = \int_{\mathbb{R}^2} (t,x)^k P_2(dx), \tag{9}$$

$k = 0, 1, \cdots, n-1$.

Denote by $P_l^{(t)}$ the marginal of $P_l (l = 1, 2)$ in the direction t and by $\varphi_l(\tau; t) (\tau \in \mathbb{R})$ its characteristic function. The support of $P_1^{(t)}$ is in the segment $[-1, 1]$. Then (9) is equivalent to

$$\varphi_1^{(k)}(\tau; t) \mid_{\tau=0} = \varphi_2^{(k)}(\tau; t) \mid_{\tau=0}, \quad k = 0, \cdots, n-1, \tag{10}$$

where $\varphi_l^{(k)}(\tau; t)$ is the kth derivative of $\varphi_l(\tau; t)$ with respect to $\tau (l = 1, 2)$ The Taylor expansion now gives

$$\begin{aligned} &\varphi_1(\tau; t) - \varphi_2(\tau; t) \\ &= \sum_{k=0}^{s-1} \frac{\varphi_1^{(k)}(0; t) - \varphi_2^{(s)}(0; t)}{k!} \tau^k + \frac{\varphi_1^{(k)}(\tilde{\tau}; t) - \varphi_2^{(s)}(\tilde{\tau}; t)}{s!} \tau^s \end{aligned} \tag{11}$$

where $\tilde{\tau}$ is some number lying between zero and τ. From (10), the first sum on the right-hand side of (11) is equal to zero. Furthermore, since s is an even number,

$$\begin{aligned} \mid \varphi_l^{(s)}(\tilde{\tau}; t) \mid &= \mid \int_{\mathbb{R}} z^s e^{i\tilde{\tau}z} P_l^{(t)}(dz) \mid \\ &\leq \int_{\mathbb{R}} z^s P_l^{(t)}(dz) = \int_{-1}^{1} z^s P_1^{(t)}(dz) \leq 1, \quad l = 1, 2. \end{aligned}$$

Thus for all $\tau \in \mathbb{R}$,

$$\mid \varphi_1(\tau; t) - \varphi_2(\tau; t) \mid \leq 2\frac{\tau^s}{s!}.$$

Now, from the condition

$$\frac{2T^s}{s} = \frac{1}{T},$$

choose $T > 0$ so that $T = (\frac{s!}{2})^{\frac{1}{s+1}}$. Therefore

$$\sup_{|\tau| \leq T} \mid \varphi_1(\tau; t) - \varphi_2(\tau; t) \mid \leq \left(\frac{2}{s!}\right)^{\frac{1}{s+1}},$$

i.e.

$$\sup_{|\tau| \leq T} \mid \int_{\mathbb{R}^2} e^{i\tau(t,x)} P_1(dx) - \int_{\mathbb{R}^2} e^{i\tau(t,x)} P_2(dx) \mid \leq \left(\frac{2}{s!}\right)^{\frac{1}{s+1}}.$$

The last inequality is equivalent to (3) since t was chosen arbitrarily on the unit circle. This proves the theorem. ∎

In particular, the proof of the theorem leads to the following corollaries:

COROLLARY 1. *Let $\theta_1, \cdots, \theta_n$ be $n \geq 2$ directions in $\mathbb{R}^2$ no two of which are collinear. Suppose that the marginals of the probabilities P_1 and P_2 with respect to the directions $\theta_1, \cdots, \theta_n$ have moments up to the even order $k \leq n-1$. Then the marginals of P_1 and P_2 with respect to any direction t have the same moments up to order k.*

COROLLARY 2. *Theorem 1 still holds if we replace the assumption that P_1 and P_2 have of coinciding marginals with respect to the directions $\theta_j (j = 1, \cdots, n)$, with the assumption that these marginals have the same moments up to order $n-1$.*

Let us now relax the condition that the support of P_1 is compact, assuming only the existence of all moments together with Carleman's conditions for definiteness of the moment problem.

For convenience, we introduce some new notation at this point. Set

$$\mu_k = \sup_{\theta \in S^1} \int_{\mathbb{R}^2} (x, \theta)^k P_1(dx), k = 0, 1, \cdots,$$

$$\beta_s = \sum_{j=1}^{(s-2)/2} \mu_{2j}^{-\frac{1}{2j}},$$

where the number s is determined from equation (2), and S^1 is the unit circle.

THEOREM 2. *Let $\theta_1, \cdots, \theta_n$ be $n \geq 2$ directions in $\mathbb{R}^2$ no two of which are collinear. Suppose that the measure P_1 has moments of any order. Suppose also that the marginals of the measures P_1 and P_2 in the directions $\theta_1, \cdots, \theta_n$ have the same moments up to order $n-1$. Then there exists an absolute constant C such that*

$$\lambda\left(P_1, P_2\right) \leq C \beta_s^{-\frac{1}{4}} \left(\mu_0 + \mu_2^{1/2}\right)^{1/4}.$$

PROOF. Let t be an arbitrary vector of the unit circle. From Corollary 1 we have that the marginals $P_1^{(t)}$ and $P_2^{(t)}$ have the same moments up to order s. From KKR (1988), p. 180, and Klebanov and Mkrtchian (1980) it follows that

$$\lambda\left(P_1^{(t)}, P_2^{(t)}\right) \leq C \sum_{j=1}^{(s-1)/2} \left[\mu_{2j}(t)\right]^{-1/(2j)} \left[\mu_0(t) + \mu_2^{1/2}(t)\right]^{1/4}, \tag{12}$$

where $\mu_k(t) = \int_{-\infty}^{\infty} u^k P_1^{(t)}(du), k = 0, 1, \cdots, s$. The theorem now follows from the obvious inequality

$$\mu_{2j}(t) \leq \mu_{2j} \quad (j = 0, 1, \cdots s/2) \qquad \blacksquare$$

THEOREM 3. *Suppose that, in addition to the conditions in Theorem 2, the characteristic function of the measure P_1 admits analytic continuation in some disc centered at the origin. Then*

$$\lambda(P_1, P_2) \leq C_{P_1}/lns, \tag{13}$$

where the constant C_{P_1} depends on the measure P_1, and not on the measure P_2 or the number of directions.

PROOF. The proof is similar to that in Theorem 2, but instead of (12) we need to use Theorem 3.2.3 in KKR (1988). ∎

Let us now consider a more realistic situation where the marginals of P_1 and P_2 in the directions $\theta_1, \cdots \theta_n$ are not the same but are close in the metric λ. We shall make use of the notation introduced in Theorem 1.

THEOREM 4. *Let $\theta_1, \cdots, \theta_n$ be $n \geq 2$ directions in $\mathbb{R}^2$, no two of which are collinear. Suppose that the supports of the measures P_1 and P_2 lie in the unit disc, where they have ε-coinciding marginals with respect to the directions $\theta_j (j = 1, \cdots, n)$, i.e.,*

$$\begin{aligned} &\lambda(P_1^{(\theta_j)}, P_2^{(\theta_j)}) \\ &= \min_{T>0} \max \left\{ \max_{|\tau| \leq T} \mid \varphi_1(\tau; \theta_j) - \varphi_2(\tau; \theta_j) \mid, 1/T \right\} \leq \varepsilon, \end{aligned} \tag{14}$$

$j = 1, 2, \cdots, n$. Then there exists a constant C depending on the directions $\theta_j (j = 1, \cdots, n)$ such that for sufficiently small $\varepsilon > 0$, we have

$$\lambda(P_1, P_2) \leq C(1/ln\left[\frac{1}{\varepsilon}\right] + 1/s), \tag{15}$$

where, as before, s is determined from equation (2).

PROOF. Let

$$\psi_j(\tau) = \varphi_1(\tau; \theta_j) - \varphi_2(\tau; \theta_j), j = 1, \cdots, n. \tag{16}$$

When $0 < \varepsilon \leq 1$ we have

$$\sup_{|\tau| \leq 1} \mid \psi_j(\tau) \mid \leq \varepsilon. \tag{17}$$

Furthermore, since the supports of the measures $P_1^{(\theta_j)}$ and $P_2^{(\theta_j)}$ are subsets of $[-1, 1]$, for any even number $k \geq 2$ we have,

$$\begin{aligned} \sup_{|\tau| \leq 1} \mid \psi_j^{(k)}(\tau) \mid &\leq \left\{ \mid \varphi_1^{(k)}(0; \theta_j) \mid + \mid \varphi_2^{(k)}(0; \theta_j) \mid \right\} / k \\ &\leq \frac{2}{k!}. \end{aligned} \tag{18}$$

Many results exist in the literature which can be used to evaluate the uniform norm of the l-th derivative of an arbitrary $k > l$ times differentiable function, through the norm of the function itself and its k-th derivative. In particular, some convenient results of that type can be found in KKR (1988). Corollary 1.5.1, in KKR (1988), states that there exist constants C_{lk} such that

$$\begin{aligned} \sup_{|\tau|\leq 1} \mid \varphi_j^{(l)}(\tau) \mid &\leq C_{lk}\Big\{ \sup_{|\tau|\leq 1} \mid \varphi_j(\tau) \mid \Big\}^{\frac{k-l}{k}} \\ &\times \Big\{ \sup_{|\tau|\leq 1} |\varphi_j^{(k)}(\tau) \mid \Big\}^{\frac{l}{k}}, l = 0, 1, \cdots, k. \end{aligned} \tag{19}$$

Choosing $k \geq 2s, l \leq s$ in our case, and taking into consideration (17) and (18), we obtain the inequality

$$\sup_{|\tau|\leq 1} \mid \varphi_j^{(l)}(\tau) \mid \leq C_s\varepsilon^{1/2}, l = 0, 1, \cdots, s; j = 1, \cdots, n, \tag{20}$$

where C_s is a new constant depending on s only. In particular, from (20) it follows that

$$\mid \varphi_1^{(l)}(0;\theta_j) - \varphi_2^{(l)}(0;\theta_j) \mid \leq C_s\varepsilon^{1/2}, l = 0, 1, \cdots, s, j = 1, \cdots, n.$$

In other words, the moments of order up to s of the marginals with respect to the directions $\theta_1, \cdots, \theta_n$ are close: they can differ by no more than $C_s\varepsilon^{1/2}$. This fact is expressed by the inequality:

$$\int_{\mathbb{R}^2} (x, \theta_j)^k P_1(dx) - \int_{\mathbb{R}^2} (x, \theta_j)^k P_2(dx) \leq C_s\varepsilon^{1/2}, \tag{21}$$

$k = 0, 1, \cdots, s; j = 1, \cdots, n$. Following the notation in Theorem 1, we can rewrite (21) in the form

$$\mid \sum_{l=0}^{k} \binom{k}{l} u_j^l v_j^{k-l} \Big[\mu_{l,k-l}(P_1) - \mu_{l,k-l}(P_2)\Big] \mid \leq C_s\varepsilon^{1/2},$$

$$k = 0, \cdots, s; j = 1, \cdots, n.$$

Thus, setting

$$\mathcal{R}_{kj} = \sum_{l=0}^{k} \binom{k}{l} z_j^{k-l} \Big[\mu_{l,k-l}(P_1) - \mu_{l,k-l}(P_2)\Big], \tag{22}$$

$k = 2, \cdots, s, j = 1, \cdots, n, z_j = v_j/u_j$, we obtain

$$\mid \mathcal{R}_{kj} \mid \leq \hat{C}\varepsilon^{1/2}, \tag{23}$$

where $\hat{C}$ depends on the directions $\theta_1, \cdots, \theta_n$ only. For any fixed $k(k = 2, \cdots, s)$ consider:

(i) the matrix A_k with elements

$$a_{lj}^{(k)} = \binom{k}{l-1} z_j^{k-(l-1)}, l = 1, \cdots, k+1, j = 1, \cdots, k+1;$$

(ii) the vector B_k with elements

$$b_l^{(k)} = \mu_{l-1,k-l+1}(P_1) - \mu_{l-1,k-l+1}(P_1), l = 1, \cdots, k+1;$$

(iii) the vector D_k with elements

$$d_j = \mathcal{R}_{kj}, j = 1, \cdots, k+1.$$

Then (22) and (23) imply the equations

$$A_k B_k = D_k \quad (k = 1, \cdots, s-1) \tag{24}$$

and the estimate

$$\| D_k \| \leq \tilde{C}\varepsilon^{1/2}. \tag{25}$$

Of course, the matrices A_k are invertible. Therefore

$$\|B_k\| \leq \|A_k^{-1}\| \|D_k\| \leq \overline{C}\varepsilon^{1/2}, \tag{26}$$

where the constant $\overline{C}$ depends on the directions $\theta_1, \cdots, \theta_n$ only. Inequality (26) shows that the first $s-1$ moments of the two-dimensional distributions are close when $\varepsilon > 0$ is sufficiently small. Such an evaluation of closeness holds for the first $s-1$ moments of the marginals corresponding to an arbitrary direction t, i.e.

$$\left| \int_{\mathbb{R}^2} (x,t)^k P_1(dx) - \int_{\mathbb{R}^2} (x,t)^k P_2(dx) \right| \leq \overline{C}\varepsilon^{1/2}, \tag{27}$$

$k = 0, \cdots, s-1$. Now we have

$$| \varphi_1(\tau;t) - \varphi_2(\tau;t) | \leq \left| \sum_{j=0}^{s-1} \frac{\varphi_1^{(j)}(0;t) - \varphi_2^{(j)}(0;t)}{j!} \tau^j \right| + \frac{2}{s!}|\tau|^s$$

$$\leq \sum_{j=0}^{s-1} \frac{\overline{C}\varepsilon^{1/2}}{j!} | \tau |^j + \frac{2 | \tau |^s}{s!} \leq \overline{C}\varepsilon^{1/2} e^{|\tau|} + \frac{2 | \tau |^s}{s!}.$$

Setting $|\tau| \leq T$ above, we find

$$T = \min\left\{ ln\left[1 + \left(\overline{C}\varepsilon^{1/2}\right)^{-1/2}\right], \left(\frac{s!}{2}\right)^{\frac{1}{s-1}} \right\},$$

and, since t is arbitrary on the unit circle, we obtain

$$\begin{aligned}\lambda(P_1, P_2) &\leq max\Big\{\overline{C}^{1/2}\varepsilon^{1/4} + \overline{C}\varepsilon^{1/2} + \Big(\frac{2}{s!}\Big)^{\frac{1}{s-1}}, 1/T\Big\} \\ &\leq C\Big[1/ln(1/\varepsilon) + 1/s\Big],\end{aligned}$$

which proves the theorem. ∎

REMARK 1. The statement in Theorem 4 still holds if instead of the ε-coincidence of the marginals as in (14), we require the ε-coincidence of the moments up to order s of these marginals. For the latter, we require that the inequalities

$$\Big| \int_{\mathbb{R}^2} (x, \theta_j)^k P_1(dx) - \int_{\mathbb{R}^2} (x, \theta_j)^k P_2(dx) \Big| \leq \varepsilon, \tag{28}$$

$j = 1, \cdots, n; k = 0, \cdots, s$, to hold.

Indeed, (28) is only a reinforcement of (21), and we can repeat the arguments of Theorem 4 which follow from the bound (21).

3. Multidimensional Generalizations.

All theorems in Section 2 admit generalizations to probability measures defined on $\mathbb{R}^m$. However we cannot choose the directions $\theta_1, \cdots, \theta_n$ in an arbitrary way. Furthermore, to obtain the same order of precision in $\mathbb{R}^m, m > 2$ corresponding to the n directions in $\mathbb{R}^2$ we need n^{m-1} directions. The results can be obtained by induction on the dimension m.

We define next the set of directions we are going to use. Choose $n \geq 2$ distinct real numbers $u_1, \cdots, u_n$, all different from zero, and first construct the set of n two-dimensional vectors

$$(1, u_1), (1, u_2), \cdots, (1, u_n).$$

Then construct n^2 three-dimensional vectors

$$(1, u_{j_1}, u_{j_2}), j_1, j_2 = 1, \cdots, n.$$

Repeating this process, by the last step, we shall have constructed n^{m-1} m-dimensional vectors

$$(1, u_{j_1}, u_{j_2}, \cdots, u_{j_{m-1}}), j_l = 1, \cdots, n, l = 1, \cdots, m-1. \tag{29}$$

Denote these m-dimensional vectors by $\theta_1, \cdots, \theta_N$, where $N = n^{m-1}$ (the choice of enumeration is irrelevant here). These inductive arguments lead to the following extensions of Theorems 1 - 4 .

PROPOSITION. *The results in Theorems 1 - 4 still hold if we consider the measures P_1 and P_2 in $\mathbb{R}^m$, and we choose as directions the $N = n^{m-1}$ vectors in (29). Further, $s = 2[(n-1)/2]$.*

To prove this, it is sufficient to note that instead of the m-dimensional vectors we can first consider a pair of one-dimensional probability distributions; the first component is the distribution of the inner product of the projections of the vector x and the vector θ_j upon the $(m-1)$-dimensional subspace, while the second is the law of the last coordinate of the vector x. This allows us to decrease the dimensionality by one. To complete the proof it is sufficient to apply inductive arguments.

4. Concluding Remarks. The above results are concerned with closeness between the probability measures P_1 and P_2 only in the case of the λ-metric. We can also consider the cases of the Lévy-Prohorov distance and the distance in variation with the additional assumptions of existence and differentiability of the densities of the relevant probabilities. To obtain the corresponding estimates, it is sufficient to use the results comparing the respective distances with the λ-metric given in KKR (1988). We do not formulate the corresponding theorems since the inequalities already obtained are far from being final. In our opinion, the estimates of closeness between the densities of smoothed distributions are more interesting.

For example, in the conditions of Theorem 2 there exists an absolute constant C such that

$$\sup_x \mid P_1 * h_\eta(x) - P_2 * h_\eta(x) \mid \leq C \frac{(\mu_o + \mu_2^{1/2})^{1/2}}{\eta \beta_s^{1/2}}, \tag{30}$$

where $h_\eta(x)$ is the two-dimensional density of the Cauchy distribution, i.e., the probability distribution whose characteristic function is $\exp\{-\eta\|x\|\}$.

The proof of this statement repeats that of Theorem 2, inequality (3.2.22), in KKR (1988).

The bounds of the deviation between probability measures with coinciding marginals offer a solution to the computer tomography paradox as stated in Gutman et al. (1991): *"It implies that for any human object and corresponding projection data there exist many different reconstructions, in particular, a reconstruction consisting only of bone and air (density 1 or 0), but still having the same projection data as the original object. Related non-uniqueness results are familiar in tomography and are usually ignored because CT machines seem to produce useful images. It is likely that the "explanation" of this apparent paradox is that point reconstruction in tomography is impossible."* Indeed, the results in the mentioned paper and our Theorem 1 show that, although the densities of the probability measures P_1 and P_2 (given that such densities exist) can be quite distant from each other for any "large" number of coinciding marginals, yet the measures P_1 and P_2 themselves are close in the weak-metric λ .

Khalfin and Klebanov (1990) have analyzed this paradox, and obtained some bounds for the closeness of probability measures with coinciding marginals for specially chosen directions for the case of uniform distance between the smoothed densities of these measures. The methods used in their work are different from those used here, and were based on evaluations of the convergence rate of interpolation processes.

In tomography the observations are, in fact, integrals of body densities along some straight lines. Using quadratic formula enables us to evaluate the moments of a set of marginals; these in turn make it possible to apply the result in Remark 1, to evaluate the precision of the reconstruction of densities. The classical theory of *moments* makes it possible to give numerical methods for reconstructing the probability measures using the moments (see, e.g., Ahiezer (1961)).

Acknowledgement. Our thanks are due to our colleague Dr. J. Gani for his very helpful comments.

References

AHIEZER, I. (1961) *Classical Moment Problems*, Fiz-Mat-Giz, Moscow.

FISHBURN, P. C., LAGARIAS, J. C., REEDS, J. A. and SHEPP, L. A. (1990). Sets uniquely determine by projections on axes. I. Continuous case. *SIAM J. Appl. Math.* **50**, 288–306.

GUTMANN, S., KEMPERMAN, J. H. B., REEDS, J. A. and SHEPP, L. A. (1991). Existence of probability measures with given marginals. *Ann. Probab.* **19**, 1781–1791.

JACOBS, K. (1978) *Measure and Integral.* Academic Press, New York.

KAKOSYAN, A. V. and KLEBANOV, L. B. (1984) On estimates of the closeness of distributions in terms of characteristic functions, *Theory Probab. Appl.* **29**, 852–853.

KAKOSYAN, A. V., KLEBANOV, L. B. and RACHEV, S. T. (1988) *Quantitative Criteria for Convergence of Probability Measures*, Ajastan, Erevan.

KARLIN, S. and STUDDEN, W. J. (1966) *Tchebycheff Systems.* Interscience, New York.

KELLERER, H. G. (1961) Funktionen und Produkträumen mit vorgegebenen Marginal-Funktionen. *Math. Ann.* **144**, 323–344.

KELLERER, H. G. (1996) Ambiguity in bounded moment problems. AMS-IMS-SIAM Joint Research Conference: *Distributions with Fixed Marginals and Related Topics.* (L. Rüschendorf, B. Schweizer, and M. Taylor, eds.). Vol. 28 Lecture Notes – Monograph Series of the IMS, 142–161.

KEMPERMAN, J. H. B. (1993). Moment problems for measures on R^n with given K-dmensional marginals. Technical Report, Rutgers University.

KHALFIN, L. A. and KLEBANOV, L. B. (1990) A solution of the computer tomography paradox and estimation of the distances between the densities of measures with the same marginals. Clarkson University, Preprint, Potsdam.

KLEBANOV, L. B. and MKRTCHIAN, S. T. (1980) Estimator of the closeness of distributions in terms of coinciding moments. In *Problems of Stability of Stochastic Models,* VNIISI, Moscow, 64-72.

LORENTZ, G. G. (1949) A problem of plane measure. *Amer. J. Math.* **71**, 417–426.

RACHEV, S. T. (1991) *Probability Metrics and the Stability of Stochastic Models,* Wiley, Chichester,

STRASSEN, V. (1965) The existence of probability measures with given marginals. *Ann. Math. Statist.* **36**, 423–439.

ZOLOTAREV, V. M. (1986) *Modern Theory for Summation of Independent Random Variables,* Nauka, Moscow.

DEPARTMENT OF STATISTICS
AND APPLIED PROBABILITY
UNIVERSITY OF CALIFORNIA
SANTA BARBARA, CA 93106-3110
rachev@bernoulli.ucsb.edu