Chapter 7

Stochastic models of environmental pollution

7.1 Introduction

Stochastic partial differential equations (SPDE's) arise from attempts to introduce randomness in a meaningful way into the study of phenomena hitherto regarded as deterministic. As examples one may cite recent research in chemical reaction-diffusions, neurophysiology or turbulence. In almost all cases, one takes as the starting point, the partial differential equations (PDE's) provided by the deterministic theories. The following PDE (given here in a somewhat simplified form) has been used in a deterministic study of pollution or water quality in a basin or reservoir:

$$D\Delta\phi - \left(V_1\frac{\partial\phi}{\partial x_1} + V_2\frac{\partial\phi}{\partial x_2}\right) - K\phi + Q = 0 \tag{7.1.1}$$

with non-conductive boundaries. Here Δ is the Laplacian operator in a bounded domain in $\mathbf{R}^2$, $\phi(x_1, x_2) \geq 0$ is the water quality or chemical concentration at the point (x_1, x_2) in the basin, D is the diffusion coefficient, V_j is the convective velocity in the x_j direction, K is the heat transfer coefficient and $Q \geq 0$ is the "load" pollutant issued from waste outfall.

The above equation is taken from a paper of T. Futagami, N. Tamai and M. Yatsuzuka [54] in which numerical methods for its solution are studied in detail. A PDE that corresponds to a transient or dynamic version of this model will be given in the next section (Eq. (7.2.1)).

Another model is the following river pollution model proposed by Kwakernaak [36] and studied by Curtain [6]: Suppose that the number of deposits in a section of the river of infinitesimal length dx (x being the distance coordinate along the river) behaves according to a Poisson process with rate parameter $\lambda(x)dx$ where $\lambda(x)$ is a given function. The number of deposits in non-overlapping sections are assumed to be independent processes and the amounts of chemical pollutant deposited each time at location x are independent random variables with given distribution F_x. The time evolution of the concentration of the chemical at a location x and at time t, $u(t,x)$ is supposed to be described by

$$\frac{\partial}{\partial t}u(t,x) = D\frac{\partial^2}{\partial x^2}u(t,x) - V\frac{\partial}{\partial x}u(t,x) + \xi(t,x) \tag{7.1.2}$$

where D is the dispersion coefficient, V is the water velocity and $\xi(t,x)$ is the (random) rate of increase of the concentration at (t,x) due to the deposits of chemical waste described above.

We consider the pollution in a domain $\mathcal{X} = [0,\ell]^d$ where the case $d = 1$ is the river pollution problem and the case $d > 1$ can be applied to problems of atmospheric pollution.

This chapter consists of our work in [30] and is organized as follows:

In Section 2, we first consider the pollution process on $\mathcal{X} = [0,\ell]^d$ as a distribution valued process u_t. Then, we show the existence of the density function $u(t,x)$ so that the pollution process can be studied as a random field. Finally, we point out the difference between the case $d = 1$ and $d > 1$, i.e. the second moment of the L^2-norm of $u(t,\cdot)$ is finite for $d = 1$ and, usually, infinite for $d > 1$.

In Section 3, we consider a more realistic model for the river pollution problem. For this model, we calculate the covariance structure of the random field $u(t,x)$ and study the limiting behavior as t tends to infinity. A relationship between the deterministic PDE (7.1.1) and the stochastic evolution equation (7.1.2) is also obtained when the leakage rate of the pollution process is strictly positive.

In Section 4, we study the river pollution model in which a tolerance level is imposed. In contrast to the models of Sections 2 and 3, this model leads to a nonlinear (or quasilinear) equation. For this model, we consider the smoothness and boundedness of the random field $u(t,x)$.

As mentioned at the end of Theorem 7.2.1, when $d > 1$, the basic result on the existence and uniqueness of solution (Theorem 7.2.2) is obtained as an application of the general theory of nuclear space valued SDE's studied in Chapter 6. We have to rely on this theory also for the quasilinear SDE treated in Section 4.

We consider the following filtering problem in Section 5: Suppose we have k stations to measure the water pollution and the observations are given by

$$Y_t^i = \int_0^t \frac{1}{2\epsilon}\int_{x_i-\epsilon_i}^{x_i+\epsilon_i} u(s,x)dxds + B_t^i, i = 1,\cdots,k$$

where $x_1,\cdots,x_k$ are the stations and $B_t^1,\cdots,B_t^k$ are independent Brownian motions.

7.2 Environmental pollution model with Poisson deposits

In this section, we first introduce the stochastic model of chemical concentration in a region assumed for simplicity to be $\mathcal{X} = [0, \ell]^d$ due to dispersion, drift, leakage of the environment and random deposits of chemicals. Then, we solve the stochastic partial differential equation (SPDE) model for $d = 1$ by making use of the Green's function. This method does not apply to the case $d > 1$, so we introduce a nuclear space Φ and consider the model as a SDE on the dual space Φ' of Φ. After the existence and uniqueness of solution of the Φ'-valued SDE are established by making use of the results of Chapter 6, we show that the solution actually lies in a suitable L^2 space by verifying Sazonov's theorem in this setup.

In the absence of random deposits, the chemical concentration $u(t, x)$ at time t and location x should satisfy the following partial differential equation

$$\frac{\partial}{\partial t}u(t,x) = Lu(t,x) - \alpha u(t,x) \tag{7.2.1}$$

with Neumann boundary conditions, where $L = D\Delta + V \cdot \nabla$, D is the dispersion coefficient, $V = (V_1, \cdots, V_d)$ is the drift vector, α is the leakage rate, $\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2}$, $\nabla = (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d})$ and $V \cdot \nabla$ denotes the inner product of V and ∇.

The chemicals are deposited at sites in $\mathcal{X}$ at random times $\tau_1(\omega) < \tau_2(\omega) < \cdots$ and locations $\kappa_1(\omega), \kappa_2(\omega), \cdots$ with positive random magnitudes $A_1(\omega), A_2(\omega), \cdots$. Taking these random deposits into account, (7.2.1) can be written formally as the SPDE

$$\frac{\partial}{\partial t}u(t,x,\omega) = Lu(t,x,\omega) - \alpha u(t,x,\omega) + \sum_j A_j(\omega)\delta_{\kappa_j(\omega)}(x)1_{\tau_j(\omega)=t} \tag{7.2.2}$$

with Neumann boundary conditions where δ_x is the Dirac measure at x.

For $A \subset \mathcal{X}$ and $B \subset \mathbf{R}_+$, let

$$N([0,t] \times A \times B) = \sum_{j:\tau_j \leq t} 1_B(A_j(\omega))1_A(\kappa_j(\omega)).$$

We make the further assumption that $\tau_1, \tau_2, \cdots$, are the jump times of a Poisson process and that $(\kappa_j, A_j), j = 1, 2, \cdots$ are *i.i.d.* random variables so that N is a Poisson random measure on $\mathbf{R}_+ \times \mathcal{X} \times \mathbf{R}_+$ with characteristic measure μ (In Kwakernaak's model $\mu(dxda) = \lambda(x)H_x(da)dx)$ on $\mathcal{X} \times \mathbf{R}_+$.

For $d = 1$, the SPDE (7.2.2) has the following meaning as an integral equation: For any continuous f on $[0, \ell]$,

$$\int_0^\ell u(t,x)f(x)dx - \int_0^\ell u(0,x)f(x)dx$$

$$= \int_0^t \int_0^\ell (D\Delta u(s,x) - (V\cdot\nabla)u(s,x) - \alpha u(s,x))f(x)dxds$$
$$+ \int_0^t \int_{\mathcal{X}} \int_0^\infty af(x)N(dsdxda). \tag{7.2.3}$$

The following theorem solves the SPDE (7.2.3) when $d = 1$.

Theorem 7.2.1 *Suppose that*

$$\int_0^\ell \int_0^\infty a\mu(dxda) < \infty,$$

then the SPDE (7.2.3) has a unique solution given by

$$u(t,x) = e^{-\alpha t}\int_0^\ell u_0(y)p(t,x,y)dy + \int_0^t \int_0^\ell \int_0^\infty ae^{-\alpha(t-s)}p(t-s,x,y)N(dsdyda) \tag{7.2.4}$$

where $p(t,x,y)$ is the Green's function of the operator $L = D\frac{\partial^2}{\partial x^2} - V\frac{\partial}{\partial x}$ with Neumann boundary conditions ($\phi'(0) = \phi'(\ell) = 0$) given by

$$p(t,x,y) = \sum_{j=0}^\infty e^{-\lambda_j^1 t}\phi_j^1(x)\phi_j^1(y)$$

and

$$\lambda_0^1 = 0, \quad \lambda_j^1 = D\left(c_1^2 + \left(\frac{j\pi}{\ell}\right)^2\right); \tag{7.2.5}$$

$$\phi_0^1(x) = \sqrt{\frac{2c_1}{1-e^{-2c_1\ell}}}, \quad \phi_j^1(x) = \sqrt{\frac{2}{\ell}}e^{c_1 x}\sin\left(\frac{j\pi}{\ell}x + \alpha_j^1\right);$$

$$\alpha_j^1 = tan^{-1}(-\frac{j\pi}{\ell c_1}), \quad c_1 = \frac{V_1}{2D}, \quad j = 1,2,\cdots.$$

In this case, $u(t,x)$ is a random field defined for each (t,x).

Proof: If $\alpha > 0$, then

$$\int_0^t \int_0^\ell \int_0^\infty ae^{-\alpha(t-s)}p(t-s,x,y)\mu(dyda)ds \tag{7.2.6}$$
$$\le \sum_{j=0}^\infty \int_0^t \int_0^\ell \int_0^\infty ae^{-\alpha(t-s)}e^{-\lambda_j^1(t-s)}|\phi_j^1(x)||\phi_j^1(y)|\mu(dyda)ds$$
$$\le \sum_{j=0}^\infty \frac{1}{\alpha+\lambda_j^1}\int_0^\ell \int_0^\infty a\mu(dyda)\sup\{|\phi_j^1(x)|^2 : j\ge 0\ x\in[0,\ell]\} < \infty.$$

For $\alpha \le 0$, since only finite many terms of $\alpha + \lambda_j^1$ can be non-positive, the conclusion in (7.2.6) still holds. Hence $u(t,x)$ in (7.2.4) is well-defined.

The SPDE (7.2.3) has at most one solution as can be seen as follows: Let $u_i(t,x)$, $i = 1,2$ be two solutions of (7.2.3) with the same initial condition. Writing $\tilde{u}(t,x) = u_1(t,x) - u_2(t,x)$ we have

$$\int_0^\ell \tilde{u}(t,x)f(x)dx = \int_0^t \int_0^\ell (D\Delta\tilde{u}(s,x) - (V\cdot\nabla)\tilde{u}(s,x) - \alpha\tilde{u}(s,x))f(x)dxds$$

which yields the PDE

$$\frac{\partial}{\partial t}\tilde{u}(t,x) = D\Delta\tilde{u}(t,x) - (V\cdot\nabla)\tilde{u}(t,x,\omega) - \alpha\tilde{u}(t,x,\omega),\ t>0, 0<x<\ell$$

$$\tilde{u}(0,x) = 0$$

and

$$\frac{\partial}{\partial x}\tilde{u}(t,0) = \frac{\partial}{\partial t}\tilde{u}(t,\ell) = 0,\ t>0.$$

The above Neumann problem has the unique solution $\tilde{u} = 0$ thus proving the assertion.

By direct verification, $u(t,x)$ given by (7.2.4) solves the SPDE (7.2.3). ∎

The method used in the above theorem is not applicable to the case of $d > 1$ since the series corresponding to (7.2.6) does not converge. To avoid such convergence difficulties, we formally regard $u(t,x)$ as an infinite dimensional process u_t determined by its action on all "smooth" functions ϕ in the sense

$$u_t[\phi] = \int_{\mathcal{X}} u(t,x)\phi(x)\rho(x)dx \tag{7.2.7}$$

where $\rho(x)dx$ is appropriately chosen such that the operator L on $H = L^2(\mathcal{X}, \rho(x)dx)$ is positive definite and self-adjoint. The appropriate form of (7.2.3) is now the integral equation

$$u_t[\phi] = u_0[\phi] + \int_0^t (u_s[-L\phi] - \alpha u_s[\phi])ds + \int_0^t \int_{\mathcal{X}} \int_0^\infty a\phi(x)\rho(x)N(dsdxda) \tag{7.2.8}$$

where u_t is a continuous random linear functional on a space Φ of "smooth" functions. It is convenient in the present context to take Φ to be the CHNS constructed below: Let

$$\rho(x) = \exp\left(-2\sum_{i=1}^d c_i x_i\right),$$

where $c_i = \frac{V_i}{2D}$, $i = 1,\cdots,d$. It is easily verified that the operator L with Neumann boundary conditions on H has eigenvalues and eigenfunctions given by

$$\lambda_{j_1\cdots j_d} = \lambda_{j_1}^1 + \cdots + \lambda_{j_d}^d$$

and

$$\phi_{j_1\cdots j_d}(x_1,\cdots,x_d) = \phi^1_{j_1}(x_1)\cdots\phi^d_{j_d}(x_d)$$

respectively where λ^i_j and $\phi^i_j(x)$ are defined by (7.2.5) with the index 1 replaced by i, for $i = 1,\cdots,d$ and $j = 1,2,\cdots$. Let T_t be the semigroup on H generated by L. For $\phi \in H$ and $r \in \mathbf{R}$, let

$$||\phi||_r^2 = \sum_{j_1\cdots j_d} < \phi, \phi_{j_1\cdots j_d} >^2 (1+\lambda_{j_1\cdots j_d})^{2r}$$

where $< \phi, \psi >$ is the inner product on H. Define

$$\Phi = \{\phi \in H : ||\phi||_r < \infty \ \forall r \in R\}$$

and let Φ_r be the completion of Φ with respect to the norm $||\cdot||_r$. From now on we shall write Φ_0 for H. It is easy to show that the canonical injection from Φ_{r+r_1} to Φ_r is Hilbert-Schmidt if we take $r_1 > \frac{d}{4}$. Hence, Φ is a countably Hilbertian nuclear space. Applying Theorems 6.2.2 and 6.3.1, we have

Theorem 7.2.2 *If there exist r_0 and r_2 such that $E||u_0||^2_{-r_0} < \infty$ and there exists a constant K such that, for any $\phi \in \Phi$,*

$$\left|\int_{\mathcal{X}}\int_0^\infty a\phi(x)\rho(x)\mu(dxda)\right|^2 + \int_{\mathcal{X}}\int_0^\infty |a\phi(x)\rho(x)|^2\mu(dxda) \le K||\phi||^2_{r_2}, \tag{7.2.9}$$

then (7.2.8) has a unique Φ_{-p_1}-valued solution where $p_1 = r_1 + \max(r_1 + r_2, r_0)$.

Proof: Take

$$A(t,v)[\phi] = -v[L\phi] - \alpha v[\phi] + \int_{\mathcal{X}}\int_0^\infty a\phi(x)\rho(x)\mu(dxda)$$

and

$$G(t,v,(x,a))[\phi] = a\phi(x)\rho(x).$$

It is easy to see that (A, G, μ) satisfies the assumptions (I) and (M) with $p_0 = r_1 + r_2$ and $q = p + 1$. Hence, by Theorem 6.2.2 and 6.3.1, (7.2.8) has a unique solution which lies in Φ_{-p_1}. ∎

(7.2.9) is an important sufficient condition in the above theorem. The next theorem gives an equivalent condition for (7.2.9).

Theorem 7.2.3 *(1°) Condition (7.2.9) holds if and only if*

$$\int_{\mathcal{X}}\int_0^\infty a(1+a)\mu(dxda) < \infty. \tag{7.2.10}$$

(2°) Suppose that (7.2.9) holds and let ν_1 and ν_2 be two finite positive measures on $\mathcal{X}$ given by

$$\nu_1(dx) = \rho(x)\int_0^\infty a\mu(dxda) \quad \text{and} \quad \nu_2(dx) = \rho(x)^2\int_0^\infty a^2\mu(dxda).$$

Then $r_2 = 0$ if and only if $\nu_1(dx) \ll dx$, $\nu_2(dx) \ll dx$, $\rho(x)^{-1}\frac{\nu_1(dx)}{dx} \in H$ and $\frac{\nu_2(dx)}{dx}$ is bounded. Here dx is d-dimensional Lebesgue measure ($d \geq 1$).

Proof: (1°) For the "only if" part, (7.2.10) follows from (7.2.9) by letting $\phi = \phi_{0\cdots 0}$ in that inequality. Now, we only need to prove the "if" part. For simplicity of notation, we assume that $d = 1$. Note that (7.2.10) implies that both ν_1 and ν_2 are finite measures. For any $\phi \in \Phi$, we have

$$\begin{aligned}
& \int_0^\ell\int_0^\infty |a\phi(x)\rho(x)|^2\mu(dxda) + \left|\int_0^\ell\int_0^\infty a\phi(x)\rho(x)\mu(dxda)\right|^2 \\
= \; & \int_0^\ell \phi(x)^2 d\nu_2([0,x]) + \left|\int_0^\ell \phi(x)d\nu_1([0,x])\right|^2 \\
= \; & \phi(\ell)^2\nu_2([0,\ell]) - 2\int_0^\ell \phi(x)\phi'(x)\nu_2([0,x])dx \\
& + \left|\phi(\ell)\nu_1([0,\ell]) - \int_0^\ell \phi'(x)\nu_1([0,x])dx\right|^2 \\
\leq \; & \nu_2([0,\ell])\left(\phi(\ell)^2 + \int_0^\ell \phi(x)^2dx + \int_0^\ell \phi'(x)^2dx\right) \\
& + 2(\nu_1([0,\ell]))^2\left(\phi(\ell)^2 + \ell\int_0^\ell \phi'(x)^2dx\right). \qquad (7.2.11)
\end{aligned}$$

Let

$$M = \max\left\{\sqrt{\frac{2c_1}{1-e^{-2c_1\ell}}}, \sqrt{\frac{2}{\ell}}e^{|c_1|\ell}, \sqrt{\frac{2}{\ell D}}e^{|c_1|\ell}\right\}$$

and note that

$$|\phi_j(x)| \leq M, \qquad |\phi_j'(x)| \leq M\sqrt{\lambda_j}.$$

From $\phi(x) = \sum < \phi_j, \phi > \phi_j(x)$, we have

$$\begin{aligned}
& \phi(\ell)^2 + \int_0^\ell \phi(x)^2dx + \int_0^\ell \phi'(x)^2dx \\
\leq \; & M^2\left(\sum_{j=0}^\infty | < \phi, \phi_j > |\right)^2 + M^2\ell\left(\sum_{j=0}^\infty | < \phi, \phi_j > |\right)^2
\end{aligned}$$

$$
\begin{aligned}
&+M^2\ell\left(\sum_{j=1}^{\infty}|<\phi,\phi_j>|\sqrt{\lambda_j}\right)^2 \\
\leq\ & M^2(1+2\ell)\left(\sum_{j=0}^{\infty}|<\phi,\phi_j>|\sqrt{1+\lambda_j}\right)^2 \leq C\|\phi\|_1^2 \qquad (7.2.12)
\end{aligned}
$$

where

$$C = M^2(1+2\ell)\sum_{j=0}^{\infty}(1+\lambda_j)^{-1}.$$

So, (7.2.9) follows from (7.2.11) and (7.2.12).
(2°) The "if" part is straightforward. For the "only if" part, suppose that $r_2 = 0$. Since

$$\int_{\mathcal{X}}\int_0^{\infty}(a\phi(x)\rho(x))^2\,\mu(dxda) \leq K\|\phi\|_0^2$$

we have

$$\int_{\mathcal{X}}\phi(x)^2\nu_2(dx) \leq K\int_{\mathcal{X}}\phi(x)^2\rho(x)dx. \qquad (7.2.13)$$

As Φ is dense in H, (7.2.13) still holds for $\phi \in H$. Hence, for any measurable subset C of $\mathcal{X}$, we have

$$\nu_2(C) \leq K\int_C \rho(x)dx.$$

So, $\nu_2(dx) \ll dx$ and $\frac{\nu_2(dx)}{dx} \leq K\rho(x)$ is bounded. For ν_1, we note that, by assumption,

$$\phi \to \int_{\mathcal{X}}\phi(x)\nu_1(dx)$$

is a linear functional on Φ and continuous with respect to the norm $\|\cdot\|_0$, this implies that there exists $h \in H$ such that

$$\nu_1(dx) = h(x)\rho(x)dx.$$

Our result follows immediately. ∎

As $r_1 > \frac{d}{4} > 0$, Theorem 7.2.2 only tells us that (7.2.8) has a unique solution u_t as a distribution valued process. The next theorem gives a sufficient condition under which the density function $u(t,x)$ exists and the expression (7.2.7) makes sense.

Theorem 7.2.4 *Suppose that $r_0 = r_2 = 0$ and μ is a finite measure on $\mathcal{X} \times \mathbf{R}_+$. Then, for any $t \geq 0$, $u_t \in \Phi_0$ a.s.*

Proof: Let $\gamma_t = e^{-\alpha t}T_t u_0$. It follows from $r_0 = 0$ that γ_t is a Φ_0-valued random variable. It is easy to see that the unique solution u_t of (7.2.8) can be represented as

$$u_t = \gamma_t + M_t$$

where M_t is a Φ'-valued random variable given by

$$M_t[\phi] = \int_0^t \int_{\mathcal{X}} \int_0^\infty a e^{-\alpha(t-s)}(T_{t-s}\phi)(x)\rho(x)N(dsdxda), \ \forall \phi \in \Phi. \quad (7.2.14)$$

It is easy to see that the characteristic function of M_t on Φ is

$$\Psi(\phi) = Ee^{iM_t[\phi]} = \exp\left(\int_0^t \int_{\mathcal{X}} \int_0^\infty \left(e^{iae^{-\alpha s}(T_s\phi)(x)\rho(x)} - 1\right)\mu(dxda)ds\right).$$

Let $\hat{\Psi}$ be a function on Φ_0 defined in the same way as Ψ with ϕ replaced by $h \in \Phi_0$. As $r_2 = 0$, it follows from Theorem 7.2.3 that $\hat{\Psi}$ is well-defined. If we can prove that $\hat{\Psi}$ is a characteristic function on Φ_0, then there exists a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ and a Φ_0-valued random variable $\hat{M}_t$ such that, $\forall \phi \in \Phi$,

$$E\exp(iM_t[\phi]) = E^{\hat{P}}\exp(i\hat{M}_t[\phi]).$$

As Φ'-valued random variables, $\hat{M}_t$ and M_t have the same distribution and hence,

$$P(\omega : M_t \in \Phi_0) = \hat{P}(\hat{\omega} : \hat{M}_t \in \Phi_0) = 1.$$

Therefore we only need to show that $\hat{\Psi}$ is a characteristic function on Φ_0.

It is easy to see that $\hat{\Psi}(0) = 1$ and $\hat{\Psi}$ is positive definite function in Φ_0. By Sazonov's theorem, we only need to prove that $\hat{\Psi}$ is continuous at 0 in the S-topology. Suppose that $\hat{\Psi}$ is not continuous at 0 in the S-topology. Then there exists $\epsilon_0 > 0$ such that, for any nuclear operator S, there exists a $\psi_S \in \Phi_0$ such that $< S\psi_S, \psi_S >_0 < 1$ and $|\hat{\Psi}(\psi_S) - 1| \geq \epsilon_0$.

Consider a sequence of mappings S_n on Φ_0 defined by

$$S_n\psi \equiv \sum_{j_1 \cdots j_d = 1}^{\infty} \frac{n}{2^{j_1 + \cdots + j_d}} < \psi, \phi_{j_1 \cdots j_d} >_0 \phi_{j_1 \cdots j_d}.$$

It is clear that S_n is a self-adjoint nonnegative definite nuclear operator on Φ_0. Therefore for this S_n, we have $\psi_n \in \Phi_0$, such that $< S_n\psi_n, \psi_n >_0 < 1$ and $|\hat{\Psi}(\psi_n) - 1| \geq \epsilon_0$.

As $|\exp(iae^{-\alpha s}(T_s\psi_n)(x)\rho(x)) - 1| \leq 2$ and μ is a finite measure, we see that

$$\lim_{n\to\infty} \log \hat{\Psi}(\psi_n) = \int_0^t \int_{\mathcal{X}} \int_0^\infty \lim_{n\to\infty} \left(e^{iae^{-\alpha s}(T_s\psi_n)(x)\rho(x)} - 1\right)\mu(dxda)ds.$$

But, for $s > 0$, we have

$$\begin{aligned}|(T_s\psi_n)(x)| &\leq \sum_{j_1\cdots j_d} e^{-\lambda_{j_1\cdots j_d}s}\left|\langle\phi_{j_1\cdots j_d},\psi_n\rangle_0\,\phi_{j_1\cdots j_d}(x)\right| \\ &\leq K_1\sum_{j_1\cdots j_d} e^{-\lambda_{j_1\cdots j_d}s}\sqrt{\frac{2^{j_1+\cdots+j_d}}{n}<S\psi_n,\psi_n>_0} \\ &\leq K_1\sum_{j_1\cdots j_d} e^{-\lambda_{j_1\cdots j_d}s}\sqrt{\frac{2^{j_1+\cdots+j_d}}{n}}\to 0,\end{aligned}$$

where

$$K_1=\sup\{|\phi_{j_1\cdots j_d}(x)|: x\in\mathcal{X}, j_1,\cdots,j_d\in\mathbf{N}\}$$

is finite. Hence, $\lim_{n\to\infty}\log\hat{\Psi}(\psi_n)=0$. This contradicts from $|\hat{\Psi}(\psi_n)-1|\geq\epsilon_0$. ∎

Under the conditions of Theorem 7.2.4, we know that u_t is Φ_0-valued and we are interested in its second moment. The following theorem tells us that in most applications the second moment of $||u_t||_0$ is finite if $d=1$ and infinite if $d>1$.

Theorem 7.2.5 *Suppose that $r_0=r_2=0$ and μ is a finite measure on $\mathcal{X}\times\mathbf{R}_+$.*
(1°) If $d=1$, then $E||u_t||_0^2<\infty$ for any t.
(2°) If $d>1$ and there exists a positive constant c such that $\frac{\nu_2(dx)}{dx}\geq c\rho(x)$, then, for any t, we have $E||u_t||_0^2=\infty$.

Proof: (1°) As in the proof of Theorem 7.2.1, we may assume that $\alpha>0$. Let M_t be given by (7.2.14). We only need to prove that $E||M_t||_0^2<\infty$. Note that

$$\begin{aligned}&E(<M_t,\phi_j>_0^2) \\ =&\int_0^t\int_{\mathcal{X}}\int_0^\infty a^2e^{-2(\alpha+\lambda_j)s}\phi_j(x)^2\rho(x)^2\mu(dxda)ds \\ &+\left(\int_0^t\int_{\mathcal{X}}\int_0^\infty ae^{-(\alpha+\lambda_j)s}\phi_j(x)\rho(x)\mu(dxda)ds\right)^2 \\ \leq& K\left(\int_0^t e^{-2(\alpha+\lambda_j)s}ds+\left|\int_0^t e^{-(\alpha+\lambda_j)s}ds\right|^2\right) \\ =& K\left(\frac{1}{2(\lambda_j+\alpha)}+\frac{1}{(\lambda_j+\alpha)^2}\right),\end{aligned}$$

here K is the constant appearing in (7.2.9). As $\lambda_j\approx j^2$, we have

$$E||M_t||_0^2 = E\int_0^\ell M(t,x)^2\rho(x)dx\leq\sum_{j=0}^\infty E(<M_t,\phi_j>_0^2)$$

$$\leq \sum_{j=0}^{\infty}\left(\frac{K}{2(\lambda_j+\alpha)}+\frac{K}{(\lambda_j+\alpha)^2}\right)<\infty.$$

The assertion (2°) of the theorem can be proved similarly. ∎

Example 7.2.1 *In Kwakernaak's model* $\mu(dxda) = \lambda(x)H_x(da)dx$. *Take* $\lambda(x) = 1$ *and* $H_x(da) = H(da)$, *then*

$$\frac{\nu_2(dx)}{dx} = \rho(x)^2\int_0^\infty a^2H(da) \geq c\rho(x)$$

and hence $E\|u_t\|_0^2 = \infty$ *in this case.*

7.3 Pollution emissions at different sites along a river

In this section, we consider a more realistic model for concentration of undesired chemicals which are released by different factories along the river. Suppose that there are r such factories located at different sites $\kappa_1,\cdots,\kappa_r \in [0,\ell]$ where the interval $[0,\ell]$ represents the river. Each of the factories deposits chemicals in terms of independent Poisson streams $N_1(t),\cdots,N_r(t)$ with random magnitudes $\{A_i^j(\omega),\ j=1,2,\cdots\}$ which are independent and have common distribution $F_i(da)$. The chemicals deposited are uniformly distributed in $(\kappa_i-\epsilon_i,\kappa_i+\epsilon_i)$, $i=1,\cdots,r$.

The model described above can be written mathematically as follows

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) &= D\frac{\partial^2}{\partial x^2}u(t,x) - V\frac{\partial}{\partial x}u(t,x) - \alpha u(t,x)\\ &\quad+\sum_{i=1}^{r}\sum_{j}\frac{A_i^j(\omega)}{2\epsilon_i}1_{(\kappa_i-\epsilon_i,\kappa_i+\epsilon_i)}(x)1_{t=\tau_i^j(\omega)} \qquad (7.3.1)\end{aligned}$$

where $\{\tau_i^j(\omega),\ j=1,2,\cdots\}$ are the jump times of $N_i(t)$. This model can be regarded as a special case of the SDE studied in Chapter 6 and be solved by applying Theorems 6.2.2 and 6.3.1 if we make the following definitions. Let Φ be defined as in the last section with $d=1$. Define a Poisson random measure N on space $\mathbf{R}_+\times[0,\ell]\times\mathbf{R}_+$ by

$$N([0,t]\times A\times B) = \sum_{i=1}^{r}1_A(\kappa_i)\sum_{j=1}^{N_i(t)}1_B(A_i^j(\omega))$$

for any $A\subset[0,\ell]$ and $B\subset\mathbf{R}_+$. Also define two measurable maps A : $\mathbf{R}_+\times\Phi'\to\Phi'$ and $G:\mathbf{R}_+\times\Phi'\times[0,\ell]\times\mathbf{R}_+\to\Phi'$ by

$$A(t,v)[\phi] = -v[L\phi] - \alpha v[\phi] + \sum_{i=1}^{r}\frac{a_i f_i}{2\epsilon_i}\int_{\kappa_i-\epsilon_i}^{\kappa_i+\epsilon_i}\phi(y)\rho(y)dy \qquad (7.3.2)$$

$$G(x,a)[\phi] = \begin{cases} \frac{a}{2\epsilon_i}\int_{\kappa_i-\epsilon_i}^{\kappa_i+\epsilon_i}\phi(y)\rho(y)dy & x=\kappa_i,\ i=1,\cdots,r \\ 0 & \text{otherwise} \end{cases}$$

where $a_i = \int_{\mathbf{R}_+} aF_i(da)$ and f_i is the parameter of the Poisson process $N_i(t)$.

It is easy to see that the characteristic measure of N is

$$\mu(A\times B) = \sum_{i=1}^{r} 1_A(\kappa_i) f_i F_j(B) \tag{7.3.3}$$

for any $A \subset [0,\ell]$ and $B \subset \mathbf{R}_+$ and (A, G, μ) given by (7.3.2) and (7.3.3) satisfies the assumptions (I) and (M) of Chapter 6. Hence, (7.3.1) has a unique Φ'-valued solution.

Because of the practical importance of the model (7.3.1), we study its various properties. First of all, we consider the regularity of the solution of (7.3.1) and calculate its mean and covariance.

Theorem 7.3.1 *$u_\cdot \in D([0,T],\Phi_0)$ and, for any $t' \geq t \geq 0$, $x, y \in [0,\ell]$,*

$$\begin{aligned} Eu(t,x) &= \sum_j e^{-(\alpha+\lambda_j)t}\phi_j(x)Eu_0[\phi_j] \\ &\quad + \sum_{i=1}^{r}\sum_j f_i a_i \frac{1-e^{-(\alpha+\lambda_j)t}}{\lambda_j+\alpha}\psi_j(\kappa_i,\epsilon_i)\phi_j(x), \end{aligned} \tag{7.3.4}$$

and

$$\begin{aligned} &Cov(u(t,x), u(t',y)) \\ &= \sum_{j,k} e^{-(\alpha+\lambda_j)t-(\alpha+\lambda_k)t'}\phi_j(x)\phi_k(y) Cov(u_0[\phi_j], u_0[\phi_k]) \\ &\quad + \sum_{j,k}\sum_{i=1}^{r} f_i b_i \frac{1-e^{-(2\alpha+\lambda_j+\lambda_k)t}}{\lambda_j+\lambda_k+2\alpha}\phi_j(x)\phi_k(y)\psi_j(\kappa_i,\epsilon_i)\psi_k(\kappa_i,\epsilon_i), \end{aligned} \tag{7.3.5}$$

where

$$\psi_j(\kappa_i,\epsilon_i) = \frac{1}{2\epsilon_i}\int_{\kappa_i-\epsilon_i}^{\kappa_i+\epsilon_i}\phi_j(y)\rho(y)dy \quad \text{and} \quad b_i = \int a^2 F_i(da). \tag{7.3.6}$$

Proof: It follows from (7.3.1), (7.3.2) and (7.3.6) that

$$\begin{aligned} u_t[\phi_j] &= u_0[\phi_j] - (\alpha+\lambda_j)\int_0^t u_s[\phi_j]ds \\ &\quad + \int_0^t\int_{\mathcal{X}}\int_0^\infty G(x,a)[\phi_j]N(dsdxda) \\ &= u_0[\phi_j] - \int_0^t\left((\alpha+\lambda_j)u_s[\phi_j] - \sum_{i=1}^{r} f_i a_i \psi_j(\kappa_i,\epsilon_i)\right)ds \\ &\quad + \int_0^t\int_{\mathcal{X}}\int_0^\infty G(x,a)[\phi_j]\tilde{N}(dsdxda). \end{aligned} \tag{7.3.7}$$

Making use of Itô's formula, we have

$$\begin{aligned} & u_t[\phi_j]^2 - u_0[\phi_j]^2 \qquad (7.3.8) \\ = & \int_0^t \left(-2(\alpha+\lambda_j)u_s[\phi_j]^2 + 2u_s[\phi_j]\sum_{i=1}^r f_i a_i \psi_j(\kappa_i,\epsilon_i) \right. \\ & \left. + \sum_{i=1}^r f_i b_i \psi_j(\kappa_i,\epsilon_i)^2 \right) ds \\ & + \int_0^t \int_{\mathcal{X}} \int_0^\infty \left((u_{s-}[\phi_j] + G(x,a)[\phi_j])^2 - u_{s-}[\phi_j]^2 \right) \tilde{N}(dsdxda). \end{aligned}$$

Hence

$$\begin{aligned} & Eu_t[\phi_j]^2 - Eu_0[\phi_j]^2 \\ \le & (2|\alpha|+1)\int_0^t Eu_s[\phi_j]^2 ds + t\sum_{i=1}^r (ra_i^2 f_i^2 + b_i f_i)\psi_j(\kappa_i,\epsilon_i)^2. \end{aligned}$$

It follows from Gronwall's inequality that, for any $t \le T$

$$Eu_t[\phi_j]^2 \le \left(Eu_0[\phi_j]^2 + T\sum_{i=1}^r (ra_i^2 f_i^2 + b_i f_i)\psi_j(\kappa_i,\epsilon_i)^2 \right) \exp((2|\alpha|+1)T).$$

Hence

$$\sum_{j=0}^\infty Eu_t[\phi_j]^2 \le \left(E\|u_0\|^2 + T\sum_{i=1}^r (ra_i^2 f_i^2 + b_i f_i) d_i \right) \exp((2|\alpha|+1)T) < \infty \qquad (7.3.9)$$

where

$$d_i = \sum_{j=0}^\infty \psi_j(\kappa_i,\epsilon_i)^2 = \left\| \frac{1}{2\epsilon_i} 1_{(\kappa_i-\epsilon_i,\kappa_i+\epsilon_i)} \right\|_0^2.$$

Note that

$$\begin{aligned} & \sum_{j=0}^\infty E\sup_{t\le T} \int_0^t \int_{\mathcal{X}} \int_0^\infty \left((u_{s-}[\phi_j] + G(x,a)[\phi_j])^2 - u_{s-}[\phi_j]^2 \right) \tilde{N}(dsdxda) \\ \le & \sum_{j=0}^\infty E\int_0^T \int_{\mathcal{X}} \int_0^\infty \left(2G(x,a)[\phi_j]^2 + u_{s-}[\phi_j]^2 \right) (N(dsdxda) + \mu(dxda)ds) \\ = & \sum_{j=0}^\infty 2\int_0^T \sum_{i=1}^r f_i \int_0^\infty \left(2a^2\psi_j(\kappa_i,\epsilon_i)^2 + Eu_s[\phi_j]^2 \right) F_i(da) ds \\ = & 4T\sum_{i=1}^r f_i b_i d_i + 2\sum_{i=1}^r f_i \int_0^T \sum_{j=0}^\infty Eu_s[\phi_j]^2 ds < \infty. \qquad (7.3.10) \end{aligned}$$

It follows from (7.3.8)-(7.3.10) that

$$\sum_{j=0}^{\infty} E \sup_{t\leq T} u_t[\phi_j]^2 < \infty. \tag{7.3.11}$$

As $u.[\phi_j] \in D([0,T],\mathbf{R})$ for each j, (7.3.11) implies that $u. \in D([0,T],\Phi_0)$. The derivations of (7.3.4) and (7.3.5) are routine and we leave them to the reader. ∎

The following theorem studies the limiting behavior of the solution u_t of (7.3.1) as t tends to infinity.

Theorem 7.3.2 *Suppose $\alpha = 0$, then, as t tends to infinity*
(1°)

$$\frac{1}{t}u_t \to \sum_{i=1}^{r} f_i a_i \psi_0(\kappa_i,\epsilon_i)\sqrt{\frac{2c}{1-e^{-2c\ell}}}\phi_0, \quad a.s. \text{ in } \Phi_0.$$

(2°) Let

$$\gamma_t = \frac{1}{\sqrt{t}}\left(u_t - t\sum_{i=1}^{r} a_i f_i \psi_0(\kappa_i,\epsilon_i)\sqrt{\frac{2c}{1-e^{-2c\ell}}}\phi_0\right).$$

Then, as Φ_0-valued random variables, γ_t converges in distribution to $\xi\phi_0$, where ξ is a real-valued Gaussian random variable with mean 0 and variance $\sum_{j=1}^{r} f_j b_j \psi_0(\kappa_j,\epsilon_j)\frac{2c}{1-e^{-2c\ell}}$.

Proof: (1°) By (7.3.7) and $u_t \in \Phi_0$, we have

$$u_t = T_t u_0 + \int_0^t \int_{\mathcal{X}} \int_0^\infty G(x,a)[\phi_0]N(dsdxda)\phi_0 + \xi_t \tag{7.3.12}$$

where

$$\xi_t = \sum_{j=1}^{\infty} \int_0^t \int_{\mathcal{X}} \int_0^\infty G(x,a)[\phi_j]e^{\lambda_j s}N(dsdxda)e^{-\lambda_j t}\phi_j.$$

Note that

$$\left\|\frac{1}{t}T_t u_0\right\|_0 \leq \left\|\frac{1}{t}u_0\right\|_0 \to 0, \ a.s.$$

As $N_i(t)/t \to f_i$ almost surely, we have

$$\begin{aligned}
&\frac{1}{t}\int_0^t \int_{\mathcal{X}} \int_0^\infty G(x,a)[\phi_0]N(dsdxda)\phi_0 \\
=\ & \sum_{i=1}^{r} \frac{N_i(t)}{t}\frac{1}{N_i(t)}\sum_{j=1}^{N_i(t)} A_i^j(\omega)\psi_0(\kappa_i,\epsilon_i)\sqrt{\frac{2c}{1-e^{-2c\ell}}}\phi_0 \\
\to\ & \sum_{i=1}^{r} f_i a_i \psi_0(\kappa_i,\epsilon_i)\sqrt{\frac{2c}{1-e^{-2c\ell}}}\phi_0, \ a.s.
\end{aligned}$$

We only need to prove that $\frac{1}{t}\xi_t$ tends to 0 a.s. Without loss of generality, we assume that $r = 1$. Then

$$\xi_t = \sum_{j\geq 1}\sum_{k=1}^{N_i(t)} A_1^k(\omega)\psi_j(\kappa_1,\epsilon_1)e^{\lambda_j(\tau_1^k(\omega)-t)}\phi_j.$$

Let $\eta_n = \tau_1^n(\omega)-\tau_1^{n-1}(\omega)$. Then $\{\eta_n\}$ is a sequence of *i.i.d.* random variables with exponential distribution with parameter f_1. Note that, $\forall k > 0$

$$\begin{aligned}
E\|\xi_{\tau_1^k(\omega)}\|_0^2 &= E\sum_{j\geq 1}\psi_j(\kappa_1,\epsilon_1)^2\left(\sum_{n=1}^k A_1^n(\omega)\exp\left(-\lambda_j\sum_{i=n+1}^k \eta_i\right)\right)^2 \\
&= \sum_{j\geq 1}\psi_j(\kappa_1,\epsilon_1)^2\left(2\sum_{k_1=1}^k\sum_{k_2=1}^{k_1-1} a_1^2\left(\frac{f_1}{f_1+\lambda_j}\right)^{k_1-k_2}\left(\frac{f_1}{f_1+2\lambda_j}\right)^{k-k_1}\right. \\
&\qquad\left. + \sum_{k_1=1}^k b_1\left(\frac{f_1}{f_1+2\lambda_j}\right)^{k-k_1}\right) \\
&\leq \sum_{j\geq 1}\psi_j(\kappa_1,\epsilon_1)^2\left(2a_1^2\frac{f_1}{\lambda_j}+b_1\right)\frac{f_1+2\lambda_j}{2\lambda_j} \\
&\leq d_1\left(2a_1^2\frac{f_1}{\lambda_1}+b_1\right)\frac{f_1+2\lambda_1}{2\lambda_1} \equiv C < \infty
\end{aligned}$$

where d_1 is given by (7.3.9). Hence

$$\sum_k P\left(\omega : \|\xi_{\tau_1^k(\omega)}\|_0^2 \geq \epsilon k^2\right) \leq \sum_k \epsilon^{-1}k^{-2}C < \infty.$$

It follows that, as k tends to infinity, $\frac{1}{k}\xi_{\tau_1^k(\omega)}$ tends to 0 a.s. and hence

$$\begin{aligned}
\left\|\frac{1}{t}\xi_t\right\|_0^2 &= \left\|\frac{1}{t}\xi_{\tau_1^{N_1(t)}(\omega)}\exp\left(-\lambda_j\left(t-\tau_1^{N_1(t)}(\omega)\right)\right)\right\|_0^2 \\
&\leq \left\|\frac{N_1(t)}{t}\frac{1}{N_1(t)}\xi_{\tau_1^{N_1(t)}(\omega)}\right\|_0^2 \to 0 \ a.s.
\end{aligned}$$

(2°) Let

$$R_t[\phi] = \frac{1}{\sqrt{t}}\int_0^t\int_{\mathcal{X}}\int_0^\infty G(x,a)[T_{t-s}\phi]\tilde{N}(dsdxda),\ \forall\phi\in\Phi.$$

Then, by (7.3.12)

$$\gamma_t = \frac{1}{\sqrt{t}}\left(T_tu_0 + \int_0^t\int_{\mathcal{X}}\int_0^\infty G(x,a)[T_{t-s}\phi]N(dsdxda)\right.$$

$$-t\sum_{i=1}^r a_i f_i \psi_0(\kappa_i,\epsilon_i)\sqrt{\frac{2c}{1-e^{-2c\ell}}}\phi_0\Bigg)$$

$$= R_t + \frac{1}{\sqrt{t}}T_t u_0 + \frac{1}{\sqrt{t}}\left(\int_0^t\int_{\mathcal{X}}\int_0^\infty G(x,a)[T_{t-s}\phi]\,\mu(dxda)ds\right.$$

$$\left. -t\sum_{i=1}^r a_i f_i \psi_0(\kappa_i,\epsilon_i)\sqrt{\frac{2c}{1-e^{-2c\ell}}}\phi_0\right)$$

$$= R_t + \frac{1}{\sqrt{t}}T_t u_0 + \sum_{i=1}^r f_i a_i \sum_{j=1}^\infty \frac{1-e^{-\lambda_j t}}{\lambda_j\sqrt{t}}\psi_j(\kappa_i,\epsilon_i)\phi_j.$$

It is easy to see that the second term tends to 0 almost surely. The third term also goes to 0 as $t \to \infty$, so that we only need to show that R_t tends to $\xi\phi_0$ in Φ_0 in distribution. We show this in two steps. First, we prove that $\{\mathcal{D}(R_t)\}$ is tight as probability measures in Φ_0. Then, we show that it has only one cluster point which is the distribution of $\xi\phi_0$. For the first step, let $P_t = \mathcal{D}(R_t) \in \mathcal{P}(\Phi_0)$ and

$$\gamma_n^2(h) = \sum_{j=n}^\infty < h,\phi_j >_0^2, \qquad \forall h \in \Phi_0.$$

Note that

$$\sup_{t>1}\int \gamma_n^2(h)P_t(dh) = \sup_{t>1} E\sum_{j=n}^\infty < R_t,\phi_j >_0^2$$

$$= \sup_{t>1} E\sum_{j=n}^\infty\left(\frac{1}{\sqrt{t}}\int_0^t\int_{\mathcal{X}}\int_0^\infty G(x,a)[T_{t-s}\phi_j]\tilde{N}(dsdxda)\right)^2$$

$$= \sup_{t>1}\sum_{j=n}^\infty \frac{1}{t}\int_0^t\int_{\mathcal{X}}\int_0^\infty (G(x,a)[T_{t-s}\phi_j])^2\mu(dxda)ds$$

$$= \sup_{t>1}\sum_{j=n}^\infty \frac{1}{t}\sum_{i=1}^r f_i b_i \frac{1-e^{-2\lambda_j t}}{2\lambda_j}(\psi_j(\kappa_i,\epsilon_i))^2$$

$$\le \sum_{j=n}^\infty\sum_{i=1}^r f_i b_i(\psi_j(\kappa_i,\epsilon_i))^2.$$

Hence

$$\lim_{n\to\infty}\sup_{t>1}\int \gamma_n^2(h)P_t(dh) = 0 \quad \text{and} \quad \sup_{t>1}\int \gamma_0^2(h)P_t(dh) < \infty.$$

It then follows from Corollary 2.3.1 that the family $\{P_t\}$ is tight. Next, let P^* be a cluster point of $\{P_t\}$. As $\mathcal{P}(\Phi_0) \subset \mathcal{P}(\Phi')$, P^* and $\mathcal{D}(\xi\phi_0)$ can also be regarded as elements in $\mathcal{P}(\Phi')$, we only need to show that they have the same characteristic function on Φ. Note that, $\forall\phi \in \Phi$,

$$\begin{aligned} E\exp(iR_t[\phi]) &= \exp\left(\sum_{j=1}^{r} f_j \int_0^t \int_0^\infty \left(\exp\left(\frac{i}{\sqrt{t}}G(\kappa_j, a)[T_s\phi]\right)\right.\right. \\ &\qquad \left.\left. -1 - \frac{i}{\sqrt{t}}G(\kappa_j, a)[T_s\phi]\right) F_j(da)ds\right). \end{aligned}$$

We then have

$$\begin{aligned} &\left| \int_0^t \int_0^\infty \left(\exp\left(\frac{i}{\sqrt{t}}G(\kappa_j, a)[T_s\phi]\right)\right.\right. \\ &\qquad \left.\left. -1 - \frac{i}{\sqrt{t}}G(\kappa_j, a)[T_s\phi] + \frac{1}{2t}G(\kappa_j, a)[T_s\phi]^2\right) F_j(da)ds \right| \\ \leq &\int_0^t \int_0^\infty \min\left(\left|\frac{i}{\sqrt{t}}G(\kappa_j, a)[T_s\phi]\right|^3, \left|\frac{i}{\sqrt{t}}G(\kappa_j, a)[T_s\phi]\right|^2\right) F_j(da)ds \\ \leq &\int_0^\infty \min\left((a\|\phi\|_0 d_j)^3, \frac{1}{\sqrt{t}}(a\|\phi\|_0 d_j)^2\right) F_j(da) \to 0 \quad \text{as } t \to \infty \end{aligned}$$

and

$$\begin{aligned} &-\sum_{j=1}^{r} f_j \int_0^t \int_0^\infty \frac{1}{2t} G(\kappa_j, a)[T_s\phi]^2 F_j(da)ds \\ \to \quad &-\frac{1}{2}\sum_{j=1}^{r} f_j \int_0^\infty G(\kappa_j, a)[< \phi, \phi_0 >_0 \phi_0]^2 F_j(da) \\ = \quad &-\frac{1}{2}\sum_{j=1}^{r} f_j b_j < \phi, \phi_0 >^2 \psi_0(\kappa_i, \epsilon_i)\frac{2c}{1 - e^{-2c\ell}}. \end{aligned}$$

Hence P^* and $\mathcal{D}(\xi\phi_0)$ have the same characteristic function on Φ and hence, $P^* = \mathcal{D}(\xi\phi_0)$. ∎

The next theorem shows the existence of the equilibrium state for the pollution process when the leakage rate is greater than zero. It also establishes the relationship between the deterministic equations of the type (7.1.1) and the stochastic evolution equation (7.1.2).

Theorem 7.3.3 *If $\alpha > 0$, then u_t converges weakly in Φ_0 to a random field u_∞. For $x \in \mathcal{X}$, let $u(x) = Eu_\infty(x)$. Then $u(x)$ is the solution of the following differential equation*

$$D\frac{d^2u(x)}{dx^2} - V\frac{du(x)}{dx} - \alpha u(x) + Q(x) = 0$$

where

$$Q(x) = \sum_{j=1}^{r} \frac{a_j f_j}{2\epsilon_j} 1_{(\kappa_j - \epsilon_j, \kappa_j + \epsilon_j)}(x).$$

Proof: It follows from the same arguments as in the proof of the previous theorem that u_t converges weakly in Φ_0 to a random field u_∞ with characteristic function

$$\begin{aligned} & E\exp(iu_\infty[h]) \qquad (7.3.13) \\ = \;& \exp\left(\sum_{j=1}^{r} f_j \int_0^\infty \int_0^\infty (\exp(ie^{-\alpha s} G(\kappa_j, a)[T_s h]) - 1) F_j(da) ds\right). \end{aligned}$$

Let $\tilde{u}_\infty$ be an Φ_0-valued random variable given by

$$\tilde{u}_\infty[h] = \int_0^\infty \int_{\mathcal{X}} \int_0^\infty e^{-\alpha s} G(x, a)[T_s h] N(dsdxda), \; \forall h \in \Phi_0.$$

Comparing the characteristic function of $\tilde{u}_\infty$ with (7.3.13) we see that $\tilde{u}_\infty$ and u_∞ have the same distribution, and hence recalling (7.3.3), we have

$$\begin{aligned} Eu_\infty[h] = E\tilde{u}_\infty[h] &= \int_0^\infty \int_{\mathcal{X}} \int_0^\infty e^{-\alpha s} G(x, a)[T_s h] \mu(dxda) ds \\ &= \sum_{j=1}^{r} f_j \int_0^\infty \int_0^\infty e^{-\alpha s} G(\kappa_j, a)[T_s h] F_j(da) ds \\ &= \sum_{j=1}^{r} f_j \int_0^\infty \int_0^\infty e^{-\alpha s} \frac{a}{2\epsilon_j} \int_{\kappa_j - \epsilon_j}^{\kappa_j + \epsilon_j} T_s h(y) \rho(y) dy F_j(da) ds. \end{aligned}$$

i.e.

$$\begin{aligned} u(x) &= Eu_\infty(x) \\ &= \sum_{j=1}^{r} a_j f_j \int_0^\infty e^{-\alpha s} T_s \left(\frac{1}{2\epsilon_j} 1_{(\kappa_j - \epsilon_j, \kappa_j + \epsilon_j)}\right)(x) ds \\ &= \int_0^\infty e^{-\alpha s} T_s Q(x) ds = (\alpha - L)^{-1} Q(x) \end{aligned}$$

where the last equality follows from (1.2.14). Hence

$$D\frac{d^2 u(x)}{dx^2} - V\frac{du(x)}{dx} - \alpha u(x) + Q(x) = 0. \qquad \blacksquare$$

7.4 Water pollution problem with a tolerance level

In this section, we suppose that there is a mechanism to clean up the river when the chemical density exceeds a fixed level $\xi(x)$. For the sake of mathematical simplicity, suppose that the changes in the chemical concentration do not depend on the locations where the chemicals are deposited. In this case, we consider the following integral equation

$$\begin{aligned} u_t[\phi] &= u_0[\phi] + \int_0^t (u_s[-L\phi] - \alpha u_s[\phi])ds \\ &\quad + \int_0^t \int_0^\infty a(\xi[\phi] - u_{s-}[\phi])N(dsda) \end{aligned} \tag{7.4.1}$$

where N is a Poisson random measure on $\mathbf{R}_+ \times \mathbf{R}_+$ with characteristic measure μ on $\mathbf{R}_+$. Let $\mathcal{X} = [0, \ell]$ and consider (7.4.1) on Φ' constructed in Section 2.

As in the previous sections, we can solve (7.4.1) by making use of Theorems 6.2.2 and 6.3.1. The next theorem considers the regularity of this solution.

Theorem 7.4.1 *Let $\xi \in \Phi_p$ and $E\|u_0\|_p^2 < \infty$. Then $u. \in D([0,\infty), \Phi_p)$.*

The proof is similar to that of Theorem 7.3.1 and is omitted.

Corollary 7.4.1 *If $\xi \in \Phi_p$ and $E\|u_0\|_p^2 < \infty$ for any $p \in \mathbf{Z}$, then $u. \in D([0,\infty), \Phi)$.*

If the Poisson deposits are bounded, then the following result shows that the magnitude of the random field $u(t,x)$ can be controlled by the magnitude of the initial random field $u(0,x)$ and the tolerance level $\xi(x)$.

Theorem 7.4.2 *Suppose that the following conditions hold.*
(1°) There exists a constant M such that $\mu\{a : a > M\} = 0$.
(2°) There exists a constant C such that $u(0,x,\omega) \le C$ for any $x \in [0,\ell]$ and $\omega \in \Omega$, where $u_0[\phi] = \int_0^\ell u(0,x)\phi(x)e^{-cx}dx$, c is given by (7.2.5) with subscript 1 dropped.
(3°) $\alpha \ge 0$ and μ is a finite measure.
Then

$$\sup_{t,x,\omega} u(t,x,\omega) \le \max\left\{ M \sup_x \xi(x), C \right\}.$$

Proof: Let $\tau_1 < \tau_2 < \cdots$ be the jump times of the Poisson random measure N and $A_1, A_2, \cdots$ be *i.i.d.* random variables with common distribution $\mu(\cdot)/\mu(\mathbf{R}_+)$. By (7.4.1), for $0 \le t < \tau_1$, we have

$$u_t[\phi] = u_0[\phi] + \int_0^t (u_s[-L\phi] - \alpha u_s[\phi])ds. \tag{7.4.2}$$

As $u_t \in H$, let $u_t[\phi] = \int_0^\ell u(t,x)\phi(x)e^{-cx}dx$. Then, by (7.4.2), $u(t,x)$ satisfies the following Kolmogorov equation

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) &= D\frac{\partial^2}{\partial x^2}u(t,x) - V\frac{\partial}{\partial x}u(t,x) - \alpha u(t,x)\\ \frac{\partial}{\partial x}u(t,0) &= \frac{\partial}{\partial x}u(t,\ell) = 0\end{aligned}$$

with initial condition $u(0,x)$. Let y_t be a $[0,\ell]$-valued Markov process generated by $D\frac{\partial^2}{\partial x^2} - V\frac{\partial}{\partial x}$ with the Neumann boundary conditions and let $p(t,x,y)$ be the Markov transition density. Then

$$u(t,x) = e^{-\alpha t}\int_0^\ell u(0,y)p(t,x,y)dy.$$

Hence

$$u(t,x) \le \sup_x u(0,x) \le C.$$

For $t = \tau_1$, we have

$$u_{\tau_1}[\phi] = u_{\tau_1 -}[\phi] + A_1(\omega)(\xi[\phi] - u_{\tau_1 -}[\phi]).$$

i.e.

$$u(\tau_1,x) = u(\tau_1-,x) + A_1(\omega)(\xi(x) - u(\tau_1-,x)). \tag{7.4.3}$$

By (1°) we know that $A_1(\omega) \le M$ a.s. Hence, by (7.4.3),

$$u(\tau_1,x) \le \begin{cases} u(\tau_1-,x) & \text{if } \xi(x) \le u(\tau_1-,x)\\ M\xi(x) & \text{if } \xi(x) > u(\tau_1-,x).\end{cases}$$

i.e

$$u(\tau_1,x) \le \begin{cases} C & \text{if } \xi(x) \le u(\tau_1-,x)\\ M\xi(x) & \text{if } \xi(x) > u(\tau_1-,x).\end{cases}$$

So

$$u(\tau_1,x) \le \max\{M\xi(x), C\} \le \max\left\{M\sup_x \xi(x), C\right\}.$$

By induction, we see that $u(t,x) \le \max\{M\sup_x \xi(x), C\}$ for any $t \ge 0$. ∎

7.5 Filtering problem

In actual practice it is not possible, in many instances, to directly observe the amount of pollution. Information about the latter is obtained through noisy observations. In such cases we have a statistical filtering problem to solve. Making the usual assumption that the noise is Gaussian we have the following observation model:

$$Y_t^i = \int_0^t u_s[\chi_i]ds + B_t^i \tag{7.5.1}$$

where B_t is a k-dimensional Brownian motion, $E(B_t^i)^2 = \sigma^2 t$ and $\chi_i \in \Phi$, $i = 1, 2, \cdots, k$. In (7.5.1) we have assumed the weighted functions to be "smooth" instead of $\frac{1}{2\epsilon_i}1_{(x_i-\epsilon_i, x_i-\epsilon_i)}$.

Our problem is to find the best linear filter for u_t based on the observed data $\{Y_s : s \le t\}$. i.e. $\forall \phi \in \Phi$ and $t > 0$, we want to find a function $K^\phi(t,s)$ with $\int_0^t |K^\phi(t,s)|^2 ds < \infty$ such that

$$\hat{u}_t^\phi = V_t^\phi + \int_0^t K^\phi(t,s)dY_s \tag{7.5.2}$$

is the best linear unbiased estimate (BLUE) of $u_t[\phi]$, where V_t^ϕ and $K^\phi(t,s)$ are deterministic. By the unbiasedness, we have

$$V_t^\phi = Eu_t[\phi] - \sum_{i=1}^k \int_0^t K^\phi(t,s)_i Eu_s[\chi_i]ds.$$

So, we only need to study the process m_t^ϕ defined as the second term on the right hand side of (7.5.2).

For any $t, s \ge 0$ and $\phi, \psi \in \Phi$, let

$$\Gamma_{ts}(\phi,\psi) \equiv Cov(u_t[\phi], u_s[\psi]). \tag{7.5.3}$$

For fixed $T > 0$, let

$$D_T = \{(t,s) : 0 \le s \le t \le T\}.$$

Lemma 7.5.1 *There exist index p and $\tilde{\Gamma} \in C(D_T \to L_{(1)}(\Phi_p))$ such that, $\forall \phi, \psi \in \Phi_p$*

$$\Gamma_{ts}(\phi,\psi) = <\tilde{\Gamma}(t,s)\phi, \psi>_p, \tag{7.5.4}$$

where $L_{(1)}(\Phi_p)$ denotes the collection of all nuclear operators on Φ_p (cf. Definition 1.2.6). Further, for s fixed, $\tilde{\Gamma}(t,s)$ is continuously differentiable with respect to $t \in [s,T]$ and

$$\sup\left\{ \left\| \frac{\partial}{\partial t}\tilde{\Gamma}(t,s)\right\|_{L_{(1)}(\Phi_p)} : (t,s) \in D_T \right\} < \infty. \tag{7.5.5}$$

Proof: We give an outline of the proof and leave the details to the reader. Let p_1 be given by Theorem 7.2.2 and $p = p_1 + 1$. For any $(t, s) \in D_T$, it follows from (7.5.3) and Theorem 7.2.2 that there exists $\tilde{\Gamma}(t, s) \in L_{(1)}(\Phi_p)$ such that (7.5.4) holds. As u_0 and N are independent and

$$u_t[\phi] = u_0[T_t\phi]e^{-\alpha t} + \int_0^t \int_{\mathcal{X}} \int_0^\infty ae^{-\alpha(t-s)}(T_{t-s}\phi)(x)\rho(x)N(dsdxda),$$

$\forall \phi \in \Phi_p$ and $(t, s) \in D_T$, we see that

$$\Gamma_{ts}(\phi, \psi) = e^{-\alpha(t+s)} \left(\Gamma^1_{ts}(\phi, \psi) + \Gamma^2_{ts}(\phi, \psi)\right) \tag{7.5.6}$$

where

$$\Gamma^1_{ts}(\phi, \psi) = Cov\left(u_0[T_t\phi], u_0[T_s\psi]\right)$$

and

$$\Gamma^2_{ts}(\phi, \psi) = \int_0^s \int_{\mathcal{X}} \int_0^\infty a^2 e^{2\alpha r}(T_{t-r}\phi)(x)(T_{s-r}\psi)(x)\rho(x)^2\mu(dxda)dr.$$

Choose the CONS of Φ_p given by

$$\phi^p_{j_1\cdots j_d} = \phi_{j_1\cdots j_d}(1 + \lambda_{j_1\cdots j_d})^{-p}.$$

For (t, s) and (t', s) in D_T, we have

$$\begin{aligned} & ||\Gamma^1_{ts} - \Gamma^1_{t's}||_{L_{(1)}(\Phi_p)} \\ \leq & \sum_{j_1\cdots j_d} \left|\left\langle\left(\Gamma^1_{ts} - \Gamma^1_{t's}\right)\phi^p_{j_1\cdots j_d}, \phi^p_{j_1\cdots j_d}\right\rangle_p\right| \\ \leq & \sum_{j_1\cdots j_d} \left|\Gamma^1_{ts}\left(\phi^p_{j_1\cdots j_d}, \phi^p_{j_1\cdots j_d}\right) - \Gamma^1_{t's}\left(\phi^p_{j_1\cdots j_d}, \phi^p_{j_1\cdots j_d}\right)\right| \\ \leq & \sum_{j_1\cdots j_d} Var\left(u_0\left[\phi^p_{j_1\cdots j_d}\right]\right)\left|e^{-\lambda_{j_1\cdots j_d}t} - e^{-\lambda_{j_1\cdots j_d}t'}\right| \end{aligned}$$

and

$$\begin{aligned} & ||\Gamma^2_{ts} - \Gamma^2_{t's}||_{L_{(1)}(\Phi_p)} \\ = & \sum_{j_1\cdots j_d} \left|\int_0^s \int_{\mathcal{X}} \int_0^\infty a^2 e^{2\alpha r}((T_{t-r} - T_{t'-r})\phi^p_{j_1\cdots j_d})(x)\right. \\ & \left.(T_{s-r}\phi^p_{j_1\cdots j_d})(x)\rho(x)^2\mu(dxda)dr\right|. \end{aligned}$$

It follows from the dominate convergence theorem that $\tilde{\Gamma}(t, s)$ is continuous in t uniformly for s in $[0, T]$. The continuity in s is proved similarly and hence $\tilde{\Gamma} \in C(D_T \to L_{(1)}(\Phi_p))$. Similarly, there exists $\tilde{\tilde{\Gamma}}(t,s) \in L_{(1)}(\Phi_p)$ such that

$$\tilde{\tilde{\Gamma}} \in C(D_T \to L_{(1)}(\Phi_p)) \quad \text{and} \quad \Gamma_{ts}((\alpha + L)\phi, \psi) = \left\langle \tilde{\tilde{\Gamma}}(t,s)\phi, \psi \right\rangle_p .$$

Making use of (7.2.8), we have

$$\Gamma_{tr}(\phi, \psi) = \Gamma_{rr}(\phi, \psi) - \int_r^t \Gamma_{sr}((\alpha + L)\phi, \psi) ds. \tag{7.5.7}$$

Then for $(t,s) \in D_T$,

$$\left\| \frac{1}{h} \left(\tilde{\Gamma}(t+h, s) - \tilde{\Gamma}(t,s) \right) - \tilde{\tilde{\Gamma}}(t,s) \right\|_{L_{(1)}(\Phi_p)} \leq \sup_{0 \leq s \leq T} E\|u_s\|^2_{-(p-1)} h.$$

Hence $\tilde{\Gamma}(t,s)$ is differentiable with respect to $t \in [s,T]$. The continuity of $\frac{\partial}{\partial t}\tilde{\Gamma}(t,s)$ and the inequality (7.5.5) can be proved in the same way. ∎

Lemma 7.5.2 *(1) For any $(t,s) \in D_T$, $\phi \in \Phi$ and $i = 1, \cdots, k$, we have*

$$\sum_{j=1}^k \int_0^t K^\phi(t,r)_j \Gamma_{rs}(\chi_j, \chi_i) dr + \sigma^2 K^\phi(t,s)_i = \Gamma_{ts}(\phi, \chi_i) \tag{7.5.8}$$

and $K^\phi(t,s)$ is uniquely determined by (7.5.8).
(2) Let p be given by Lemma 7.5.1, then there exists $\tilde{K} \in C(D_T \to \mathbf{R}^k \otimes \Phi_{-p})$ such that

$$K^\phi(t,s) = \tilde{K}(t,s)[\phi], \qquad \forall \phi \in \Phi_p. \tag{7.5.9}$$

Furthermore, for s fixed, $\tilde{K}(t,s)$ is continuously differentiable with respect to $t \in [s,T]$.

Proof: By the definition of m^ϕ, for $s < t$ we have

$$Cov\left(u_t[\phi] - m_t^\phi, \int_0^s u_r[\chi_i]dr + B_s^i\right) = 0.$$

Hence

$$\begin{aligned}
&\int_0^s \Gamma_{tr}(\phi, \chi_i) dr \\
= &\sum_{j=1}^k Cov\Big(\int_0^t K^\phi(t,r)_j u_r[\chi_j] dr + \int_0^t K^\phi(t,r)_j dB_r^j, \\
&\qquad \int_0^s u_r[\chi_i]dr + B_s^i \Big) \\
= &\sum_{j=1}^k \int_0^s \int_0^t K^\phi(t,r)_j \Gamma_{rr'}(\chi_j, \chi_i) dr dr' + \sigma^2 \int_0^s K^\phi(t,r)_i dr.
\end{aligned}$$

Taking the derivative with respect to s on both sides, it is clear that (7.5.8) holds. It follows from the positivity of the kernel function $\{\Gamma_{tr}(\chi_j, \chi_i)\}$ that $K^{\phi}(t,s)$ is uniquely determined by (7.5.8).

For $f \in C(D_T \to \mathbf{R}^k \otimes \Phi_{-p})$, let

$$(\Pi f)(t,s)_i = \sum_{j=1}^{k} \int_0^t \Gamma_{sr}(\chi_i, \chi_j) f(t,r)_j dr.$$

Then Π is a continuous linear operator on $C(D_T \to \mathbf{R}^k \otimes \Phi_{-p})$ and its norm is smaller than a finite constant C_1 given by

$$C_1 = \sup\left\{ \sum_{j=1}^{k} \int_0^T |\Gamma_{sr}(\chi_i, \chi_j)| ds + |\Gamma_{tr}(\chi_i, \chi_j)| : r \in [0,T] \ \ 1 \le i \le d \right\}.$$

Furthermore, for $f \in C(D_T \to \mathbf{R}^k \otimes \Phi_{-p})$ which is continuously differentiable with respect to $t \in [s,T]$ and such that

$$M_f = \sup\left\{ \left\| \frac{\partial}{\partial t} f(t,s)_j \right\|_{-p}, \|f(t,s)_j\|_{-p} : (t,s) \in D_T, j = 1, \cdots, k \right\} < \infty, \tag{7.5.10}$$

it is easy to see that, for any $s \ge 0$ fixed, $\Pi f(t,s)$ is continuously differentiable with respect to $t \in [s,T]$ and $M_{\Pi f} \le C_1 M_f$. Without loss of generality, we may assume that $C_1 < 1$ as we can multiply Y_i, χ_i and B_i, $i = 1, \cdots, k$, by a small constant without changing the observation equation (7.5.1).

Let $f(t,s)_i[\phi] = \Gamma_{ts}(\phi, \chi_i)$. Then by Lemma 7.5.1, $f \in C(D_T \to \mathbf{R}^k \otimes \Phi_{-p})$ and satisfies (7.5.10). Let

$$\tilde{K}(t,s) = \sum_{n=0}^{\infty} (-1)^n (\Pi^n f)(t,s)$$

and $\tilde{K}^{\phi}(t,s) = \tilde{K}(t,s)[\phi]$. One can then show that $\tilde{K}^{\phi}(t,s)$ solves the equation (7.5.8) and hence (7.5.9) holds. It follows from Lemma 7.5.1 and (7.5.10) that, for s fixed, $\tilde{K}(t,s)$ is continuously differentiable with respect to $t \in [s,T]$. ∎

Finally, we state and prove the main result of this section. For simplicity of notation, we take $\sigma = 1$ and $\alpha = 0$.

For $t \in [0,T]$ and $\phi, \psi \in \Phi$, let

$$\gamma_t(\phi, \psi) = Cov\left(u_t[\phi] - \hat{u}_t^{\phi}, u_t[\psi] - \hat{u}_t^{\psi} \right).$$

Theorem 7.5.1 *(1°) Let p be given in Lemma 7.5.1. Then there exists* $\tilde{\gamma} \in C([0,T], L_{(1)}(\Phi_p))$ *such that*

$$\gamma_t(\phi, \psi) =< \tilde{\gamma}_t\phi, \psi >_p, \qquad \forall \phi, \psi \in \Phi$$

and $\tilde{\gamma}_t$ *is the unique solution of the following equation on* $L_{(1)}(\Phi_p)$*:*

$$\tilde{\gamma}_t = \tilde{\gamma}_0 - \int_0^t \left(\tilde{\gamma}_s L + L\tilde{\gamma}_s + \sum_{i=1}^k (\tilde{\gamma}_s\chi_i) \otimes (\tilde{\gamma}_s\chi_i) \right) ds + t\mathcal{G}_p \qquad (7.5.11)$$

where $\mathcal{G}_p$ *is a nonnegative operator on* Φ_p *given by*

$$\langle \mathcal{G}_p\phi, \psi \rangle_p = \mathcal{G}(\phi, \psi) = \int_{\mathcal{X}} \int_0^\infty a^2\phi(x)\psi(x)\mu(dxda).$$

(2°) There exists a continuous Φ'*-valued process* m_t *such that* $m_t^\phi = m_t[\phi]$ *for any* $\phi \in \Phi$ *where* m_t *is the unique solution of the following SDE on* Φ'

$$\begin{aligned} m_t[\phi] &= -\int_0^t \left(m_s[L\phi] + \sum_{i=1}^k \gamma_s(\phi, \chi_i) m_s[\chi_i] \right) ds \qquad (7.5.12) \\ &\quad + \sum_{i=1}^k \int_0^t \gamma_s(\phi, \chi_i) dY_s^i, \qquad \forall \phi \in \Phi. \end{aligned}$$

Proof: It follows from (7.5.7), (7.5.8) and standard Hilbert space techniques that, $\forall (t,s) \in D_T$, $\phi \in \Phi$ and $i = 1, \cdots, k$,

$$\frac{\partial}{\partial t} K^\phi(t,s)_i - K^{-L\phi}(t,s)_i + \sum_{j=1}^k K^\phi(t,t)_j K^{\chi_j}(t,s)_i = 0.$$

Then

$$\begin{aligned} m_t[\phi] &= \sum_{j=1}^k \int_0^t K^\phi(t,s)_j dY_s^j \qquad (7.5.13) \\ &= \sum_{j=1}^k \int_0^t \left(\int_s^t \frac{\partial}{\partial r} K^\phi(r,s)_j dr \right) dY_s^j + \sum_{j=1}^k \int_0^t K^\phi(s,s)_j dY_s^j \\ &= \sum_{j=1}^k \int_0^t \left(\int_0^r \frac{\partial}{\partial r} K^\phi(r,s)_j dY_s^j \right) dr + \sum_{j=1}^k \int_0^t K^\phi(s,s)_j dY_s^j \\ &= \sum_{j=1}^k \int_0^t \int_0^r \left(K^{-L\phi}(r,s)_j - \sum_{i=1}^k K^\phi(r,r)_i K^{\chi_i}(r,s)_j \right) dY_s^j dr \\ &\quad + \sum_{j=1}^k \int_0^t K^\phi(s,s)_j dY_s^j \\ &= \int_0^t \left(m_s[-L\phi] - \sum_{i=1}^k K^\phi(s,s)_i m_s[\chi_i] \right) ds + \sum_{j=1}^k \int_0^t K^\phi(s,s)_j dY_s^j. \end{aligned}$$

Letting $s = t$ in (7.5.8), we get

$$\begin{aligned} K^{\phi}(t,t)_i &= \Gamma_{tt}(\phi, \chi_i) - \sum_{j=1}^{k} \int_0^t K^{\phi}(t,r)_j \Gamma_{rt}(\chi_j, \chi_i) dr \\ &= Cov(u_t[\phi], u_t[\chi_i]) - \sum_{j=1}^{k} \int_0^t K^{\phi}(t,r)_j Cov(u_r[\chi_j], u_t[\chi_i]) dr \\ &= Cov(u_t[\phi] - \sum_{j=1}^{k} \int_0^t K^{\phi}(t,r)_j u_r[\chi_j] dr, u_t[\chi_i]) \\ &= Cov(u_t[\phi] - \hat{u}_t^{\phi}, u_t[\chi_i]) = Cov(u_t[\phi] - \hat{u}_t^{\phi}, u_t[\chi_i] - \hat{u}_t^{\chi_i}) \\ &= \gamma_t(\phi, \chi_i). \end{aligned} \tag{7.5.14}$$

Hence, it follows from (7.5.13) and (7.5.14) that m_t satisfies (7.5.12).

Note that

$$\begin{aligned} \gamma_t(\phi, \psi) &= Cov\left(u_t[\phi] - m_t^{\phi}, u_t[\psi] - m_t^{\psi}\right) \\ &= \Gamma_{tt}(\phi, \psi) - Cov(m_t[\phi], m_t[\psi]). \end{aligned}$$

By (7.5.6), we have

$$\begin{aligned} &\Gamma_{tt}(\phi, \psi) - \Gamma_{00}(T_t\phi, T_t\psi) \\ &= \int_0^t \int_{\mathcal{X}} \int_0^{\infty} a^2 (T_{t-r}\phi)(x)(T_{t-r}\psi)(x)\rho(x)^2 \mu(dxda)dr. \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{dt}\Gamma_{tt}(\phi, \psi) &= -\Gamma_{00}(T_t L\phi, T_t\psi) - \Gamma_{00}(T_t\phi, T_t L\psi) \\ &\quad + \int_{\mathcal{X}} \int_0^{\infty} a^2 \phi(x)\psi(x)\rho(x)^2 \mu(dxda) \\ &\quad - \int_0^t \int_{\mathcal{X}} \int_0^{\infty} a^2 (T_{t-r}L\phi)(x)(T_{t-r}\psi)(x)\rho(x)^2 \mu(dxda)dr \\ &\quad - \int_0^t \int_{\mathcal{X}} \int_0^{\infty} a^2 (T_{t-r}\phi)(x)(T_{t-r}\psi)(x)\rho(x)^2 \mu(dxda)dr \\ &= -\Gamma_{tt}(L\phi, \psi) - \Gamma_{tt}(\phi, L\psi) + \mathcal{G}(\phi, \psi). \end{aligned}$$

On the other hand

$$\begin{aligned} Cov(m_t[\phi], m_t[\psi]) &= Cov(m_t[\phi], u_t[\psi]) \\ &= \sum_{i=1}^{k} \int_0^t K^{\phi}(t,s)_i \Gamma_{st}(\chi_i, \psi) ds. \end{aligned}$$

Hence

$$\begin{aligned}
&\frac{d}{dt}Cov(m_t[\phi], m_t[\psi]) \\
= &\sum_{i=1}^{k} K^{\phi}(t,t)_i \Gamma_{tt}(\chi_i, \psi) + \sum_{i=1}^{k}\int_0^t \frac{\partial}{\partial t}K^{\phi}(t,s)_i \Gamma_{st}(\chi_i,\psi)ds \\
&+ \sum_{i=1}^{k}\int_0^t K^{\phi}(t,s)_i \frac{\partial}{\partial t}\Gamma_{st}(\chi_i,\psi)ds \\
= &\sum_{i=1}^{k}\gamma_t(\phi,\chi_i)\Gamma_{tt}(\chi_i,\psi) - \sum_{i=1}^{k}\int_0^t K^{\phi}(t,s)_i\Gamma_{st}(\chi_i, L\psi)ds \\
&+ \sum_{i=1}^{k}\int_0^t \left\{ K^{-L\phi}(t,s)_i - \sum_{j=1}^{k}\gamma_t(\phi,\chi_j)K^{\chi_j}(t,s)_i \right\}\Gamma_{st}(\chi_i,\psi)ds \\
= &\sum_{i=1}^{k}\gamma_t(\phi,\chi_i)\gamma_t(\psi,\chi_i) - Cov(m_t[L\phi], m_t[\psi]) - Cov(m_t[\phi], m_t[L\psi]).
\end{aligned}$$

Therefore

$$\frac{d}{dt}\gamma_t(\phi,\psi) = -\gamma_t(L\phi,\psi) - \gamma_t(\phi, L\psi) + \mathcal{G}(\phi,\psi) - \sum_{i=1}^{k}\gamma_t(\phi,\chi_i)\gamma_t(\psi,\chi_i).$$

Now we prove the uniqueness for the solution of (7.5.11). Let $\tilde{\gamma}'$ be another solution of (7.5.11), $\tilde{\alpha} = \tilde{\gamma} - \tilde{\gamma}'$ and

$$M \equiv \sup\left\{ \sum_{i=1}^{k} \|\chi_i\|_p^2 \left(\|\tilde{\gamma}_t\|_{L(\Phi_p)} + \|\tilde{\gamma}_t'\|_{L(\Phi_p)} \right) : 0 \le t \le T \right\}.$$

As

$$\begin{aligned}
\frac{d}{dt}\alpha_t(\phi,\psi) &= -\alpha_t(L\phi,\psi) - \alpha_t(\phi, L\psi) \\
&\quad - \sum_{i=1}^{k}\left\{ \alpha_t(\phi,\chi_i)\gamma_t(\psi,\chi_i) + \gamma_t'(\phi,\chi_i)\alpha_t(\psi,\chi_i) \right\},
\end{aligned}$$

it is easy to see that

$$\begin{aligned}
&\alpha_t(\phi,\psi) \\
= &-\sum_{i=1}^{k}\int_0^t \left\{ \alpha_s(T_{t-s}\phi,\chi_i)\gamma_s(T_{t-s}\psi,\chi_i) + \gamma_s'(T_{t-s}\phi,\chi_i)\alpha_s(T_{t-s}\psi,\chi_i) \right\} ds.
\end{aligned}$$

Therefore

$$\|\tilde{\alpha}_t\|_{L(\Phi_p)} \le M\int_0^t \|\tilde{\alpha}_s\|_{L(\Phi_p)}ds,$$

and hence $\tilde{\alpha}_t = 0$ for all $t \in [0, T]$. The uniqueness for the solution of (7.5.12) is verified in a similar fashion. ∎

Remark 7.5.1 *One can show that $V_t^\phi = V_t[\phi]$ for $V_t \in \Phi'$ and, moreover that*

$$\hat{u}_t = V_t + m_t \in \Phi_{-p}, \qquad a.s.$$