Distributions with Fixed Marginals and Related Topics
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Ambiguity in Bounded Moment Problems

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Let $(X, \mathcal{A}, \mu)$ be a finite measure space and $\mathcal{K}$ be a linear subspace of $\mathcal{L}_1(\mu)$ (e.g. generated by the one-dimensional projections, if X is a product space). The following inverse problem is treated: To what extent is a set $A \in \mathcal{A}$ "$\mathcal{K}$-determined" within the class of all (fuzzy sets) $g \in \mathcal{L}_\infty(\mu)$ satisfying $0 \leq g \leq 1$, i.e. which lower and upper bounds A_* and A^* for A can be derived from knowing the integrals $\int_A f d\mu$, $f \in \mathcal{K}$ – thus generalizing the uniqueness problem $(A_* = A^*)$.

Introduction. This is the natural extension of a paper entitled "Uniqueness in bounded moment problems," Kellerer (1993). The questions treated there had their origin in a central problem of tomography: which n-dimensional objects can be reconstructed from the measures of all their $(n-1)$-dimensional sections orthogonal to the different axes? In this form the planar case was studied by Kuba and Volcic (1988), while the extension to higher dimensions was carried out by Fishburn et al. (1990, 1991).

As was done by Kemperman (1990, 1991), the author in Kellerer (1993) subsumed this classical case under the following "bounded moment problem": given a measure space $(X, \mathcal{A}, \mu)$ and a family $\mathcal{K}$ of integrable test functions, which sets $A \in \mathcal{A}$ are – up to null sets – uniquely determined by the integrals $\int_A f d\mu$, $f \in \mathcal{K}$? In the weak version A is compared with ordinary sets only, while in the strong version this comparison takes place in the class $\mathcal{G}$ of all fuzzy sets (i.e. functions attaining their values in the unit interval). Since both models coincide in important situations, but methods of convex analysis cannot be applied to the weak version, in the sequel the strong version will be emphasized.

Now the search for uniquely determined sets is in fact a very restricted view of this kind of inverse problems. As considered by Kuba and Volcic (1993) in the planar case, also in the general case two sets of uniqueness are associated

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with any set $A \in \mathcal{A}$: $U_1(A)$ resp. $U_0(A)$ are defined as the intersections of the sets $\{g = 1\}$ resp. $\{g = 0\}$, taken over all functions $g \in \mathcal{G}$ satisfying

$$\int_X fg d\mu = \int_A f d\mu \text{ for } f \in \mathcal{K}.$$

It is one of the main results in Kellerer (1993) that uniqueness of A is closely connected with the existence of a representation

$$(1) \qquad A \underset{\mu}{=} \{f > 0\} \text{ and } X \backslash A \underset{\mu}{=} \{f < 0\}$$

for some $f \in \mathcal{K}$. Under this assumption the set $\{f = 0\}$, containing no information about A via the integral $\int_A f d\mu$, has to be a null set. Thus it is only natural to conjecture that in the general case the representation (1) has to be replaced by

$$(2) \qquad U_1(A) \underset{\mu}{=} \{f > 0\} \text{ and } U_0(A) \underset{\mu}{=} \{f < 0\}$$

for some $f \in \mathcal{K}$. Under this assumption A is known except for variations inside the set $V(A) \underset{\mu}{=} \{f = 0\}$.

However, as follows from Kellerer (1993), a representation of type (1) for uniquely determined sets is (a) restricted to special situations or (b) valid only in a weakened version. It is not surprising that this holds even more in the general case. Roughly speaking, Section 2 of this paper is concerned with aspect (b), while Section 3 is devoted to aspect (a). For a brief survey of these sections see the following two paragraphs.

Interpreting the reconstruction of a set $A \in \mathcal{A}$ from the associated integrals $\int_A f d\mu$, $f \in \mathcal{K}$, as an extension of some linear functional suggests an application of the Hahn-Banach theorem. This provides in Theorem 2.1 a characterization of the sets $U_1(A)$, $U_0(A)$ and their relative complements that turns out to be crucial for all that follows. It yields in particular Theorem 2.2, solving the problem whenever the family $\mathcal{K}$ is of finite dimension. This in turn implies Theorem 2.3, ensuring a representation of type (2) whenever the basic space X is finite.

As a corollary this yields Proposition 3.1, extending known results on $(0, 1)$-matrices to more general arrays. The final applications concern the classical (two-dimensional) marginal problem. First a representation of type (2) is provided in the discrete case by Proposition 3.3 without any assumption on the underlying measure μ. This result then is carried over to the continuous case by Proposition 3.5, where, however, μ has to have product form. It remains an open problem, whether there is a common extension of these two propositions.

Before collecting the needed prerequisites in Section 1, some notation has to be fixed. Given the measure space $(X, \mathcal{A}, \mu)$, the function spaces $\mathcal{L}_1(\mu)$ – with its norm $\|\cdot\|$ – and $\mathcal{L}_\infty(\mu)$ have their usual meaning. Thus, in particular, functions that agree modulo μ are identified and the same convention holds for sets in $\mathcal{A}$. Since suppressing μ in equations and inequalities, however, can be misleading, the symbols $\underset{\mu}{=}$ and $\underset{\mu}{\leq}$ resp. $\underset{\mu}{\subset}$ are used whenever null sets may intervene. Finally, since μ is always assumed to be $(\sigma-)$ finite, μ-essential intersection and union are well defined for each family $\mathcal{A}_0 \subset \mathcal{A}$; they will be denoted by μ-inf $\mathcal{A}_0$ resp. μ-sup $\mathcal{A}_0$.

1. Preliminaries. Throughout the paper the following conventions hold:

(a) $(X, \mathcal{A}, \mu)$ is a finite measure space,

(b) $\mathcal{K}$ is a linear subspace of $\mathcal{L}_1(\mu)$.

Concerning (a), the σ-finite case is covered as well, multiplying the measure μ by a strictly positive function $h \in \mathcal{L}_1(\mu)$ and all functions in $\mathcal{K}$ by $1/h$. Concerning (b), the linearity assumption is no real restriction, as is immediate from the central notion:

DEFINITION 1.1. The set

$$\mathcal{G} = \{g \in \mathcal{L}_\infty(\mu) : 0 \underset{\mu}{\leq} g \underset{\mu}{\leq} 1\}$$

is endowed with the equivalence relation

$$g_1 \underset{\mathcal{K}}{\sim} g_2 \Leftrightarrow \int_X f g_1 d\mu = \int_X f g_2 d\mu \quad \text{for all} \quad f \in \mathcal{K}.$$

Clearly, this relation is compatible not only with equality modulo μ but also with the linear structure in $\mathcal{G}$.

A thorough study of ambiguity with respect to this equivalence relation requires an extended list of sets of uniqueness and variability:

DEFINITION 1.2. For $A \in \mathcal{A}$

(a) the sets $U_1(A), \cdots, V(A)$ are defined by

$$\begin{aligned}
U_1(A) &= \mu - \inf\{\{g = 1\} : \mathcal{G} \ni g \underset{\mathcal{K}}{\sim} 1_A\},\\
U_0(A) &= \mu - \inf\{\{g = 0\} : \mathcal{G} \ni g \underset{\mathcal{K}}{\sim} 1_A\},\\
V_1(A) &= A \backslash U_1(A),\\
V_0(A) &= (X \backslash A) \backslash U_0(A),\\
U(A) &= U_1(A) \cup U_0(A),\\
V(A) &= V_1(A) \cup V_0(A);
\end{aligned}$$

(b) the sets $U_1^*(A), \cdots, V^*(A)$ are defined similarly, restricting $g \in \mathcal{G}$ to indicator functions.

In accordance with the underlying equivalence relation, these mappings from $\mathcal{A}$ to $\mathcal{A}$ have to be understood modulo μ. They have some simple properties, which will be used in the sequel without further mention:

LEMMA 1.3. *The sets introduced in Definition 1.2 satisfy*

$$(a) \qquad U_1(A) \subset A \ \text{ and } \ U_0(A) \subset X \backslash A$$

with analogous relations for V_i and U_i^*, V_i^*;

$$(b) \qquad U_i(X \backslash A) = U_{1-i}(A) \ \text{ for } \ i = 0, 1$$

with analogous equations for V_i and U_i^*, V_i^*;

$$(c) \qquad U(X \backslash A) = U(A)$$

with analogous equations for V and U^*, V^*;

$$(d) \qquad U_i(A) \subset U_i^*(A) \ \text{ and } \ V_i(A) \supset V_i^*(A) \ \text{ for } \ i = 0, 1$$

with analogous relations for U, V.

PROOF. (a) and (d) are trivial, (b) and (c) follow from

$$\mathcal{G} \ni g \underset{\mathcal{K}}{\sim} 1_A \ \text{ if and only if } \ \mathcal{G} \ni 1 - g \underset{\mathcal{K}}{\sim} 1_{X \backslash A}. \qquad \blacksquare$$

Clearly, A being fixed, there is a monotone dependence on $\mathcal{K}$: the sets of uniqueness increase and the sets of variability decrease, if the subspace $\mathcal{K}$ is enlarged. Moreover, for any $A \in \mathcal{A}$, the set

$$N = \mu - \inf\{\{f = 0\} : f \in \mathcal{K}\}$$

is contained in $V(A)$ and $V^*(A)$.

As already mentioned in the introduction, the weak notions $U_i^*, \cdots$ appear only at some places, while the emphasis in general is on the strong notions $U_i, \cdots$. Here, both the extreme cases deserve special names:

DEFINITION 1.4. A set $A \in \mathcal{A}$ is called

(a) "uniquely determined," if $V(A) \underset{\mu}{=} \emptyset$,

(b) "totally undetermined," if $V(A) \underset{\mu}{=} X$.

Thus, for instance, all sets are totally undetermined in the case $\mathcal{K} = \{0\}$ and uniquely determined in the case $\mathcal{K} = \mathcal{L}_1(\mu)$.

The simplest nontrivial case is easily dealt with:

LEMMA 1.5. *Let $\mathcal{K}$ be spanned by a single function f and $A \in \mathcal{A}$ be an arbitrary set.*

(a) If $f \mid A \underset{\mu}{\geq} 0$ and $f \mid X \backslash A \underset{\mu}{\leq} 0$, then

$$U_1(A) \underset{\mu}{=} \{f > 0\} \text{ and } U_0(A) \underset{\mu}{=} \{f < 0\};$$

(b) if $f \mid A \underset{\mu}{\leq} 0$ and $f \mid X \backslash A \underset{\mu}{\geq} 0$, then

$$U_1(A) \underset{\mu}{=} \{f < 0\} \text{ and } U_0(A) \underset{\mu}{=} \{f > 0\};$$

(c) in all other cases A is totally undetermined.

PROOF. (a) The functional I defined by

$$I(g) = \int_X fg d\mu \text{ for } g \in \mathcal{G}$$

transforms the condition $g \underset{\mathcal{K}}{\sim} 1_A$ into the equation $I(g) = I(1_A)$ and attains its maximum $I(1_A)$ at g if and only if

$$\{g = 1\} \underset{\mu}{\supset} \{f > 0\} \text{ and } \{g = 0\} \underset{\mu}{\supset} \{f < 0\}.$$

(b) Replace f by $-f$ in (a).

(c) Due to $V(A) = V(X \backslash A)$ there are essentially two cases:

$$\text{(1)} \qquad \int_A f^+ d\mu > 0 \text{ and } \int_A f^- d\mu > 0$$

yields constants $\gamma, \vartheta \in]0,1[$ such that

$$\gamma \int_{X\setminus A} f d\mu = \vartheta\left(-\int_A f^- d\mu\right) + (1-\vartheta)\int_A f^+ d\mu,$$

and this implies

$$(*) \qquad g \underset{\mathcal{K}}{\sim} 1_A \text{ and } \{g = 1_A\} = \emptyset$$

for the function

$$g = \gamma 1_{X\setminus A} + \vartheta 1_{A\cap\{f>0\}} + (1-\vartheta)1_{A\cap\{f<0\}} \in \mathcal{G};$$

$$(2) \qquad \int_A f^+ d\mu > 0 \text{ and } \int_{X\setminus A} f^+ d\mu > 0$$

yields constants $\gamma, \vartheta \in]0,1[$ such that

$$\int_A f^- d\mu - \gamma \int_X f^- d\mu = \vartheta\left(-\int_{X\setminus A} f^+ d\mu\right) + (1-\vartheta)\int_A f^+ d\mu,$$

because the left-hand side varies between the values $\int_A f^- d\mu \geq 0$ and $-\int_{X\setminus A} f^- d\mu \leq 0$, and this implies $(\star)$ for the function

$$g = \gamma 1_{\{f>0\}} + \vartheta 1_{\{f<0\}} \in \mathcal{G}.$$

∎

This result allows the following reduction process:

LEMMA 1.6. *Let $A \in \mathcal{A}$ and $f \in \mathcal{K}$ satisfy*

$$f \mid A \underset{\mu}{\geq} 0 \text{ and } f \mid X\setminus A \underset{\mu}{\leq} 0$$

and define

$$X' = \{f = 0\} \text{ and } A' = A \cap X'.$$

Then, with the notations U_i' and V' for the problem resulting from the restriction of μ and $\mathcal{K}$ to X', the following equations hold:

$$U_1(A) \underset{\mu}{=} \{f > 0\} \cup U_1'(A'),$$

$$U_0(A) \underset{\mu}{=} \{f < 0\} \cup U_0'(A'),$$

$$V(A) \underset{\mu}{=} V'(A').$$

PROOF. If g' denotes the restriction of $g \in \mathcal{G}$ to X', then it follows as in the proof of Lemma 1.5(a) that an equivalence $g \underset{\mathcal{K}}{\sim} 1_A$ can be split into the conditions

(a) $\{g = 1\} \underset{\mu}{\subset} \{f > 0\}$ and $\{g = 0\} \underset{\mu}{\subset} \{f < 0\}$,

(b) $g' \underset{\mathcal{K}'}{\sim} 1_{A'}$,

where $\mathcal{K}'$ denotes the restriction of $\mathcal{K}$ to X'. Thus the assertion is an immediate consequence of the definition of U_i' and V'. ∎

This result yields first bounds for the sets U_i (and thus for the sets V_i as well): if $\mathcal{K}_A$ denotes the family of all functions $f \in \mathcal{K}$ satisfying the condition in Lemma 1.6, it follows that

$$U_1(A) \underset{\mu}{\supset} \mu - \sup\{\{f > 0\} : f \in \mathcal{K}_A\} = A_*,$$

$$U_1(A) \underset{\mu}{\supset} \mu - \inf\{\{f \geq 0\} : f \in \mathcal{K}_A\} = A^*$$

with analogous bounds for $U_0(A)$.

It should be pointed out here that these bounds are attained for some $f \in \mathcal{K}_A$, provided $\mathcal{K}$ is a closed subspace of $\mathcal{L}_1(\mu)$. Indeed, choosing functions $f_n \in \mathcal{K}_A$ with

$$\bigcup_{n\in\mathbb{N}} \{f_n > 0\} \underset{\mu}{=} A_* \quad \text{and} \quad \bigcap_{n\in\mathbb{N}} \{f_n \geq 0\} \underset{\mu}{=} A^*$$

and guaranteeing $\sum_{n\in\mathbb{N}} \|f_n\| < \infty$ by appropriate scalar factors, the function $f = \sum_{n\in\mathbb{N}} f_n \in \mathcal{K}$ belongs to $\mathcal{K}_A$ and satisfies

$$\{f > 0\} \underset{\mu}{=} A_* \quad \text{and} \quad \{f \geq 0\} \underset{\mu}{=} A^*.$$

2. General Results. The main result of Kellerer (1993), i.e. the criterion for sets to be uniquely determined, stated there in Theorem 2.1, has the following extension:

THEOREM 2.1. *For fixed $A \in \mathcal{A}$ let U_i and V_i stand for $U_i(A)$ and $V_i(A)$, respectively. Then the functional*

$$D : f \to \int_{U_1} (1 - f)^+ d\mu + \int_{U_0} (1 + f)^+ d\mu + \int_{V_1} (-f)^+ d\mu + \int_{V_0} (+f)^+ d\mu$$

on $\mathcal{L}_1(\mu)$ satisfies

$$\inf\{D(f) : f \in \mathcal{K}\} = 0.$$

Proof. 1. Due to the duality $\mathcal{L}_1^*(\mu) = \mathcal{L}_\infty(\mu)$, there is a bijection between the functions $g \in \mathcal{L}_\infty(\mu)$ such that

$$g \underset{\mathcal{K}}{\sim} 1_A, \tag{1}$$

$$0 \underset{\mu}{\leq} g \underset{\mu}{\leq} 1 \tag{2}$$

and the continuous linear functionals I on $\mathcal{L}_1(\mu)$ such that

$$I(f) = \int_A f d\mu \ \text{ for } f \in \mathcal{K}, \tag{1'}$$

$$0 \leq I(f) \leq \int_X f d\mu \ \text{ for } 0 \leq f \in \mathcal{L}_1(\mu), \tag{2'}$$

where the continuity of I is actually a consequence of condition (2′). By means of the functional

$$J(f) = \int_X f^+ d\mu \ \text{ for } f \in \mathcal{L}_1(\mu),$$

which is obviously positively homogeneous and subadditive, condition (2′) can be replaced by

$$I(f) \leq J(f) \ \text{ for all } f \in \mathcal{L}_1(\mu).$$

Indeed, this inequality implies (2′) due to

$$I(f) \leq J(f) = 0 \ \text{ whenever } f \leq 0,$$

while the converse is trivial.

2. By the definition of the sets U_i each J-dominated linear functional I on $\mathcal{L}_1(\mu)$ extending

$$I_A(f) = \int_A f d\mu \ \text{ for } f \in \mathcal{K}$$

has to satisfy

$$I(f) = \int_A f d\mu \ \text{ for } \{f \neq 0\} \underset{\mu}{\subset} U_1,$$
$$I(f) = 0 \ \text{ for } \{f \neq 0\} \underset{\mu}{\subset} U_0.$$

Due to the positivity of I, these two conditions can be subsumed under the sole requirement

$$I(f_0) = \mu(U_1) \ \text{ for } f_0 = 1_{U_1} - 1_{U_0}.$$

3. Now start the proof of the Hahn-Banach theorem by adjoining to $\mathcal{K}$ in the first step the function f_0. Then it turns out that the admissible values

of $I(f_0)$ reduce to $\mu(U_1)$ if and only if

$$\sup_{f\in\mathcal{K}}(I_A(f) - J(f - f_0)) = \mu(U_1) = \inf_{f\in\mathcal{K}}(J(f + f_0) - I_A(f)).$$

By inserting J and I_A explicitly the first equation is easily transformed into the condition $\inf_{f\in\mathcal{K}} D(f) = 0$ (while the second one becomes

$$\inf_{f\in\mathcal{K}}(\int_{U_1}(1+f)^- d\mu + \int_{U_0}(1-f)^- d\mu + \int_{V_1}(+f)^- d\mu + \int_{V_0}(-f)^- d\mu) = 0$$

and is seen to be no condition at all by choosing $f = 0$). ∎

To get a first insight into this result assume the infimum of the functional D to be attained for some function $f \in \mathcal{K}$. This forces all integrals involved to vanish and thus yields the inequalities

$$f \mid U_1(A) \underset{\mu}{\geq} 1 \text{ and } f \mid U_0(A) \underset{\mu}{\leq} -1,$$
$$f \mid V_1(A) \underset{\mu}{\geq} 0 \text{ and } f \mid V_0(A) \underset{\mu}{\leq} 0.$$

According to Lemma 1.6, however, the sets $\{f > 0\}$ and $\{f < 0\}$ are contained in $U_1(A)$ and $U_0(A)$, respectively, and this implies

$$U_1(A) \underset{\mu}{=} \{f > 0\}, V(A) \underset{\mu}{=} \{f = 0\}, U_0(A) \underset{\mu}{=} \{f < 0\}.$$

As already mentioned in the introduction, such a representation in general cannot be expected even for uniquely determined sets. In order to obtain a reasonable representation at least in the case of a finite-dimensional subspace $\mathcal{K}$, lexicographic order "$\prec$" has to be used. Specialized to $x \in \mathbb{R}^k$ it is defined by $x \prec 0$ resp. $x \succ 0$ meaning $x \neq 0$ with the first nonzero coordinate being negative resp. positive.

With this notation Proposition 3.8 in Kellerer (1993) has the following natural extension (with a less geometric proof than given there):

THEOREM 2.2. *If $0 < k =$ dim $\mathcal{K} < \infty$, then for any set $A \in \mathcal{A}$ there exist functions $f_i \in \mathcal{K}$ such that*

$$U_1(A) \underset{\mu}{=} \{(f_1, \cdots, f_k) \succ 0\},$$
$$V(A) \underset{\mu}{=} \{(f_1, \cdots, f_k) = 0\},$$
$$U_0(A) \underset{\mu}{=} \{(f_1, \cdots, f_k) \prec 0\}.$$

PROOF. With the functional D from Theorem 2.1 and a base $h_i, 1 \leq i \leq k$, of $\mathcal{K}$ define

$$d(a_1, \cdots, a_k) = D(\sum_{1 \leq i \leq k} a_i h_i) \text{ for } a_1, \cdots, a_k \in \mathbb{R}.$$

Then Theorem 2.1 provides coefficients a_i^n such that

$$d(a_1^n, \cdots, a_k^n) \to 0 \text{ for } n \to \infty, \tag{1}$$

and there are two possibilities:

Case 1: If the sequence $(a_1^n, \cdots, a_k^n)$, $n \in \mathbb{N}$, is bounded, by taking a subsequence, it can be assumed that

$$a_i^n \to a_i \in \mathbb{R} \text{ for } 1 \leq i \leq k.$$

By the continuity of d this yields

$$D(f) = 0 \text{ for } f = \sum_{1 \leq i \leq k} a_i h_i \in \mathcal{K},$$

and by the remark following the proof of Theorem 2.1 this means

$$U_1(A) \underset{\mu}{=} \{f > 0\}, \ V(A) \underset{\mu}{=} \{f = 0\}, \ U_0(A) \underset{\mu}{=} \{f < 0\},$$

i.e. the functions $f_1 = f$ and $f_2 = \cdots = f_k = 0$ are appropriate.

Case 2: If the sequence $(a_1^n, \cdots, a_k^n)$, $n \in \mathbb{N}$, is unbounded, after taking subsequences (and changing the indices and/or the sign of h_1, if necessary), it can be assumed that

$$|a_1^n| \geq |a_i^n| \text{ for } 1 \leq i \leq k \text{ and } n \in \mathbb{N}, \tag{2}$$

$$a_1^n \to \infty, \tag{3}$$

$$a_i^n / a_1^n \to a_i \in [-1, +1] \text{ for } 1 \leq i \leq k. \tag{4}$$

This yields the representation

$$\sum_{1 \leq i \leq k} a_i^n h_i = a_1^n (f + \sum_{1 < i \leq k} b_i^n h_i),$$

where

$$f = h_1 + \sum_{1 < i \leq k} a_i h_i \neq 0,$$

$$b_i^n = a_i^n / a_1^n - a_i \to 0 \text{ for } 1 < i \leq k.$$

By (1) and the definition of D this implies

$$f \mid U_1 \cup V_1 \underset{\mu}{\geq} 0 \text{ and } f \mid U_0 \cup V_0 \underset{\mu}{\leq} 0,$$

which, in the notation of Lemma 1.6, means

$$\begin{aligned} U_1(A) &\underset{\mu}{=} \{f > 0\} \cup U_1'(A'), \\ U_0(A) &\underset{\mu}{=} \{f < 0\} \cup U_0'(A'), \\ V(A) &\underset{\mu}{=} V'(A'). \end{aligned}$$

In view of $f \neq 0$ the restriction $\mathcal{K}'$ of $\mathcal{K}$ to $\{f = 0\}$ satisfies $\dim \mathcal{K}' = \dim \mathcal{K} - 1$, and the assertion follows by induction on k, starting with Lemma 1.6. ∎

As shown in Kellerer (1993), the number of functions f_i required in general cannot be decreased below the dimension of $\mathcal{K}$ even for uniquely determined sets. An exception, however, is the case where

$$\{f = 0\} \underset{\mu}{=} \emptyset \quad \text{or} \quad X \text{ for all } f \in \mathcal{K};$$

here obviously only f_1 is needed and thus any set $A \in \mathcal{A}$ is either uniquely determined or totally undetermined.

It is a natural conjecture that the statement of Theorem 2.2 extends to subspaces $\mathcal{K}$ of infinite dimension, if the finite sequence $f_1, \cdots, f_k$ is replaced by a sequence f_n, $n \in \mathbb{N}$ (with the corresponding lexicographic order). This, however, is disproved in Kellerer (1993), where instead it is shown that a uniquely determined set $A \in \mathcal{A}$ has a representation

$$A \underset{\mu}{=} \{f^* > 0\} \text{ and } X \backslash A \underset{\mu}{=} \{f^* < 0\}$$

for some function $f^* \in \mathcal{K}^*$. Here, $\mathcal{K}^*$ denotes the hull of $\mathcal{K}$ with respect to an extended mode of weak convergence (for details see Section 2 of Kellerer (1993)). While it is easily seen that the sets $\{f^* > 0\}$ and $\{f^* < 0\}$ are again contained in $U_1(A)$ and $U_0(A)$, respectively, it is an open problem, whether this result carries over to the general case, i.e. whether it is true that

$$U_1(A) \underset{\mu}{=} \{f^* > 0\}, \; V(A) \underset{\mu}{=} \{f^* = 0\}, \; U_0(A) \underset{\mu}{=} \{f^* < 0\}$$

for some function $f^* \in \mathcal{K}^*$.

To conclude this section, instead of a generalization, a special case of Theorem 2.2 will be considered. The following result is the natural extension of Corollary 4.10 in Kellerer (1993):

THEOREM 2.3. *If X is finite, then for any set $A \in \mathcal{A}$ there exists a function $f \in \mathcal{K}$ such that*

$$U_1(A) \underset{\mu}{=} \{f > 0\}, \;\; V(A) \underset{\mu}{=} \{f = 0\}, \;\; U_0(A) \underset{\mu}{=} \{f < 0\}.$$

PROOF. Since $k = \dim \mathcal{K} < \infty$, Theorem 2.2 applies and provides suitable functions $f_1, \cdots, f_k$. With the notation

$$T_i = \{f_i \neq 0\} \text{ for } 1 \leq i < k$$

the functions $f_{k-1}, \cdots, f_1$ (in this order) can be replaced by appropriate positive multiples such that

$$\min_{x \in T_i} |f_i(x)| > \max_{x \in X} | \sum_{i < j \leq k} f_j(x)| \text{ for } i = k-1, \cdots, 1.$$

This leads to the equivalence

$$(f_1(x), \cdots, f_k(x)) \succ / = / \prec 0 \Leftrightarrow \sum_{1 \leq i \leq k} f_i(x) > / = / < 0,$$

i.e. the function $f = \sum_{1 \leq i \leq k} f_i \in \mathcal{K}$ solves the problem. ∎

3. Marginal Problems. As already mentioned in the introduction, the classical case of a bounded moment problem concerns the reconstruction of an n-dimensional set, given the measures of all its $(n-1)$-dimensional sections orthogonal to the different axes. Here, even for uniquely determined sets, it makes an essential difference whether $n = 2$ or $n > 2$. While in the higher-dimensional case counterexamples predominate, the results in the two-dimensional case are quite satisfactory (for both see Section 6 of Kellerer (1993)). It is the aim of this section to show this to be true also in the general setting of this paper.

To fix the notation, the marginal problem is introduced by

$$(M) \begin{cases} X = X_1 \times X_2 \text{ and } \mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2, \\ \mathcal{K} = \{f_1 \circ \pi_1 + f_2 \circ \pi_2 : f_i \in \mathcal{L}_1(\mu_i)\}, \end{cases}$$

where π_i denotes the projection from X to X_i and μ_i the associated marginal measure of μ.

The first result takes up Theorem 2.3 to study $(0,1)$-matrices, as is done in an extensive literature (see Ryser (1960) and Brualdi (1980) for surveys and Kuba (1989) for some recent results). In this case strong and weak notions coincide:

PROPOSITION 3.1. *Let the problem (M) be specified by*

$(\star)$ $\quad X_i$ *finite,* $\mathcal{A}_i = \mathcal{P}(X_i)$, $\mu(\{x\}) = 0$ *or 1 for all* $x \in X$.

Then for any set $A \in \mathcal{A}$ *there exists a function* $f \in \mathcal{K}$ *such that*

$$U_1(A) \underset{\mu}{=} U_1^*(A) \underset{\mu}{=} \{f > 0\},$$
$$V(A) \underset{\mu}{=} V^*(A) \underset{\mu}{=} \{f = 0\},$$
$$U_0(A) \underset{\mu}{=} U_0^*(A) \underset{\mu}{=} \{f < 0\}.$$

PROOF. 1. According to Theorem 2.3 and due to the symmetry in A and $X \backslash A$ it is enough to show that for fixed $y \in X$ to a function $g \in \mathcal{G}$ with

$$g \underset{\mathcal{K}}{\sim} 1_A \text{ and } g(y) < 1 \tag{1}$$

there corresponds a set $B \in \mathcal{A}$ with

$$1_B \underset{\mathcal{K}}{\sim} 1_A \text{ and } y \notin B.$$

Since X is finite, it is no restriction to assume in addition

$$C = \{x \in X : 0 < g(x) < 1\} \text{ minimal under condition (1).} \tag{2}$$

Thus the assertion is established by proving $C = \emptyset$.

2. Since all sums $\sum_{x_1} g(x_1, x_2)$ and $\sum_{x_2} g(x_1, x_2)$ are integer-valued, the hypothesis $C \neq \emptyset$ is easily seen to yield a "loop"

$$(x_1^0, x_2^0), (x_1^0, x_2^1), (x_1^1, x_2^1), \cdots, (x_1^\ell, x_2^\ell) \in C$$

with $\ell > 1$ and $x_i^0, \cdots, x_i^{\ell-1}$ pairwise distinct, but $x_i^\ell = x_i^0$ for $i = 1, 2$. Then, by alternate addition of appropriate values $-a$ and $+a$ to the values of g at the vertices of the loop, the cardinality of C can be decreased without violating assumption (1) – contradicting, however, assumption (2). ∎

It should be pointed out that by this result in particular known characterizations can be extended to more general arrays – as, for instance, triangular matrices – by proper choice of the points $x \in X$ with $\mu(\{x\}) = 0$.

It will be shown next that in the marginal situation Theorem 2.3 extends from the finite to the countable case. This requires, however, some preparation:

LEMMA 3.2. *Let problem (M) be specified by*

$(*)$ $\quad X_i$ *countable and* $\mathcal{A}_i = \mathcal{P}(X_i)$.

Then a set $A \in \mathcal{A}$ is totally undetermined if and only if

$$(a) \qquad A_1 \times A_2 \underset{\mu}{\subset} A \text{ and } (X_1 \backslash A_1) \times (X_2 \backslash A_2) \underset{\mu}{\subset} X \backslash A$$

admits only the trivial solution

$$(b) \qquad A_1 \times A_2 \underset{\mu}{=} \emptyset \text{ and } (X_1 \backslash A_1) \times (X_2 \backslash A_2) \underset{\mu}{=} \emptyset.$$

PROOF. 1. To prove the condition to be necessary, assume $V(A) \underset{\mu}{=} X$ and let the sets A_i satisfy (a). Then the function

$$f = 1_{A_1} \circ \pi_1 + 1_{A_2} \circ \pi_2 - 1 \in \mathcal{K}$$

fulfills

$$f \mid A \underset{\mu}{\geq} 0 \text{ and } f \mid X \backslash A \underset{\mu}{\leq} 0.$$

Therefore by Lemma 1.6

$$A_1 \times A_2 = \{f > 0\} \underset{\mu}{\subset} U_1(A) \underset{\mu}{=} \emptyset,$$
$$(X_1 \backslash A_1) \times (X_2 \backslash A_2) = \{f < 0\} \underset{\mu}{\subset} U_0(A) \underset{\mu}{=} \emptyset,$$

i.e. the sets A_i satisfy (b).

2. To prove the condition to be sufficient, consider the pairs (A_1, A_2) as points of the compact space

$$Y = \{0,1\}^{X_1} \times \{0,1\}^{X_2}.$$

For fixed $x^0 = (x_1^0, x_2^0) \in X$ with $\mu(\{x^0\}) > 0$ the function

$$d(A_1, A_2) = \mu((X \backslash A) \cap (A_1 \times A_2)) + \mu(A \cap ((X_1 \backslash A_1) \times (X_2 \backslash A_2)))$$

is continuous and by assumption strictly positive on the closed subset

$$Y^0 = \{(A_1, A_2) \in Y : x_i^0 \in A_i \text{ for } i = 1, 2\}.$$

Thus there exists $\varepsilon > 0$ such that

$$d(A_1, A_2) \geq \varepsilon \text{ for } (A_1, A_2) \in Y^0,$$

while obviously

$$d(A_1, A_2) \geq 0 \text{ for } (A_1, A_2) \in Y \backslash Y^0.$$

Assuming in addition $\varepsilon \leq \mu(\{x^0\})$ define a measure $\overline{\nu}$ by

$$\overline{\nu}(B) = \mu(B) - \varepsilon 1_B(x^0) \text{ for } B \in \mathcal{A},$$

and let ν_1, ν_2 denote the marginal measures of the restriction of μ to A. Then, by the elementary identity

$$d(A_1, A_2) = \mu(A_1 \times A_2) + \mu(A) - \mu(A \cap (A_1 \times X_2)) - \mu(A \cap (X_1 \times A_2)),$$

the hypotheses

$$\nu_1(X_1) = \gamma = \nu_2(X_2),$$
$$\nu_1(A_2) + \nu_2(A_2) \leq \overline{\nu}(A_1 \times A_2) + \gamma \text{ for } A_i \in \mathcal{A}_i$$

of Satz 4.4 in Kellerer (1964a) are fulfilled (with $\underline{\nu} = 0$). This yields the existence of a measure $\nu \leq \overline{\nu}$ with marginals ν_i. It is a simple consequence that its density $g = d\nu/d\mu \in \mathcal{G}$ satisfies

$$g \underset{\mathcal{K}}{\sim} 1_A \text{ and } g(x^0) < 1.$$

Therefore $x^0 \notin U_1(A)$ and thus, x^0 being essentially arbitrary, $U_1(A) \underset{\mu}{=} \emptyset$. Since the assertion is symmetric in A and $X \backslash A$, finally $U_0(A) \underset{\mu}{=} \emptyset$ holds as well. ∎

Now the desired representation can be established:

PROPOSITION 3.3. *Let problem (M) be specified as in Lemma 3.2. Then for any set $A \in \mathcal{A}$ there exists a function $f \in \mathcal{K}$ such that*

$$U_1(A) \underset{\mu}{=} \{f > 0\}, \;\; V(A) \underset{\mu}{=} \{f = 0\}, \;\; U_0(A) \underset{\mu}{=} \{f < 0\}.$$

PROOF. 1. Denote by $\mathcal{R}$ the family of all pairs (A_1, A_2) satisfying

$$A_1 \times A_2 \underset{\mu}{\subset} A \text{ and } (X_1 \backslash A_1) \times (X_2 \backslash A_2) \underset{\mu}{\subset} X \backslash A.$$

Since X is countable, $\mathcal{R}$ contains a sequence (A_1^n, A_2^n) $n \in \mathbb{N}$, such that

$$\bigcup\{A_1 \times A_2 : (A_1, A_2) \in \mathcal{R}\} = \bigcup_{n \in \mathbb{N}} A_1^n \times A_2^n,$$

$$\bigcup\{(X_1 \backslash A_1) \times (X_2 \backslash A_2) : (A_1, A_2) \in \mathcal{R} = \bigcup_{n \in \mathbb{N}} (X_1 \backslash A_1^n) \times (X_2 \backslash A_2^n).$$

The functions

$$f_i = \sum_{n \in \mathbb{N}} 2^{-n}(1_{A_i^n} - 1_{X_i \backslash A_i^n}) \text{ for } i = 1, 2$$

are bounded by ± 1, hence the function

$$f = f_1 \circ \pi_1 + f_2 \circ \pi_2$$

belongs to $\mathcal{K}$ and is appropriate, as will now be shown.

2. Since obviously

$$f \mid A \underset{\mu}{\geq} 0 \text{ and } f \mid X \backslash A \underset{\mu}{\leq} 0,$$

it follows from Lemma 1.6 that

$$\{f > 0\} \underset{\mu}{\subset} U_1(A) \text{ and } \{f < 0\} \underset{\mu}{\subset} U_0(A).$$

Therefore it remains to show

$$\{f = 0\} \underset{\mu}{\subset} V(A),$$

where $\{f = 0\}$ is a countable union of products

$$X' = X_1' \times X_2' \text{ with } X_i' = \{f_i = \gamma_i\} \text{ and } \gamma_1 + \gamma_2 = 0.$$

3. To this end let $A_i' \subset X_i'$ satisfy

(a′) $$A_1' \times A_2' \underset{\mu}{\subset} X' \cap A \text{ and } (X_1' \backslash A_1') \times (X_2' \backslash A_2') \underset{\mu}{\subset} X' \backslash A.$$

Then the sets

$$A_i = A_i' \cup \{f_i > \gamma_i\} \text{ for } i = 1, 2$$

satisfy

$$A_1 \times A_2 \underset{\mu}{\subset} A,$$

because f is strictly positive on $(A_1 \times A_2) \backslash (A_1' \times A_2')$, and similarly

$$(X_1 \backslash A_1) \times (X_2 \backslash A_2) \underset{\mu}{\subset} X \backslash A.$$

Therefore $(A_1, A_2) \in \mathcal{R}$, hence by the definition of f in particular

$$A_1' \times A_2' \underset{\mu}{\subset} \{f > 0\} \text{ and } (X_1' \backslash A_1') \times (X_2' \backslash A_2') \underset{\mu}{\subset} \{f < 0\}.$$

Since the sets on the left-hand sides are as well subsets of $X' \subset \{f = 0\}$, in fact

(b′) $$A_1' \times A_2' \underset{\mu}{=} \emptyset \text{ and } (X_1' \backslash A_1') \times (X_2' \backslash A_2') \underset{\mu}{=} \emptyset.$$

Thus Lemma 3.2 applies locally to X' and yields, in a self-explaining notation,

$$V(A) \supset V'(X' \cap A) \underset{\mu}{=} X',$$

as had to be shown. ∎

According to this result the set $V(A)$ has the special form

$$V(A) \underset{\mu}{=} \bigcup_{n\in\mathbb{N}} (A_1^n \times A_2^n)$$

with disjoint sets A_i^n, $n \in \mathbb{N}$, for $i = 1, 2$.

The final application concerns essentially the classical geometric situation. As is easily checked, the proof of Lemma 3.2 makes substantial use of the discrete structure and cannot be carried over to the continuous situation. Apart from an isomorphism argument, the proof of the following analogue of Lemma 3.2 is therefore based on the main result of Kuba and Volcic (1993):

LEMMA 3.4. *Let problem (M) be specified by*

$$(*) \qquad \mu_i \text{ nonatomic and } \mu = \mu_1 \otimes \mu_2.$$

Then the equivalence in Lemma 3.2 holds as well.

PROOF. 1. It follows as in the proof of Lemma 3.2 that a totally undetermined set $A \in \mathcal{A}$ satisfies the condition stated there. In proving the converse, the σ-algebras $\mathcal{A}_i$ may be reduced to separable, i.e. countably generated, σ-algebras $\mathcal{A}_i^0$, due to the following reasons:

(1) $\mathcal{A}$ is the union of all products $\mathcal{A}^0 = \mathcal{A}_1^0 \otimes \mathcal{A}_2^0$ with separable σ-algebras $\mathcal{A}_i^0 \subset \mathcal{A}_i$;
(2) adjoining to $\mathcal{A}_i^0$ a sequence of finite partitions $X_i = \bigcup_k A_{ik}^n$ into sets $A_{ik}^n \in \mathcal{A}_i$ with $\mu(A_{ik}^n) < \frac{1}{n}$ for $n \in \mathbb{N}$ keeps μ_i nonatomic;
(3) when $\mathcal{G}$ in Definition 1.2 is reduced to the subclass $\mathcal{G}^0$ of $\mathcal{A}^0$-measurable functions, then the sets $U_i(A)$ are enlarged and thus the assertion, A to be totally undetermined, is strengthened;
(4) when $\mathcal{K}$ is similarly reduced to $\mathcal{K}^0$, then the relations $g \underset{\mathcal{K}}{\sim} 1_A$ and $g \underset{\mathcal{K}^0}{\sim} 1_A$ are equivalent for $g \in \mathcal{G}^0$, as is seen by smoothing $f \in \mathcal{K}$ to $f^0 \in \mathcal{K}^0$ via Radon-Nikodym.

2. Since it is no restriction to assume in addition $\mu_i(X_i) = 1$, it follows by an appropriate isomorphism theorem (see for instance Satz 1.4 in Kellerer (1964b)) that it is in fact enough to treat the case where X is the unit interval, $\mathcal{A}_i$ its Borel σ-algebra, and μ_i the restriction of Lebesgue measure. This, however, is done in Kuba and Volcic (1993), where the problem is first reduced, using "monotone rearrangements" as in Lorentz (1949), to sets with decreasing "cross functions" $x_1 \to \mu_2(A_{x_1})$ and $x_2 \to \mu_1(A_{x_2})$ and then is settled in (the course of the proof of) Theorem 4.3 by actually establishing the stronger result $V^*(A) \underset{\mu}{=} \emptyset$. ∎

Now the product situation can be treated, where again strong and weak notions coincide:

PROPOSITION 3.5. *Let problem (M) be specified as in Lemma 3.4. Then for any set $A \in \mathcal{A}$ there exists a function $f \in \mathcal{K}$ such that*

$$U_1(A) \underset{\mu}{=} U_1^*(A) \underset{\mu}{=} \{f > 0\},$$
$$V(A) \underset{\mu}{=} V^*(A) \underset{\mu}{=} \{f = 0\},$$
$$U_0(A) \underset{\mu}{=} U_0^*(A) \underset{\mu}{=} \{f < 0\}.$$

PROOF. 1. The proof of Proposition 3.3, concerning the sets $U_i(A)$ and $V(A)$, requires only two modifications:

(1) the defining equations for the sequence (A_1^n, A_2^n), $n \in \mathbb{N}$, have to hold only modulo μ – and can clearly be satisfied;

(2) the representation of the set $\{f = 0\}$ as a countable union of products $X_1' \times X_2'$ has to hold only modulo μ – and can be obtained, because it follows from Fubini's theorem that the defining values γ_i may be restricted to the countable sets

$$\Gamma_i = \{\gamma : \mu_i(\{f_i = \gamma\}) > 0\}.$$

2. To prove the equations concerning the sets $U_i^*(A)$ and $V^*(A)$ it is enough to verify the inclusion $V(A) \subset V^*(A)$, or equivalently, to show

$$B = \{0 < g < 1\} \underset{\mu}{\subset} V^*(A) \text{ for } g \underset{\mathcal{K}}{\sim} 1_A.$$

To this end split B into the sets

$$B_1 = B \cap U_1^*(A) \text{ and } B_2 = B \backslash B_1$$

and, replacing in Satz 1.7 in Kellerer (1964b) the inequality $0 \leq f \leq g$ by $0 \leq 1_{B_i} g \leq 1_{B_i}$, choose sets $C_i \in \mathcal{A}$ such that

$$C_i \subset B_i \text{ and } 1_{C_i} \underset{\mathcal{K}}{\sim} 1_{B_i} g \text{ for } i = 1, 2.$$

Then the set

$$C = C_1 \cup C_2 \cup \{g = 1\}$$

satisfies $1_C \underset{\mathcal{K}}{\sim} 1_A$ and thus includes $U_1^*(A)$. This yields the equation $C_1 \underset{\mu}{=} B_1$, which in view of $1 \in \mathcal{K}$ implies

$$\mu(B_1) = \int_X 1_{C_1} d\mu = \int_X 1_{B_1} g d\mu = \int_{B_1} g d\mu.$$

Since the last integrand is strictly less than 1, necessarily

$$\mu(B \cap U_1^*(A)) = 0$$

and similarly

$$\mu(B \cap U_0^*(A)) = 0.$$

Together this yields $B \underset{\mu}{\subset} V^*(A)$, as had to be shown. ∎

It is a consequence of this result that not only the set $V(A)$ but also $U_1(A)$ and $U_0(A)$ are simply countable unions of product sets.

In conclusion, the main open problem has to be pointed out: Extend Lemma 3.4 to the case where μ, instead of being a product measure, is only assumed to be absolutely continuous with respect to such a product. This would make it possible to deal with the geometric situation under the additional information that the unknown set A is contained in some fixed (non-product) subset X_0 of the plane. The crucial point lies in the fact that the proof in Kuba and Volcic (1993) cannot be extended to this situation, because it makes use of monotone rearrangements in both coordinates separately.

References

BRUALDI, R. A. (1980). Matrices of zeros and ones with fixed row and column sum vectors. *Linear Algebra* **33**, 159–231.

FISHBURN, P. C., J. C. LAGARIAS, J. A. REEDS, and L. A. SHEPP. (1990). Sets uniquely determined by projections on axes. I. Continuous case. *SIAM J. Appl. Math.* **50**, 288–306.

—— (1991). Sets uniquely determined by projections on axes. II. Discrete case. *Discrete Math.* **91**, 149–159.

KELLERER, H. G. (1964a). Maßtheoretische Marginalprobleme. *Math. Ann.* **153**, 168–198.

—— (1964b). Schnittmaß – Funktionen in mehrfachen Produkträumen. *Math. Ann.* **155**, 369–391.

—— (1993). Uniquenes in bounded moment problems. *Trans. Amer. Math. Soc.* **336**, 727–757.

KEMPERMAN, J. H. B. (1990). On sets that are uniquely determined by a restricted set of integrals. *Trans. Amer. Math. Soc.* **322**, 417–458.

—— (1991). Sets of uniqueness and systems of inequalities having a unique solution. *Pacific J. Math.* **148**, 275–301.

KUBA, A. (1989). Determination of the structure of the class $\mathcal{A}(R, S)$ of (0,1)-matrices. *Acta Cybernet.* **9**, 121–132.

KUBA, A. and A. VOLCIC (1988). Characterization of measurable plane sets which are reconstructable from their two projections. *Inverse Problems* **4**, 513–527.

—— (1993). The structure of the class of non-uniquely reconstructable sets. *Acta Sci. Math.* **58**, 363–388.

LORENTZ, G. G. (1949). A problem of plane measure. *Amer. J. Math.* **71**, 417–426.

RYSER, H. J. (1960). Matrices of zeros and ones. *Bull. Amer. Math. Soc.* **66**, 442–464.

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