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FAMILIES OF m-VARIATE DISTRIBUTIONS WITH GIVEN MARGINS AND $m(m-1)/2$ BIVARIATE DEPENDENCE PARAMETERS

BY HARRY JOE
University of British Columbia

A class of m-variate distributions with given margins and $m(m-1)/2$ dependence parameters, which is based on iteratively mixing conditional distributions, is derived. The family of multivariate normal distributions is a special case. The motivation for the class is to get parametric families that have $m(m-1)/2$ dependence parameters and properties that the family of multivariate normal distributions does not have. Properties of the class are studied, with details for (i) conditions for bivariate tail dependence and non-trivial limiting multivariate extreme value distributions and (ii) range of dependence for a bivariate measure of association such as Kendall's tau.

1. Introduction. The main purpose of this paper is to derive and study a class of m-variate distributions with given margins and $m(m-1)/2$ dependence parameters, one parameter corresponding to each bivariate margin. Of the parameters, $m-1$ can be interpreted as dependence parameters and the remainder can be interpreted as conditional dependence parameters. The multivariate normal family is a special case with the parametrization in terms on $m-1$ correlations and $(m-1)(m-2)/2$ partial correlations, each (independently) being in the range $(-1,1)$. The class considered here includes multivariate families with different amounts of bivariate tail dependence for different bivariate margins; the multivariate normal family does not have tail dependence, a property which is important for extreme value behavior.

The class of multivariate distributions is defined in Section 2. Properties studied there include: (a) partial closure under taking of margins, (b) simulation from the class, (c) bivariate tail dependence, (d) ordering by concordance, and (e) range of dependence.

Because tail dependence properties are one reason for studying non-multivariate normal families, conditions for bivariate tail dependence are stud-

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ied in some detail in Section 3 and the form of the limiting extreme value copula is obtained when there is tail dependence. In Section 4, the range of dependence of the multivariate family as measured by bivariate measures, such as Kendall's tau or tail dependence, is studied, particularly in the 3-dimensional case. For purposes of statistical modelling, a wide range of dependence is important when one has no reason to assume a special dependence pattern.

2. Multivariate Families with a Parameter for Each Bivariate Margin. The class of multivariate distributions to be studied is given in (2.1)–(2.3) below, after some notation.

F or $F_{1,\cdots,m}$ denotes a multivariate distribution, with continuous univariate margins $F_1,\cdots,F_m$. The higher order margins are denoted as F_S where S is a subset of $\{1,2,\cdots,m\}$ with cardinality at least 2. The densities, when they exist, are denoted as f_S and the survival functions are denoted as $\overline{F}_S$. The notation $\mathbf{y}_S$ is equivalent to $\{y_i : i \in S\}$. For specific subsets S, a simplifying notation used is that the braces for the subset are omitted. C denotes a bivariate copula (with uniform (0,1) margins). With subscripts, C_{jk}, $j < k$, is associated with the bivariate margin F_{jk} if $k = j+1$; otherwise if $k > j+1$, it is associated with the conditional distribution of the j^{th} and k^{th} variables given those indexed strictly between j and k. For S being a subset of $\{1,2,\cdots,m\}$ with cardinality less than m and $j \notin S$, $F_{j|S}$ denotes the conditional distribution of variable j given those whose indices are in S. That is, if $S_j = S \cup \{j\}$ and all densities (with respect to Lebesgue measure) exist, then $F_{j|S}(z_j|\mathbf{y}_S) = \int_{-\infty}^{z_j} f_{j|S}(y_j|\mathbf{y}_S)dy_j$ where $f_{j|S}(y_j|\mathbf{y}_S) = f_{S_j}(\mathbf{y}_{S_j})/f_S(\mathbf{y}_S)$.

We suppose that $C(u_1,u_2;\theta)$ is a family of bivariate copulas which include independence and the Fréchet upper bound, such as one of the families listed in Joe (1993). We suppose that the parametrization is such that the dependence in C increases as θ increases and that $\theta = \theta_I$ for independence, $\theta = \theta_U$ for the Fréchet upper bound. For $j < k$, there is a parameter $\theta_{jk} = \theta_{kj}$ and copula $C_{jk} = C(\cdot;\theta_{jk})$. For $1 \le j < m$, the $(j,j+1)$ bivariate margin of F is $F_{j,j+1}(y_j,y_{j+1}) = C_{j,j+1}(F_j(y_j),F_{j+1}(y_{j+1}))$. Based on these bivariate margins and the copulas C_{jk}, the higher-ordered margins $F_{1\cdots k},\cdots,F_{m-k+1\cdots m}$, $3 \le k < m$, will be defined by recursion and at the last stage $F = F_{1\ldots m}$ is defined.

For $m = 3$, the trivariate family is

$$F_{123}(y_1,y_2,y_3;\theta_{12},\theta_{13},\theta_{23}) = \int_{-\infty}^{y_2} C_{13}(F_{1|2}(y_1|z_2;\theta_{12}),F_{3|2}(y_3|z_2;\theta_{23}))F_2(dz_2), \tag{2.1}$$

where $F_{1|2},F_{3|2}$ are conditional cumulative distribution functions (cdfs) obtained from F_{12},F_{23}. By construction, (2.1) is a proper trivariate distribution with univariate margins F_1, F_2, F_3 and (1,2) bivariate margin F_{12}, and (2,3) bivariate margin F_{23}. The third parameter θ_{13} can be interpreted as a conditional dependence parameter (conditional dependence of the first and third univariate margin given the second) with $\theta_{13} = \theta_I$ corresponding to conditional independence and $\theta_{13} = \theta_U$ corresponding to perfect conditional dependence. The (1,3) bivariate margin of (2.1) can be obtained as $F_{13}(y_1, y_3; \theta_{12}, \theta_{23}, \theta_{13}) = F_{123}(y_1, \infty, y_3; \theta_{12}, \theta_{23}, \theta_{13})$. Note that this depends on all of the dependence parameters. In general, it will not be the same as $C(F_1, F_3; \theta)$ for some θ; see the examples given below.

For $m = 4$, define F_{234} in a similar way to F_{123} (by adding 1 to all subscripts in (2.1)). Note that both F_{123}, F_{234} have a common bivariate margin F_{23}. Then the 4–variate family is

$$\begin{aligned} F_{1234}(y_1, y_2, y_3, y_4; \theta_{12}, \cdots, \theta_{34}) = \int_{-\infty}^{y_2}\int_{-\infty}^{y_3} C_{14}(F_{1|23}(y_1|z_2, z_3; \theta_{12}, \theta_{13}, \theta_{23}), \\ F_{4|23}(y_4|z_2, z_3; \theta_{23}, \theta_{24}, \theta_{34}))F_{23}(dz_2, dz_3; \theta_{23}), \end{aligned} \tag{2.2}$$

where $F_{1|23}, F_{4|23}$ are conditional cdfs obtained from F_{123}, F_{234}. The parameters θ_{13}, θ_{24} are interpreted as for (2.1). The θ_{14} parameter is a conditional dependence parameter (conditional dependence of the first and fourth univariate margin given the second and third). As for (2.1), one can get the margins F_{14}, F_{124}, F_{134} by letting appropriate variables go to ∞ in (2.2).

It should be clear that this can be extended recursively and inductively. Assuming $F_{1\cdots m-1}, F_{2\cdots m}$ have been defined with a common $(m-2)$–dimensional margin $F_{2\cdots m-1}$, the m–variate family is

$$\begin{aligned} &F_{1\cdots m}(y_1, \cdots, y_m; \theta_{12}, \cdots, \theta_{1m}) \\ &= \int_{-\infty}^{y_2} \cdots \int_{-\infty}^{y_{m-1}} C_{1m}(F_{1|2\cdots m-1}(y_1|z_2, \cdots, z_{m-1}; \theta_{12}, \cdots, \theta_{m-2,m-1}), \\ &\quad F_{m|2\cdots m-1}(y_m|z_2, \cdots, z_{m-1}; \theta_{23}, \cdots, \theta_{m-1,m})) \\ &\quad \cdot F_{2\cdots m-1}(dz_2, \cdots, dz_{m-1}; \theta_{23}, \cdots, \theta_{m-2,m-1}), \end{aligned} \tag{2.3}$$

where $F_{1|2\cdots m-1}, F_{m|2\cdots m-1}$ are conditional cdfs obtained from $F_{1\cdots m-1}, F_{2\cdots m}$.

Similar to (2.1)–(2.3), one can define a family of m–variate distributions through survival functions, $\overline{F}_S$. Let $\overline{F}_j = 1 - F_j$ be the univariate survival functions. The bivariate margins with consecutive indices are $\overline{F}_{j,j+1}(y_j, y_{j+1}; \theta_{j,j+1}) = C^*_{j,j+1}(\overline{F}_j(y_j), \overline{F}_{j+1}(y_{j+1}); \theta_{j,j+1})$, where $C^*_{j,j+1}$ is the copula linking the univariate survival functions to the bivariate survival function. The m-variate case is like (2.3) with all F's replaced by $\overline{F}$'s and the integrals having lower limits y_j, $j = 2, \cdots, m-1$ and upper limits ∞. This leads to

$$\overline{F}_{1\cdots m}(y_1, \cdots, y_m; \theta_{12}, \cdots, \theta_{m-1,m}) =$$
$$\int_{y_2}^{\infty} \cdots \int_{y_{m-1}}^{\infty} C^*_{1m}(\overline{F}_{1|2\cdots m-1}(y_1|z_2, \cdots, z_{m-1}; \theta_{12}, \cdots, \theta_{m-2,m-1}),$$
$$\begin{aligned} &\overline{F}_{m|2\cdots m-1}(y_m|z_2, \cdots, z_{m-1}; \theta_{23}, \cdots, \theta_{m-1,m})) \\ &\cdot F_{2\cdots m-1}(dz_2, \cdots, dz_{m-1}; \theta_{23}, \cdots, \theta_{m-2,m-1}), \end{aligned} \tag{2.3$'$}$$

It is straightforward to show that this family is the same as that from (2.1)–(2.3) with $C^*_{jk}(u, v) = u + v - 1 + C_{jk}(1-u, 1-v)$ or $C_{jk}(u, v) = u + v - 1 + C^*_{jk}(1-u, 1-v)$.

Models (2.3) and (2.3$'$) are a unifying method for constructing multivariate distributions with a parameter for each bivariate margin. Note that these models can be generalized into a nonparametric family with C_{jk}, $j < k$, as the indices. The standard multivariate normal family is a special case of (2.3) and two other special cases, which have been used previously by the author, are given in Example 2 below.

EXAMPLE 1. Let $F_j = \Phi$, $j = 1, \cdots, m$, where Φ is the standard normal cdf. The bivariate normal copulas are $C(u_1, u_2; \theta) = \Phi_\theta(\Phi^{-1}(u_1), \Phi^{-1}(u_2))$, $-1 \le \theta \le 1$, where Φ_θ is the bivariate normal cdf with correlation θ, means 0 and variances 1. Then for (2.3), with $k - j > 1$, $\theta_{jk} = \rho_{jk\cdot(j+1,\cdots,k-1)}$ is the partial correlation of variables j and k given variables $j+1, \cdots, k-1$. [The proof of this is given in the Appendix.] For this parametrization of the multivariate normal family, all values of $\{\theta_{jk}, j < k\}$ in $(-1, 1)^{m(m-1)/2}$ are possible. The margins F_{jk} with $k - j > 1$ are also bivariate normal.

EXAMPLE 2. Joe (1994) used the extreme value limits of (2.3) to get families of multivariate extreme value distributions that have a parameter for each bivariate margin. The starting families of copulas were

$$C(u_1, u_2; \theta) = 1 - [(1-u_1)^\theta + (1-u_2)^\theta - (1-u_1)^\theta(1-u_2)^\theta]^{1/\theta}, \qquad \theta \ge 1, \tag{2.4}$$

and

$$C(u_1, u_2; \theta) = u_1 + u_2 - 1 + [(1-u_1)^{-\theta} + (1-u_2)^{-\theta} - 1]^{-1/\theta}, \qquad \theta \ge 0. \tag{2.5}$$

Some properties and discussion of these families are given in Joe (1994) and will not be repeated here. In (2.4), $\theta = 1$ corresponds to independence and $\theta \to \infty$ corresponds to the Fréchet upper bound; in (2.5), the corresponding values are 0 and ∞. There is no known multivariate family of copulas with $m(m-1)/2$ distinct bivariate dependence parameters that have (2.4) or (2.5) for all bivariate margins.

EXAMPLE 3. A special case consists of the multivariate distributions arising from a first order Markov chain based on a copula C and a marginal distribution F. That is, $C_{j,j+1} = C$ for all j and C_{jk} corresponds to independence if $k - j > 1$. In this case, for $m \geq 4$, (2.3) can be more simply written as

$$F_{1,\cdots,m}(y_1,\cdots,y_m) = \int_{-\infty}^{y_2} \cdots \int_{-\infty}^{y_{m-1}} F_{1|2}(y_1|z_2)F_{m|m-1}(y_m|z_{m-1}) \\ F_{2,\cdots,m-1}(dz_2,\cdots,dz_{m-1}),$$

where the transition distribution is $F_{i|i-1}(x_i|x_{i-1}) = B(F(x_{i-1}), F(x_i))$ and $B(u,v) = \partial C(u,v)/\partial u$.

EXAMPLE 4. This example is given to show a case where all bivariate margins belong to the same family; however the multivariate family is very limited in its range of dependence. Let $C(u_1,u_2;\theta) = u_1u_2(1+\theta(1-u_1)(1-u_2))$, $-1 \leq \theta \leq 1$, be the Morgenstern (1956) family. Let $C_{j,j+1}$ have parameter $\theta_{j,j+1}$ in this family and let C_{jk} correspond to independence ($\theta_{jk} = 0$) for $k - j > 1$. Also let all of the univariate margins be uniform on (0,1). Then for (2.1) to (2.3), the bivariate margins have copulas F_{jk} in the Morgenstern family with parameters α_{jk}, which satisfy $\alpha_{j,j+2} = \theta_{j,j+1}\theta_{j+1,j+2}/3$ and $\alpha_{j,j+\ell} = \theta_{j,j+1}\theta_{j+\ell-1,j+\ell}\alpha_{j+1,j+\ell-1}/9$ for $\ell \geq 3$. A short proof of this is given in the Appendix.

Note that it is not essential that the copulas C_{jk} all belong to the same family if one is concerned only with constructing some multivariate distributions and not with whether one can get a parametric family with a wide range of dependencies.

A few definitions used in the remainder of the article are given next.

DEFINITION. *Concordance ordering.* Let F, F' be bivariate distributions with univariate margins F_1, F_2. The F' is larger in concordance (or positive quadrant dependence) than F if $F'(x,y) \geq F(x,y)$ for all x, y. This definition is from Yanagimoto and Okamoto (1969) and Tchen (1980).

DEFINITION. *Bivariate tail dependence.* A bivariate copula $C(u,v)$ has *(upper) tail dependence* if $\overline{C}(u,u)/(1-u)$ converges to a constant δ in (0,1] as $u \to 1$, where $\overline{C}$ is the survival distribution corresponding to C. This definition is from Joe (1993).

DEFINITION. *Positive and negative quadrant dependence.* Let F be a bivariate distribution with univariate margins F_1, F_2. F is positive (negative) quadrant dependent if it is larger (smaller) in concordance than F_1F_2, that is, $F(x,y) \geq F_1(x)F_2(y)$ ($F(x,y) \leq F_1(x)F_2(y)$) for all x, y.

DEFINITION. *Kendall's tau.* Let F be a bivariate continuous cdf and let $(X_1, Y_1), (X_2, Y_2)$ be independent random pairs with distribution F. Then $\tau = 2\Pr((X_1 - X_2)(Y_1 - Y_2) > 0) - 1 = 4\int F\,dF - 1$.

DEFINITION. *Bivariate Fréchet bounds.* Consider the family of bivariate cdfs with univariate margins F_1, F_2. The Fréchet upper bound is $\min\{F_1(x), F_2(y)\}$ and the Fréchet lower bound is $\max\{0, F_1(x) + F_2(y) - 1\}$.

Next we obtain some properties of (2.3).

PROPERTY 1. *Partial closure.* Marginal densities of the form $F_{j,j+1,\cdots,j+k}$ with $k > 1$ have form (2.3). This follows directly from the recursive definition and shifting of indices by adding $j-1$ to all indices in (2.3) and letting $m = k - j + 1$. $F_{j,j+1,\cdots,j+k}$ has parameters $\theta_{j,j+1}, \cdots, \theta_{k-1,k}$.

PROPERTY 2. *Densities without integrals.* Assume the F_j are differentiable with densities f_j, $j = 1, \cdots, m$, and that the family of copulas have second order derivatives and hence densities $c(u_1, u_2; \theta)$. Then the family (2.3) of cdfs have densities. Although (2.3) has a representation in terms on integrals, the density $f_{1\cdots m}$ of (2.3) does not involve integrals. (The density f_{1m} however does involve integrals.) The proof of this is based on induction/recursion and the following observations.

Let $c_{jk}(u_1, u_2) = c(u_1, u_2; \theta_{jk})$. Then, omitting the parameters θ_{jk},

(a) $F_{1|2}(y_1|y_2) = \frac{\partial F_{12}(y_1,y_2)}{\partial y_2} / f_2(y_2) = \frac{\partial}{\partial u_2} C_{12}(F_1(y_1), F_2(y_2))$, $f_{1|2}(y_1|y_2) = \frac{\partial}{\partial y_1} F_{1|2}(y_1|y_2) = c_{12}(F_1(y_1),\ F_2(y_2)) f_1(y_1)$ do not involve integrals and similarly for $F_{3|2}, f_{3|2}$;

(b) $F_{1|23} = \frac{\partial}{\partial u_2} C_{13}(F_{1|2}, F_{3|2})$, $F_{3|12} = \frac{\partial}{\partial u_1} C_{13}(F_{1|2}, F_{3|2})$, $f_{123} = \frac{\partial^3 F_{123}}{\partial y_1 \partial y_2 \partial y_3}$ $= c_{13}(F_{1|2}(y_1|y_2), F_{3|2}(y_2|y_3)) \cdot f_{1|2}(y_1|y_2) f_{3|2}(y_3|y_2) f_2(y_2)$ do not involve integrals;

(c) for $m \geq 4$, with the inductive assumption that $F_{1|2\cdots m-1}$, $F_{m|2\cdots m-1}$, $f_{1\cdots m-1}$, $f_{2\cdots m}, f_{2\cdots m-1}$ do not involve integrals, then $f_{1|2\cdots m-1}, f_{m|2\cdots m-1}$ do not involve integrals, and

$$F_{1|2\cdots m} = \frac{\partial}{\partial u_2} C_{1m}(F_{1|2\cdots m-1}, F_{m|2\cdots m-1}),$$

$$f_{1\cdots m} = \frac{\partial^m F_{1\cdots m}}{\partial y_1 \cdots \partial y_m} = c_{1m}(F_{1|2\cdots m-1}, F_{m|2\cdots m-1}) f_{1|2\cdots m-1} f_{m|2\cdots m-1} f_{2\cdots m-1}$$

do not involve integrals. Similarly, $F_{m+1|2\cdots m}$ and $f_{2\cdots m+1}$ do not involve integrals.

PROPERTY 3. *Simulation.* The procedure to simulate a random vector from (2.3) is to first simulate $(x_2, \cdots, x_{m-1})$ from $F_{2,\cdots,m-1}$ and second to simulate a bivariate uniform random pair (u_1, u_m) from the copula C_{1m}.

Then x_1, x_m can be obtained as $G_\ell^{-1}(u_\ell)$, $\ell = 1, m$ respectively, where $G_\ell = F_{\ell|2,\cdots,m-1}$. The resulting $(x_1, \cdots, x_m)$ is then appropriate. Simulating from a bivariate copula is straightforward for the known families using the conditional approach (see below) or a representation such as a mixture of conditionally independent distributions (Marshall and Olkin, 1988). Note that G_1, G_m have closed forms if the C_{jk}'s have closed forms; however the functional inverses need not have closed form. If G_ℓ does not have closed form, then x_ℓ can be obtained numerically as the root of the equation $G_\ell(x_\ell) = u_\ell$, say using the Newton-Raphson method. Simulation from $F_{2,\cdots,m-1}$ can be based on the ideas in the preceding three sentences since its density $f_{2,\cdots,m-1}$ can be decomposed as a product of conditional densities, for example, $f_2 f_{3|2} \cdots f_{m-1|1,\cdots,m-2}$.

PROPERTY 4. *Concordance ordering.* As C_{jk} increases in concordance (as θ_{jk} increases) with other bivariate margins held fixed, then from property 1 and (2.3), $F_{j\cdots k}(y_j, \cdots, y_k)$ is increasing in θ_{jk} for all y_i, $i = j, \cdots, k$, and hence $F_{jk}(y_j, y_k)$ is increasing in θ_{jk} for all y_j, y_k.

It can be checked (for example, with the multivariate normal family) that a stronger concordance property such as "F_{13} increases in concordance as C_{12} increases in concordance" does not hold.

Now to get closer to one of the goals of this paper, we mention some tail dependence properties associated with (2.3).

PROPERTY 5. *Tail dependence.* For the trivariate case given in (2.1), if C_{12} and C_{23} have upper tail dependence and some regularity conditions hold, then F_{13} has upper tail dependence. For the general m-dimensional case in (2.3), if $C_{j,j+1}$, $j = 1, \cdots, m-1$, have upper tail dependence and some regularity conditions hold, then F_{jk}, $k - j > 1$, all have upper tail dependence.

The precise statement of this result and its proof are given in Section 3 together with some necessary lemmas on tail dependence.

PROPERTY 6. *Range of dependence.* How close does family (2.3) come to covering all theoretical $\{\tau_{ij}\}$ or $\{\delta_{ij}\}$? Some answers are provided in Section 4. Here τ_{ij} is Kendall's tau for the (i,j) bivariate margin, and δ_{ij} is the (upper) tail dependence parameter for the (i,j) bivariate margin.

Because of Property 1, we make the following final remark for this section. The family (2.3) has only partial closure in that $F_{j,\cdots,j+k}$, $k > 1$, have the same form but other margins have different forms. Hence the use of the family requires the decision of how to assign variables to the indices if there is no natural order to the variables. Some numerical checks for the families in Example 2 in Joe (1994) suggest that bivariate margins $F_{j,j+k}$, $k > 1$, are close to those in (2.4) or (2.5) if the bivariate margins $F_{j,j+1}$ are the ones with the most dependence.

3. Tail Dependence Results. In this section, the quantities needed for analyzing tail dependence of multivariate distributions are given and used to prove tail dependence of (2.3) under some conditions. The same quantities are then used for deriving the limiting multivariate extreme value copula of (2.3) when all bivariate margins have upper tail dependence.

To stress ideas and concepts, we assume the existence of derivatives and other regularity conditions as needed. Some equivalent conditions for bivariate tail dependence are given first. Sometimes it is more convenient to work with exponential margins than uniform margins. For a bivariate copula C, let

$$G(x,y) = C(1-e^{-x}, 1-e^{-y}). \tag{3.1}$$

The definition of upper tail dependence in Section 2 becomes

$$e^{x}\overline{G}(x,x) \to \delta \in (0,1], \quad x \to \infty. \tag{3.2}$$

Now assuming that G has derivatives to second order, let $G_{1|2}(x|y) = e^{y}\frac{\partial G(x,y)}{\partial y}$ and $\overline{G}_{1|2} = 1 - G_{1|2}$. Then

$$\begin{aligned} e^{x}\overline{G}(x,x) &= e^{x}\int_{x}^{\infty} \overline{G}_{1|2}(x|y)e^{-y}dy \\ &= \int_{-\infty}^{0} \overline{G}_{1|2}(x|x-v)e^{v}dv = \int_{0}^{\infty} \overline{G}_{1|2}(x|x+v)e^{-v}dv. \end{aligned}$$

Assuming that $e^{x}\overline{G}(x,x)$ converges as $x \to \infty$ and that

$$\overline{G}_{1|2}(x|x+v) \to a(v) \tag{3.3}$$

for all v ($v < 0$ is needed below), where a is continuous and $a \le 1$, then by the Bounded Convergence Theorem,

$$e^{x}\overline{G}(x,x) \to \int_{0}^{\infty} a(v)e^{-v}dv. \tag{3.4}$$

Tail dependence holds if and only if a is not identically 0 (a.s.) on $(0,\infty)$.

Now let g be the density of G. Then

$$\begin{aligned} e^{x}\overline{G}(x,x) &= e^{x}\int_{x}^{\infty}\int_{x}^{\infty} g(y_1,y_2)dy_1dy_2 \\ &= \int_{0}^{\infty}\int_{0}^{\infty} e^{x}g(x+v_1,x+v_2)dv_1dv_2. \end{aligned} \tag{3.5}$$

Assuming that

$$e^{x}g(x+v_1,x+v_2) \to b(v_1,v_2) \tag{3.6}$$

and that the Lebesgue Dominated Convergence Theorem can be used in (3.5),

$$e^x\overline{G}(x,x) \to \int_0^\infty \int_0^\infty b(v_1,v_2)dv_1dv_2$$

and tail dependence holds if and only if b is not identically 0 (a.s.) on $(0,\infty)^2$.

A relation between a and b (when both are non-zero) is as follows: $a(v) = \lim_x e^{x+v}\int_x^\infty g(z,x+v)dz = \lim_x e^{x+v}\int_{-v}^\infty g(x+v+w,x+v)dw = \int_{-v}^\infty b(w,0)dw$. Hence $a(v)$ is increasing in v and under some regularity conditions, $a(v)\to 1$ as $v\to\infty$. This requires that $b(\cdot,0)$ is a density on $(-\infty,\infty)$ but $b(\cdot,0)$ is just the limit of the conditional densities $g_{2|1}(x+\cdot|x)$. Note that the condition of $a(v)$ increasing in v is closely related to the condition $\overline{G}_{1|2}(x|x+v)$ increasing in v for all x; this latter condition is the stochastically increasing positive dependence condition.

EXAMPLE 5. For illustration, we give these quantities for two families of bivariate copulas which are listed in Joe (1993); one family has upper tail dependence and the other does not have tail dependence.

(i) Consider the family $C(u,v;\theta) = 1-[(1-u)^\theta+(1-v)^\theta-(1-u)^\theta(1-v)^\theta]^{1/\theta}$, $\theta\ge 1$. Independence obtains when $\theta=1$ and upper tail dependence holds when $\theta>1$. Let $G, G_{1|2}, g$ be as defined above. It is straightforward to verify the following: $e^x\overline{G}(x,x)\to 2-2^{1/\theta}$ as $x\to\infty$, $\overline{G}_{1|2}(x|x+v)\to 1-(1+e^{\theta v})^{-1+1/\theta} = a(v)$, $-\infty<v<\infty$, as $x\to\infty$, and $e^x g(x+v_1,x+v_2)\to(\theta-1)e^{-\theta(v_1+v_2)}(e^{-\theta v_1}+e^{-\theta v_2})^{-2+1/\theta} = b(v_1,v_2)$ as $x\to\infty$. Note that $a(v)$ is increasing in v and that $a(v)\to 1$ as $v\to\infty$.

(ii) A family due to Frank (1979) is $C(u,v;\theta) = -\theta^{-1}\log([\eta-(1-e^{-\theta u})(1-e^{-\theta v})]/\eta)$, $-\infty<\theta<\infty$, where $\eta = 1-e^{-\theta}$. It is straightforward to show that as $x\to\infty$: $e^x\overline{G}(x,x)\to 0$, $\overline{G}_{1|2}(x|x+v)\sim\theta e^{-x}/[(1-e^{-\theta})(1+\theta e^{-x}+\theta e^{-x-v})]\to 0$, $e^x g(x+v_1,x+v_2)\sim\theta e^{-x-v_1-v_2}/(1-e^{-\theta})\to 0$.

THEOREM 3.1. *Suppose that C_{12}, C_{23} have upper tail dependence and that the Lebesgue dominated convergence theorem can be applied to (3.7) below. Then F_{13} in (2.1) has upper tail dependence and the tail dependence parameter is given in (3.8).*

PROOF. Let F_{12}, F_{23} be defined as in (3.1) with C_{12}, C_{23} respectively. Let a be defined as in (3.3) with subscripts 12 or 32 for the (1,2) or (2,3) bivariate margin respectively. Putting exponential margins in (2.1) and omitting the parameters θ_{jk} leads to

$$F_{123}(y_1,y_2,y_3) = \int_0^{y_2} C_{13}(F_{1|2}(y_1|z_2), F_{3|2}(y_3|z_2))e^{-z_2}dz_2$$

and

$$\begin{aligned}\overline{F}_{13}(x,x) &= 1 - F_1(x) - F_3(x) + F_{13}(x,x)\\ &= 1 - \int_0^\infty F_{1|2}(x|z)e^{-z}dz - \int_0^\infty F_{3|2}(x|z)e^{-z}dz\\ &+ \int_0^\infty C_{13}(F_{1|2}(x|z), F_{3|2}(x|z))e^{-z}dz\\ &= \int_0^\infty \overline{C}_{13}(F_{1|2}(x|z), F_{3|2}(x|z))e^{-z}dz.\end{aligned}$$

Hence

$$e^x\overline{F}_{13}(x,x) = \int_{-x}^\infty \overline{C}_{13}(F_{1|2}(x|x+v), F_{3|2}(x|x+v))e^{-v}dv \tag{3.7}$$

$$\to \int_{-\infty}^\infty \overline{C}_{13}(1-a_{12}(v), 1-a_{32}(v))e^{-v}dv \tag{3.8}$$

assuming the Lebesgue dominated convergence theorem can be used and a_{12}, a_{32} are the limits for $\overline{F}_{1|2}, \overline{F}_{3|2}$ as in (3.3). ∎

REMARKS. 1. The condition of C_{12}, C_{23} having upper tail dependence is essentially necessary. For example, let $C_{23}(u,v) = C_{13}(u,v) = uv$, $C_{12}(u,v) = 1 - [(1-u)^\theta + (1-v)^\theta - (1-u)^\theta(1-v)^\theta]^{1/\theta}$, $\theta > 1$. Then $e^x\overline{F}_{13}(x,x) \sim e^{-x}\int_{-x}^\infty [1-(1+e^{\theta v})^{-1+1/\theta}]e^{-v}dv \to 0$ as $x \to \infty$.

2. For the result of Theorem 3.1, positive dependence for the copula C_{13} is not necessary. Even if C_{13} corresponds to the Fréchet lower bound, F_{13} can have upper tail dependence - under regularity conditions, the tail dependence parameter is $\int_{-\infty}^\infty e^{-v}\max\{a_{12}(v) + a_{32}(v) - 1, 0\}dv$.

3. If C_{13} has upper tail dependence and C_{12}, C_{23} do not, then it is possible that F_{13} has tail dependence. (A sufficient condition is that C_{12}, C_{23} satisfy the positive dependence condition, $\overline{C}_{k|2}(u|v) \geq u$ for all $v \geq v_0$ for some $v_0 \in (0,1)$, $k = 1,3$.) This case is not so interesting for the goal of getting tail dependence for every bivariate margin.

THEOREM 3.2. *Suppose that $C_{j,j+1}$, $j = 1,\cdots,m-1$, have upper tail dependence and that all copulas C_{jk} have densities. For (2.3) with exponential univariate margins, suppose that for j,k, with $k > j$, that the following pointwise convergences hold as $x \to \infty$:*

(i) $\overline{F}_{j|j+1,\cdots,k}(x|x+v_{j+1},\cdots,x+v_k) \to a_{j,j+1,\cdots,k}(v_{j+1},\cdots,v_k)$,

(ii) $\overline{F}_{k|j,\cdots,k-1}(x|x+v_j,\cdots,x+v_{k-1}) \to a_{k,j+1,\cdots,k-1}(v_j,\cdots,v_{k-1})$,

(iii) $e^x f_{j,\cdots,k}(x+v_j,\cdots,x+v_k) \to b_{j,\cdots,k}(v_j,\cdots,v_k)$,

and that the functions on the right hand sides of (i), (ii), (iii) are not identically 0 (a.s.). Also assume that the Lebesgue dominated convergence theorem *applies for the integrals in (3.9), (3.10), (3.11), and that C_{jk} $(k-j>1)$ is such that (3.12) is positive, then F_{jk}, $k-j>1$, all have upper tail dependence.*

PROOF. This is by induction starting with the result in Theorem 3.1. We assume as in the proof of Theorem 3.1 that all univariate margins are exponential distributions. Note from property 2 in Section 2 that all multivariate densities exist. The trivariate argument works to show that $F_{j,j+2}$, $j=1,\cdots,m-2$, have tail dependence. Suppose that it has been shown that $F_{j,j+i}$ has upper tail dependence for all $2\le i\le \ell$, $\ell\ge 2$. For $2\le k-j\le \ell$,

$$\begin{aligned} e^x\overline{F}_{j,j+1}(x,x) &= e^x\overline{F}_{j,\cdots,k}(x,x,0,\cdots,0)\\ &= \int_0^\infty\int_0^\infty\int_{-x}^\infty\cdots\int_{-x}^\infty e^x f_{j,\cdots,k}(x+v_j,\cdots,x+v_k)dv_j\cdots dv_k. \end{aligned}\tag{3.9}$$

From condition (iii) and assuming that the Lebesgue dominated convergence theorem applies in (3.9) over the region $(0,\infty)^2\times(-\infty,\infty)^{k-j-1}$, the right hand side of (3.9) converges to $\int_0^\infty\int_0^\infty\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty b_{j,\cdots,k}(v_j,\cdots,v_k)\,dv_j\cdots dv_k$, and $b_{j,\cdots,k}$ must not be identically 0 (a.s.) since by (3.2), the left hand side of (3.9) is positive. For $2\le k-j\le\ell$, we also have

$$\begin{aligned} e^x\overline{F}_{j,j+1}(x,x) = \int_0^\infty\int_{-x}^\infty\cdots\int_{-x}^\infty &\overline{F}_{j|j+1,\cdots,k}(x|x+v_{j+1},\cdots,v_k)\\ &\cdot e^x f_{j+1,\cdots,k}(x+v_{j+1},\cdots,x+v_k)dv_{j+1}\cdots dv_k. \end{aligned}\tag{3.10}$$

From conditions (i) and (iii) and assuming that the Lebesgue dominated convergence theorem applies in (3.10) over the region $(0,\infty)\times(-\infty,\infty)^{k-j-1}$, the right hand side of (3.10) converges to

$$\int_0^\infty\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty a_{j,j+1,\cdots,k}(v_{j+1},\cdots,v_k)b_{j+1,\cdots,k}(v_{j+1},\cdots,v_k)dv_{j+1}\cdots dv_k,$$

and $a_{j,j+1,\cdots,k}$ cannot be identically 0 (a.s.). With similar assumptions,

$$\begin{aligned} e^x\overline{F}_{k-1,k}(x,x)\to\int_0^\infty\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty & a_{k,j,\cdots,k-1}(v_j,\cdots,v_{k-1})\\ & b_{j,\cdots,k-1}(v_j,\cdots,v_{k-1})dv_j\cdots dv_{k-1}, \end{aligned}$$

and $a_{k,j,\cdots,k-1}$ cannot be identically 0 (a.s.).

Now, for $k=j+\ell+1$,

$$\begin{aligned} e^x\overline{F}_{jk}(x,x) = \int_{-x}^\infty\cdots\int_{-x}^\infty &\overline{C}_{jk}(F_{j|j+1,\cdots.k-1}(x|x+v_{j+1},\cdots,x+v_{k-1}),\\ &F_{k|j+1,\cdots,k-1}(x|x+v_{j+1},\cdots,x+v_{k-1}))\\ &\cdot e^x f_{j+1,\cdots,k-1}(x+v_{j+1},\cdots,x+v_{k-1})dv_{j+1}\cdots dv_{k-1} \end{aligned}\tag{3.11}$$

$$\to \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \overline{C}_{jk}(1 - a_{j,j+1,\cdots,k-1}(v_{j+1}, \cdots, v_{k-1}),$$

$$1 - a_{k,j+1,\cdots,k-1}(v_{j+1}, \cdots, v_{k-1})) \cdot b_{j+1,\cdots,k-1}(v_{j+1}, \cdots, v_{k-1}) dv_{j+1} \cdots dv_{k-1} \tag{3.12}$$

assuming that the Lebesgue dominated convergence theorem applies to (3.11) in $(-\infty, \infty)^{k-j-1}$. If $\overline{C}_{jk}(u_1, u_2) > 0$ when $u_1, u_2 < 1$ (for example, if C_{jk} has positive quadrant dependence, $\overline{C}_{jk}(u_1, u_2) > (1 - u_1)(1 - u_2)$), then (3.12) is positive. (3.12) in general will be positive unless zero values of $\overline{C}_{jk}$ occur only when $a_{j,j+1,\cdots,k-1}$ and $a_{k,j+1,\cdots,k-1}$ are positive. ∎

REMARK. The assumptions given in Theorems 3.1 and 3.2 are not really too strong since they do hold in special cases such as those in Example 2 of Section 2.

Next we derive the multivariate extreme value copula associated with (2.3) when all bivariate margins $C_{j,j+1}$ have upper tail dependence. Some details and background are given in the Appendix of Joe (1994).

For a multivariate distribution with exponential univariate margins, the multivariate extreme value limit which has the Gumbel margin $H(x_j) = \exp\{-e^{x_j}\}$, $j = 1, \cdots, m$, is

$$\lim_{n\to\infty} F^n_{1,\cdots,m}(x_1 + \log n, \cdots, x_m + \log n).$$

This is equivalent to

$$\exp\{-\lim n[1 - F_{1,\cdots,m}(x_1 + \log n, \cdots, x_m + \log n)]\}. \tag{3.13}$$

The limit in the exponent of (3.13) can be calculated in more than one way depending on the form of the copulas C_{jk}. Two different forms are described briefly with the limits being expressed in terms of the functions a and b in Theorem 3.2. The formulas below are generalizations of the special cases given in Joe (1994), which result from the families given in Example 2 of Section 2. The use of the families of copulas in (2.4) (respectively (2.5)) leads to an extreme value limit with exponent given in (3.16) (respectively (3.18)).

The first form below is convenient if $1 - C_{jk}$ is simple (for all j, k). From (2.3),

$$1 - F_{1,\cdots,m}(x_1, \cdots, x_m) = \int_0^{x_2} \cdots \int_0^{x_{m-1}} [1 - C_{1m}(F_{1|2,\cdots,m-1}, F_{m|2,\cdots,m-1})] \, dF_{2,\cdots,m-1} + 1 - F_{2,\cdots,m-1}(x_2, \cdots, x_{m-1}) \tag{3.14}$$

so that a recursion formula is possible for the limit in the exponent of (3.13). Let $M = \log n$. Then n times the integrand in (3.14) is:

$$
\begin{aligned}
n \int_0^{x_2+M} & \cdots \int_0^{x_{m-1}+M} [1 - C_{1m}(F_{1|2,\cdots,m-1}(x_1 + M|z_2, \cdots, z_{m-1}), \\
& F_{m|2,\cdots,m-1}(x_m + M|z_2, \cdots, z_{m-1}))] \\
& \cdot f_{2,\cdots,m-1}(z_2, \cdots, z_{m-1}) dz_2 \cdots dz_{m-1} \\
= & \int_{-M}^{x_2} \cdots \int_{-M}^{x_{m-1}} [1 - C_{1m}(F_{1|2,\cdots,m-1}(x_1 + M|v_2 + M, \cdots, v_{m-1} + M), \\
& F_{m|2,\cdots,m-1}(x_m + M|v_2 + M, \cdots, v_{m-1} + M))] \\
& \cdot e^M f_{2,\cdots,m-1}(v_2 + M, \cdots, v_{m-1} + M) dv_2 \cdots dv_{m-1} \\
\to & \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_{m-1}} [1 - C_{1m}(1 - a_{1,2\cdots,m-1}(v_2 - x_1, \cdots, v_{m-1} - x_1), \\
& 1 - a_{m,2\cdots,m-1}(v_2 - x_m, \cdots, v_{m-1} - x_m))] \\
& \cdot b_{2,\cdots,m-1}(v_2, \cdots, v_{m-1}) dv_2 \cdots dv_{m-1}
\end{aligned}
\tag{3.15}
$$

under conditions similar to those in Theorem 3.2. Let $\eta_{jk}(x_j, \cdots, x_k)$, with $j < k$, $k - j > 1$, be the similar limit to (3.15) starting with C_{jk} instead of C_{1m} and let $\eta_{j,j+1}(x_j, x_{j+1}) = \lim n[1 - F_{j,j+1}(x_j + \log n, x_{j+1} + \log n)]$. Then $\lim n[1 - F_{1,\cdots,m}(x_1 + \log n, \cdots, x_m + \log n)]$, in the exponent of (3.13), is

$$
\begin{cases}
\sum_{i=1}^{m/2} \eta_{i,m+1-i}(x_i, \cdots, x_{m+1-i}) & \text{if } m \text{ is even} \\
\sum_{i=1}^{(m-1)/2} \eta_{i,m+1-i}(x_i, \cdots, x_{m+1-i}) + \exp(-x_{(m+1)/2}) & \text{if } m \text{ is odd.}
\end{cases}
\tag{3.16}
$$

The second form is based on the identity

$$
\begin{aligned}
1 - \Pr(X_1 \le x_1, \cdots, X_m \le x_m) = & \sum_{i=1}^{m} \Pr(X_i > x_i) \\
& - \sum_{j<k} \Pr(X_j > x_j, X_k > x_k, X_i \le x_i, j < i < k).
\end{aligned}
$$

The limit in the exponent of (3.13) is then based on $\lim n \ \exp(-x_i - \log n) =$

$\exp(-x_i)$, $\lim n[1 - F_{j,j+1}(x_j + \log n, x_{j+1} + \log n)] = \zeta_{j,j+1}(x_j, x_{j+1})$, and

$$\begin{aligned}
&\lim n \Pr(X_j > x_j + \log n, X_k > x_k + \log n, X_i \leq x_i + \log n, j < i < k) \\
&= \int_{-M}^{x_{j+1}} \cdots \int_{-M}^{x_{k-1}} \overline{C}_{jk}(F_{j|j+1,\cdots,k-1}(x_j + M|v_{j+1} + M, \cdots, v_{k-1} + M), \\
&\quad F_{k|j+1,\cdots,k-1}(x_k + M|v_{j+1} + M, \cdots, v_{k-1} + M)) \\
&\quad \cdot e^M f_{j+1,\cdots,k-1}(v_{j+1} + M, \cdots, v_{k-1} + M) dv_{j+1} \cdots dv_{k-1} \\
&\to \int_{-\infty}^{x_{j+1}} \cdots \int_{-\infty}^{x_{k-1}} \overline{C}_{jk}(1 - a_{j,j+1,\cdots,k-1}(v_{j+1} - x_j, \cdots, v_{k-1} - x_j), \\
&\quad 1 - a_{k,j+1,\cdots,k-1}(v_{j+1} - x_k, \cdots, v_{k-1} - x_k)) \\
&\quad \cdot b_{j+1,\cdots,k-1}(v_{j+1}, \cdots, v_{k-1}) dv_{j+1} \cdots dv_{k-1}
\end{aligned} \tag{3.17}$$

under conditions similar to those in Theorem 3.2, for $k-j > 1$ and $(X_j, \cdots, X_k)$ having distribution $F_{j,\cdots,k}$ with form (2.3). Let the limit in (3.17) be denoted by $\zeta_{jk}(x_j, \cdots, x_k)$. Then $\lim n[1 - F_{1,\cdots,m}(x_1 + \log n, \cdots, x_m + \log n)]$ is

$$\sum_i \exp(-x_i) - \sum_{j<k} \zeta_{jk}(x_j, \cdots, x_k). \tag{3.18}$$

The final result of this section is to show that for a bivariate copula with tail dependence, the extreme value limit (3.13) has the same tail dependence parameter δ.

THEOREM 3.3. *Let C be a bivariate copula and let $F(x_1, x_2) = C(1 - e^{-x_1}, 1 - e^{-x_2})$. Suppose $\lim_{u\to 1} \overline{C}(u, u)/(1 - u) = \delta$, where $\delta \in (0, 1]$ and $\lim_n F^n(x_1 + \log n, x_2 + \log n) = H(x_1, x_2) = \exp\{-\eta(x_1, x_2)\}$ with univariate margins $\exp\{-e^{-x_j}\}$, $j = 1, 2$. Let $C^*(u_1, u_2) = H(-\log[-\log u_1], -\log[-\log u_2])$. Then $\lim_{u\to 1} \overline{C}^*(u, u)/(1 - u) = \delta$.*

PROOF. From (3.13) and (3.2),

$$\begin{aligned}
\eta(x, x) &= \lim n[1 - F(x + \log n, x + \log n)] \\
&= \lim n[e^{-x-\log n} + e^{-x-\log n} - \overline{F}(x + \log n, x + \log n)] \sim 2e^{-x} - \delta e^{-x}.
\end{aligned}$$

Now,

$$\begin{aligned}
\overline{C}^*(u, u)/(1 - u) &= [1 - 2u + \exp\{\eta(-\log[-\log u], -\log[-\log u])\}]/(1 - u) \\
&\sim [1 - 2u + u^{2-\delta}]/(1 - u) \to \delta
\end{aligned}$$

as $u \to 1$. ∎

4. Range of Dependence. We should get a "wide" range of dependence from (2.3), especially if the family of copulas include both the Fréchet upper and lower bounds. Let us measure the range of dependence via possible sets of $\{\zeta_{ij}, i < j\}$, where ζ_{ij} is a measure of dependence for the (i, j) bivariate margin. We study the trivariate case in more detail and make generalizations where possible to higher dimensions.

Three-dimensional case. This case is easier to study than the general multivariate case since two of the three bivariate margins can be considered arbitrary, and then the third bivariate margin has constraints given the other two. Also this case provides insight into what is possible for the multivariate case.

We will follow (2.1) in thinking of the (1,3) bivariate margin as depending on the (1,2) and (2,3) margins. One can argue that (2.1) has a full range of dependence if C_{12}, C_{23} have a full range of dependence from the Fréchet lower bound to the Fréchet upper bound and likewise for C_{13} (because then perfect positive and negative conditional dependence are possible). This argument should extend to the multivariate case in (2.3). The analysis of the range of dependence is not so straightforward when one is trying to have positively dependent distributions only (C_{12}, C_{23}, C_{13} between independence and the Fréchet upper bound) such as in the case of multivariate distributions with upper tail dependence. Hence we consider looking at the range of possible $(\zeta_{12}, \zeta_{23}, \zeta_{13})$ where ζ_{jk} is a measure of dependence for the (j, k) bivariate margin. The Pearson correlation coefficient is not suitable because it is not invariant with respect to the choice of the univariate margins F_i; for example, the correlation coefficient for the Fréchet upper bound with margins F_1, F_2 is not 1 unless $F_1 = F_2$. Instead we use ζ equal to Kendall's tau for a measure of monotone dependence invariant with respect to the univariate margins, and $\zeta = \delta$, the upper tail dependence parameter given in Section 2 ($\delta_{jk} = \lim_{u \to 1} \overline{C}_{jk}(u, u)/(1 - u)$).

For the bivariate normal distribution, $\tau = (2/\pi)\arcsin(\rho)$ (see, for example, Kepner, Harper and Keith, 1989). Hence, for the trivariate normal distributions, the constraint $-1 \leq \rho_{13\cdot2} \leq 1$ is the same as

$$\rho_{12}\rho_{23} - [(1 - \rho_{12}^2)(1 - \rho_{23}^2)]^{1/2} \leq \rho_{13} \leq \rho_{12}\rho_{23} + [(1 - \rho_{12}^2)(1 - \rho_{23}^2)]^{1/2}$$

or

$$-\cos(0.5\pi(\tau_{12} + \tau_{23})) \leq \sin(0.5\pi\tau_{13}) \leq \cos(0.5\pi(\tau_{12} - \tau_{23}))$$

or

$$-1 + |\tau_{12} + \tau_{23}| \leq \tau_{13} \leq 1 - |\tau_{12} - \tau_{23}|. \tag{4.1}$$

As shown in Theorem 4.1 below, the bounds in (4.1) are bounds for general F_{23} that are compatible with F_{12}, F_{23}, that is, within the trivariate normal family, all of the possible values of $(\tau_{12}, \tau_{13}, \tau_{23})$ for trivariate distributions are attainable.

THEOREM 4.1. *Let F_{123} be a continuous trivariate distribution with bivariate margins F_{12}, F_{23}, F_{13}. Let τ_{jk} be the value of Kendall's tau for the (j,k) bivariate margin. Then inequality (4.1) is satisfied and the bounds are sharp.*

PROOF. Let (X_{i1}, X_{i2}, X_{i3}), $i = 1,2$, be independent random vectors from the distribution F_{123}. Then $\tau_{jk} = 2\eta_{jk} - 1$, $1 \le j < k \le 3$, where $\eta_{jk} = \Pr((X_{1j} - X_{2j})(X_{1k} - X_{2k}) > 0)$. Then $\eta_{13} = \Pr((X_{11} - X_{21})(X_{12} - X_{22})^2(X_{13} - X_{23}) > 0) = \Pr((X_{11} - X_{21})(X_{12} - X_{22}) > 0, (X_{12} - X_{22})(X_{13} - X_{23}) > 0) + \Pr((X_{11} - X_{21})(X_{12} - X_{22}) < 0, (X_{12} - X_{22})(X_{13} - X_{23}) < 0)$. Hence an upper bound for η_{13} is $\min\{\eta_{12}, \eta_{23}\} + \min\{1 - \eta_{12}, 1 - \eta_{23}\}$ and a lower bound is $\max\{0, \eta_{12} + \eta_{23} - 1\} + \max\{0, (1 - \eta_{12}) + (1 - \eta_{23}) - 1\}$. After substituting for τ_{jk} and simplifying, inequality (4.1) results. The sharpness follows from the special trivariate normal case. ∎

From the construction of (2.1), one might expect the upper (lower) bound in (4.1) to be attained for any family $C(\cdot;\theta)$ that includes the Fréchet upper (lower) bound. This is proved in the next theorem under some conditions on F_{12} and F_{23}.

THEOREM 4.2. *Let F_{123} be defined as in (2.1). Let $\tau_{12}, \tau_{13}, \tau_{23}$ be the values of the Kendall's tau for the three bivariate margins. If C_{13} in (2.1) is the Fréchet upper bound and $F_{3|2}^{-1}(F_{1|2}(y_1|y_2)|y_2)$ is (strictly) increasing in y_2, then $\tau_{13} = 1 - |\tau_{12} - \tau_{23}|$. Similarly, if C_{13} is the Fréchet lower bound and $F_{3|2}^{-1}(1 - F_{1|2}(y_1|y_2)|y_2)$ is (strictly) increasing in y_2, then $\tau_{13} = -1 + |\tau_{12} + \tau_{23}|$.*

PROOF. Let (X_{i1}, X_{i2}, X_{i3}), $i = 1,2$, be independent random vectors from the distribution F_{123}. The proof in Theorem 4.1 for (4.1) is based on

$$\begin{aligned}\max\{0, \Pr(E_1) + \Pr(E_2) - 1\} &+ \max\{0, \Pr(E_1^c) + \Pr(E_2^c) - 1\} \\ &\le \Pr(E_1 \cap E_2) + \Pr(E_i^c \cap E_2^c) \\ &\le \min\{\Pr(E_1), \Pr(E_2)\} + \min\{\Pr(E_1^c), \Pr(E_2^c)\},\end{aligned} \tag{4.2}$$

where the events E_1, E_2 are $\{(X_{11} - X_{21})(X_{12} - X_{22}) > 0\}$ and $\{(X_{13} - X_{23})(X_{12} - X_{22}) > 0\}$. The upper bound in (4.2) is attained if $E_1 \subset E_2$ or $E_2 \subset E_1$. The lower bound in (4.2) is attained if $E_1 \subset E_2^c$ or $E_2 \subset E_1^c$ or $E_1^c \subset E_2$ or $E_2^c \subset E_1$ [equivalently, $E_1 \cap E_2 = \varphi$ or $E_1^c \cap E_2^c = \varphi$].

For C_{13} being the Fréchet upper and lower bound respectively, (2.1) becomes $F_U(y_1, y_2, y_3) = \int_{-\infty}^{y_2} \min\{F_{1|2}(y_1|z), F_{3|2}(y_3|z)\}F_2(dz)$ and $F_L(y_1, y_2, y_3) = \int_{-\infty}^{y_2} \max\{F_{1|2}(y_1|z) + F_{3|2}(y_3|z) - 1, 0\}F_2(dz)$. For F_U, representations for the two vectors are $X_{13} = r(X_{11}, X_{12})$ and $X_{23} = r(X_{21}, X_{22})$ where $r(x_1, x_2) = F_{3|2}^{-1}(F_{1|2}(x_1|x_2)|x_2)$. The function r is increasing in x_1, and if r is also increasing in x_2, then $(X_{11} - X_{21})(X_{12} - X_{22}) > 0$ implies $(X_{13} - X_{23})(X_{12} - X_{22}) > 0$ or $E_1 \subset E_2$, and the upper bound in (4.1) is attained.

The condition on r is an ordering on F_{12}, F_{32} that is studied in detail in Fang and Joe (1992). For F_L, representations are $X_{13} = s(X_{11}, X_{12})$ and $X_{23} = s(X_{21}, X_{22})$ where $s(x_1, x_2) = F_{3|2}^{-1}(1 - F_{1|2}(x_1|x_2)|x_2)$. If s is increasing in x_2, then $(X_{11}-X_{21})(X_{12}-X_{22}) < 0$ implies $(X_{13}-X_{23})(X_{12}-X_{22}) > 0$. That is, $E_1^c \subset E_2$, and hence the lower bound in (4.2) is attained. A sufficient condition for s to satisfy the given condition is that both $F_{1|2}(\cdot|y)$ and $F_{3|2}(\cdot|y)$ are stochastically increasing in y (or cdfs decreasing in y). More generally, the condition on s is equivalent to an ordering on F_{12}^* and F_{32}, where $F_{12}^*(x, y) = F_2(y) - F_{12}(F_1^{-1}(1 - F_1(x)), y)$. ∎

Theorem 4.2 applies to the families of bivariate copulas in Examples 2 and 5, using results in Fang and Joe (1992).

Now, we go on to discussion of δ_{jk} which is relevant only if all of the bivariate margins F_{jk} of (2.1) are positively dependent. (If, for example, $F_{12}(u, v) \le uv$ for all u, v near 1, then $\overline{F}_{12}(u, u)/(1-u) \le (1-u)^2/(1-u) \to 0$ as $u \to 1$; that is, the tail dependence parameter δ is 0.) From Section 3, we can assume that C_{12}, C_{23} have upper tail dependence so that $\delta_{12}, \delta_{23} > 0$. The value of δ_{13} is then given in Theorem 3.1. Unlike Kendall's tau, it does not appear possible to obtain a closed form sharp bound on the range of δ_{13} given δ_{12}, δ_{23}. Let $\delta_{13}^U(\delta_{12}, \delta_{23})$ and $\delta_{13}^L(\delta_{12}, \delta_{23})$ be the largest and smallest possible values of δ_{13} given δ_{12}, δ_{23}. It is not hard to get ranges that δ_{13}^U and δ_{13}^L must fall into. These are given in the next theorem.

THEOREM 4.3. *Let Φ be the standard normal cumulative distribution function. Then*

$$2[1-\Phi(|\Phi^{-1}(1-0.5\delta_{12})-\Phi^{-1}(1-0.5\delta_{23})|)] \le \delta_{13}^U(\delta_{12}, \delta_{23}) \le 1-|\delta_{12}-\delta_{23}| \quad (4.3)$$

and

$$\max\{0, \delta_{12}+\delta_{23}-1\} \le \delta_{13}^L(\delta_{12}, \delta_{23}) \le 2[1-\Phi(\Phi^{-1}(1-0.5\delta_{12})+\Phi^{-1}(1-0.5\delta_{23}))]. \quad (4.4)$$

PROOF. Let (U_1, U_2, U_3) have a trivariate distribution with uniform univariate margins. Then $\Pr(U_3 > u|U_1 > u) = \Pr(U_3 > u, U_2 > u|U_1 > u) + \Pr(U_3 > u, U_2 \le u|U_1 > u) = \Pr(U_3 > u, U_1 > u|U_2 > u) + \Pr(U_3 > u, U_2 \le u|U_1 > u) \le \min\{\Pr(U_1 > u|U_2 > u), \Pr(U_3 > u|U_2 > u)\} + 1 - \Pr(U_2 > u|U_1 > u)$. By taking limits as $u \to 1$, $\delta_{13} \le \min\{\delta_{12}, \delta_{23}\} + 1 - \delta_{12}$. Similarly by interchanging the subscripts 1 and 3, $\delta_{13} \le \min\{\delta_{12}, \delta_{23}\} + 1 - \delta_{23}$. From these two upper bounds on δ_{13}, $\delta_{13}^U \le 1 - |\delta_{12} - \delta_{23}|$. For the lower bound on δ_{13}, $\Pr(U_3 > u|U_1 > u) = \Pr(U_3 > u, U_1 > u)/\Pr(U_2 > u) \ge \Pr(U_3 > u, U_1 > u|U_2 > u) \ge \max\{0, \Pr(U_3 > u|U_2 > u) + \Pr(U_1 > u|U_2 > u) - 1\}$. By taking a limit as $u \to 1$, the lower bound in (4.4) obtains.

An analysis of the inequalities in the preceding paragraph suggests that they are not tight. Hence we provide a lower bound for δ_{13}^U and an upper bound for δ_{13}^L from a specific trivariate family with bivariate upper tail dependence. This family is the Hüsler-Reiss (1989) family and it will not be repeated here because its form is not simple. (The Hüsler-Reiss family is an extreme value family with a bivariate dependence parameter for each bivariate margin and it is closed under all margins; it obtains as a certain limit of the multivariate normal family and does not fit into the class in this paper.) The bivariate margins have copulas in the family $C(u_1, u_2; \lambda) = \exp\{-(-\log u_2)\Phi(\lambda + 0.5\lambda^{-1}\log[(-\log u_2)/(-\log u_1)]) - (-\log u_1)\Phi(\lambda + 0.5\lambda^{-1}\log[(-\log u_1) /(-\log u_2)])\}$, $\lambda \geq 0$, where Φ is the standard normal distribution function and $\lambda = 0$ corresponds to the Fréchet upper bound and $\lambda \to \infty$ corresponds to independence. The relation between the tail dependence parameter δ and λ is $\lambda = \Phi^{-1}(1 - 0.5\delta)$. For the trivariate family, the constraints on the three parameters $\lambda_{12}, \lambda_{13}, \lambda_{23}$ are $-1 \leq g(\lambda_{12}, \lambda_{13}, \lambda_{23}), g(\lambda_{13}, \lambda_{23}, \lambda_{12}), g(\lambda_{12}, \lambda_{23}, \lambda_{13}) \leq 1$, where $g(a, b, c) = (a^2 + b^2 - c^2)/(2ab)$. Each of these inequalities reduce to $|\lambda_{23} - \lambda_{12}| \leq \lambda_{13} \leq \lambda_{12} + \lambda_{23}$, which then yield the upper limit in (4.3) and the lower limit in (4.4). ∎

REMARKS. Numerical calculations for the families in Example 2 show that the bounds from the Hüsler-Reiss family can be improved (but there is no closed form formula). As an example, we take $\delta_{12} = 0.3, \delta_{23} = 0.7$. For the family (2.3′) with $m = 3$ and (1,2) and (2,3) margins with copulas in the family (2.5), the Fréchet lower and upper bounds for C_{13} lead to $\delta_{13} = 0.150, 0.521$ respectively. The corresponding bounds from the Hüsler-Reiss family are 0.155, 0.515. The corresponding nonparametric bounds from Theorem 4.3 are 0.0, 0.6. The bounds from the first paragraph of the above proof appear to be better as δ_{12}, δ_{23} increase.

Four dimensions and higher. Similar to other problems for multivariate dependence concepts, results get harder to prove in higher dimensions. As in the three-dimensional case, if the family $C(\cdot;\theta)$ has the range of dependence from the Fréchet lower bound to the Fréchet upper bound, then (2.3) has a wide range of dependence structure. However the analysis of this range through τ_{ij} or δ_{ij} is harder in that we cannot get simple bounds that are sharp. One could get inequalities for τ_{jk} in terms of $\tau_{ii'}$ with $j \leq i < i' \leq k$, $(i, i') \neq (j, k)$, but these would not be as easy to picture as in the three dimensional case.

For example, for $m = 4$, using (4.1), bounds for τ_{14} given $\tau_{12}, \tau_{23}, \tau_{34}, \tau_{13}, \tau_{24}$ are

$$-1 + \max\{|\tau_{13} + \tau_{34}|, |\tau_{12} + \tau_{24}|\} \leq \tau_{14} \leq 1 - \max\{|\tau_{13} - \tau_{34}|, |\tau_{12} - \tau_{24}|\}. \quad (4.5)$$

These bounds do not depend on τ_{23}; so far we have not found nonparametric bounds involving τ_{23} that can be better than (4.5). In numerical examples for families in Examples 1,2,5, there is a substantial effect on the bounds for τ_{14} when τ_{23} is varied and the remaining τ's are held fixed. For 4–variate normal distributions, using $\tau = (2\pi)\arcsin(\rho)$ and $-1 \le \rho_{14\cdot 23} \le 1$ yields bounds for τ_{14} given the other τ's. These bounds are not the same as (4.5,) implying that, unlike for $m = 3$, the nonparametric bounds of (4.5) are not sharp for the 4–variate normal family. From numerical simulations for the family in Example 5 (ii), the bounds from the 4–variate normal family are not bounds in general. For example, with $\tau_{12} = \tau_{34} = 0.5$, $\tau_{23} = 0.3$, $\tau_{13} = \tau_{24} = 0.4$, the upper bound for τ_{14} from the 4–variate normal family is 0.854, the upper bound from Example 5 (ii) is 0.862 and (4.5) leads to 0.9.

Similarly, (non–sharp) bounds for δ_{14} given $\delta_{12}, \delta_{23}, \delta_{34}, \delta_{13}, \delta_{24}$, which extend (4.3), (4.4), can be obtained. These are

$$\max\{0, \delta_{12} + \delta_{24} - 1, \delta_{13} + \delta_{34} - 1\} \le \delta_{14} \le \min\{1 - |\delta_{12} - \delta_{24}|, 1 - |\delta_{13} - \delta_{34}|\}. \tag{4.6}$$

As for Kendall's tau, τ_{14}, we have not found improved bounds that make use of the (2,3) margin.

The bounds in (4.5) and (4.6) can be extended to higher dimensions. The hard problem is to get improved bounds that are not so simple.

5. Discussion. We have studied a method of iteratively mixing conditional distributions to get families of multivariate distributions, including the multivariate normal family, with one dependence parameter for each bivariate margins and with some of the parameters having an interpretation for conditional dependence. Appropriate choices of copulas lead to multivariate distributions with bivariate tail dependence, a property that the multivariate normal family does not have. However, in general, the new families do not have all marginal distributions in the same family. Also permutation symmetric copulas do not result as a special case. There are some other possibly undesirable properties but the important property that does hold is the wide range of possible dependence, as studied in Section 4. There are applications where one may need more than $m(m-1)/2$ dependence or multivariate parameters for a m–variate distribution; in these cases, it is important that the parameters are interpretable.

It appears that when studying parametric families of multivariate distributions, as opposed to bivariate distributions, that there must be some unsatisfactory properties. The properties of simplicity in form and breadth of dependence structure are not compatible. For example, the families of multivariate distributions given in Section 4 of Joe (1993) have closed–form cdfs and all bivariate margins in the same parametric family but do not have much breadth of dependence structure. Further research includes finding other methods for constructing parametric families of multivariate distributions that have a wide range of dependence. For some applications involving (ordinal) categorical variables, it would be desirable to have a multivariate family with closed form cdf's and a wide range of dependence.

Appendix

1. Proof of Multivariate Normal Result in Example 1: Starting with $F_{j,j+1}$ bivariate normal, we show that if $F_{j,\cdots,j+m-2}$ $(m > 2)$ are $(m-1)$–dimensional multivariate normal, then $F_{j,\cdots,j+m-1}$ are m–dimensional multivariate normal. It suffices to show that $F_{1,\cdots,m}$ in (2.3) is multivariate normal assuming that $F_{1,\cdots,m-1}$ and $F_{2,\cdots,m}$ are multivariate normal, for $m \geq 3$.

Let Φ_Ω, ϕ_Ω respectively denote the multivariate normal cdf and pdf with zero mean vector and covariance matrix Ω. Let

$$R = \begin{bmatrix} 1 & \rho_{1m} \\ \rho_{1m} & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \Sigma_{22} & \Sigma_{2m} \\ \Sigma_{m2} & 1 \end{bmatrix}$$

be the covariance matrices associated with C_{1m}, $F_{1,\cdots,m-1}$ and $F_{2,\cdots,m}$ respectively. Also let $a_{11} = [1 - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}]^{1/2}$, $a_{mm} = [1 - \Sigma_{m2}\Sigma_{22}^{-1}\Sigma_{2m}]^{1/2}$ and $\zeta_2 = (z_2, \cdots, z_{m-1})'$, $\zeta = (z_1, \cdots, z_m)'$. With bivariate normal copulas and univariate standard normal margins, (2.3) simplifies to

$$\int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_{m-1}} \Phi_R\left(\frac{x_1 - \Sigma_{12}\zeta_2}{a_{11}}, \frac{x_m - \Sigma_{m2}\zeta_2}{a_{mm}}\right) \phi_{\Sigma_{22}}(\zeta_2) d\zeta_2. \tag{A1}$$

Writing Φ_R as an integral, (A1) becomes

$$(a_{11}a_{mm})^{-1} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_m} \phi_R\left(\frac{z_1 - \Sigma_{12}\zeta_2}{a_{11}}, \frac{z_m - \Sigma_{m2}\zeta_2}{a_{mm}}\right) \phi_{\Sigma_{22}}(\zeta_2) d\zeta. \tag{A2}$$

Clearly, the integrand of (A2) is a constant multipied by the exponential of a quadratic form in $z_1, \cdots, z_m$, so that (A2) corresponds to a m–dimensional multivariate normal cdf. Let the covariance matrix of the resulting multivariate normal distribution be denoted by

$$\begin{bmatrix} 1 & \Sigma_{12} & \sigma_{1m} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{2m} \\ \sigma_{1m} & \Sigma_{m2} & 1 \end{bmatrix}.$$

The squared reciprocal in (A2) of $(2\pi)^{m/2}$ times the constant is $|\Sigma_{22}|(1 -$

$\rho_{1m}^2)a_{11}^2a_{mm}^2$; it is also equal to

$$|\Sigma_{22}| \cdot | \begin{bmatrix} 1 & \sigma_{1m} \\ \sigma_{m1} & 1 \end{bmatrix} - \begin{bmatrix} \Sigma_{12} \\ \Sigma_{m2} \end{bmatrix} \Sigma_{22}^{-1} [\, \Sigma_{21} \quad \Sigma_{2m} \,] |.$$

Hence $(1-\rho_{1m}^2)a_{11}^2a_{mm}^2 = a_{11}^2a_{mm}^2 - (\sigma_{1m} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{2m})^2$ or $\rho_{1m}^2 = (\sigma_{1m} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{2m})/[a_{11}a_{mm}])^2$. Since (by Property 4) σ_{1m} must be increasing as ρ_{1m} increases, $\rho_{1m} = (\sigma_{1m} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{2m})/[a_{11}a_{mm}]$, which is the partial correlation of the variables 1 and m given variables $2, \cdots, m-1$.

2. Proof of Result in Example 4 for Morgenstern's Copula. It suffices to prove the result for the (1,3) and (1,4) margin. The general case obtains by changing indices and using induction. The conditional distributions are $F_{j|2}(x_j|x_2) = x_j + \theta_{j2}x_j(1-x_j)(1-2x_2)$, $j = 1, 3$, etc. Hence

$$F_{13}(x_1, x_3) = \int_0^1 F_{1|2}(x_1|z_2)F_{3|2}(x_3|z_2)dz_2 = x_1x_3 + \theta_{12}\theta_{23}x_1(1-x_1)x_3(1-x_3)/3$$

by direct calculation and $\alpha_{13} = \theta_{12}\theta_{23}/3$. Next with $\alpha_{23} = \theta_{23}$,

$$\begin{aligned} F_{14}(x_1, x_4) &= \int_0^1 \int_0^1 F_{1|23}(x_1|z_2, z_3)F_{4|23}(x_4|z_2, z_3)f_{23}(z_2, z_3)dz_2dz_3 \\ &= \int_0^1 \int_0^1 F_{1|2}(x_1|z_2)F_{4|3}(x_4|z_3)f_{23}(z_2, z_3)dz_2dz_3 \\ &= x_1x_4 + \theta_{12}\theta_{34}\alpha_{23}x_1(1-x_1)x_4(1-x_4)/9, \end{aligned}$$

and $\alpha_{14} = \theta_{12}\theta_{34}\alpha_{23}/9$. ∎

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DEPARTMENT OF STATISTICS
UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, BC, CANADA V6T 1Z2
harry@stat.ubc.ca