ON ESTIMATING THE PERIOD OF A CYCLIC POISSON PROCESS

ROELOF HELMERS
CWI

I WAYAN MANGKU
Bogor Agricultural University

We propose and investigate a simple nonparametric estimator of the period of a cyclic Poisson process. It is assumed that only a single realization of the Poisson process is observed in a bounded window. We prove consistency and establish a rate of convergence of the proposed estimator when the size of the window expands.

1. Introduction and main result

Let X denote a cyclic Poisson point process defined on a probability space $(\Omega, \mathcal{A}, P)$, with absolutely continuous σ-finite mean measure μ w.r.t. Lebesgue measure ν and with (unknown) locally integrable intensity function $\lambda\colon \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$, i.e., for any bounded Borel set B, we have $\mu(B) = \int_B \lambda(s)\,ds < \infty$. In addition, λ is cyclic (with period τ), i.e., for some $\tau \in R^+$

$$\lambda(s + k\tau) = \lambda(s) \tag{1.1}$$

for all $s \in \mathbb{R}$ and $k \in \mathbb{Z}$. The period τ is assumed to be unknown.

Suppose that, for some $\omega \in \Omega$, a single realization $X(\omega)$ of the Poisson point process X is observed, though only in a bounded interval (called window) $W \subset \mathbb{R}$. Since λ is locally integrable, the Poisson point process X always places only a finite number of points in any bounded subset of $\mathbb{R}$. In order to investigate the consistency of an estimator of τ we let the window W depend on "time" $n = 1, 2, \ldots$, in such a way that $|W_n| \to \infty$, as $n \to \infty$, where $|W_n|$ denotes the size (or Lebesgue measure) of W_n. In this set-up, a necessary condition for the existence of a consistent estimator (of τ) is that $\int_{\mathbb{R}} \lambda(s)\,ds = \mathrm{E}\,X(\mathbb{R}) = \infty$, which implies that P almost surely the point pattern $X(\omega)$ contains infinitely many points (cf. Rathbun and Cressie, 1994). Note that for cyclic λ the requirement $\int_{\mathbb{R}} \lambda(s)\,ds = \infty$ is automatically satisfied, provided the global intensity $\theta = \tau^{-1} \int_0^\tau \lambda(s)\,ds$ of the process X is positive. Therefore we will assume throughout that $\theta > 0$.

The aim of this paper is to propose and investigate a simple nonparametric estimator $\hat{\tau}_n$ of the period τ of a cyclic Poisson process X, using a single realization $X(\omega)$ of X, observed in the window W_n. Let Θ denote the parameter space, $\tau \in \Theta$, and let Θ be a bounded open interval in $\mathbb{R}^+$, such that no multiple of τ is contained in Θ. Our estimator $\hat{\tau}_n$ of τ is obtained as follows: for any $\delta \in \Theta$, define

$$Q_n(\delta) = \frac{1}{|W_n|} \sum_{i=1}^{N_{n\delta}} \left(X(U_{\delta,i}) - \frac{1}{N_{n\delta}} \sum_{j=1}^{N_{n\delta}} X(U_{\delta,j}) \right)^2 \tag{1.2}$$

with $N_{n\delta} = [|W_n|/\delta]$, denoting the (maximum) number of adjacent disjoint intervals $U_{\delta,i}$ of length δ in the window W_n; for convenience we denote by a_n and b_n the left- and right-endpoint of W_n, that is $W_n = [a_n, b_n]$, and require that the $U_{\delta,i}$'s are intervals of the form $[a_n + r + (i-1)\delta, a_n + r + i\delta)$ for some $r \in [0, (|W_n| - \delta N_{n\delta})]$. Otherwise the specific choice of r is free and basically of no importance (cf. condition (E)). Now we define

$$\hat{\tau}_n = \arg\min_{\delta\in\Theta} Q_n(\delta) \tag{1.3}$$

The parameter τ can also be estimated as follows: first estimate $k\tau$, for some positive integer k, satisfying $k = o(|W_n|)$, by $k\hat{\tau}_{n,k}$, which is given by

$$k\hat{\tau}_{n,k} = \arg\min_{\delta\in\Theta_k} Q_n(\delta) \tag{1.4}$$

and let $\hat{\tau}_{n,k}$ denote the resulting estimator of τ. Here $\Theta_k = (\tau_{k,0}, \tau_{k,1})$ is an open interval, such that no other multiple of τ than $k\tau$ is contained in Θ_k. Of course $\Theta_1 = \Theta$ and $\hat{\tau}_{n,1} = \hat{\tau}_n$. The restriction on the parameter space Θ_k of course requires some prior information about the value of $k\tau$. Note that $\hat{\tau}_{n,k}$ is not uniquely determined by (1.4), because $Q_n(\delta)$ may have flat parts. However, it can be checked that whatever specific choice of $\hat{\tau}_{n,k}$ is made, our results remain valid.

In practice one may obtain the estimator $\hat{\tau}_{n,k}$ by inspection of the graph of $Q_n(\delta)$. Because of relation (A.2) in the Appendix and the remark following Theorem 1.1 we know that typically a single realization of $Q_n(\delta)$ will behave approximately like a quadratic in a neighborhood of $k\tau$, provided $k = k_n \sim |W_n|^c$, for some $0 \le c < \frac{1}{3}$, whereas $Q_n(\delta)$ will attain larger values elsewhere (cf. relation (A.1) in the Appendix). Hence, one may as well obtain the $k\hat{\tau}_{n,k}$'s by minimising $Q_n(\delta)$ for $\delta \in (0, D_n)$, where D_n is of the order $|W_n|^{1/3}$; the requirement on Θ_k becomes superfluous.

Throughout we also assume that λ satisfies condition (E):

(E) *If there exists a $t \in (0,\tau)$ such that, for each $n \ge 1$, $\int_{U_{t,i}} \lambda(s)\,ds = t\theta$ with $U_{t,i} = [a_n + r + (i-1)t, a_n + r + it])$, for $i = 1, \ldots, N_{nt}$, then*

$$\nu\left\{r : \int_{U_{t,i}} \lambda(s)\,ds = t\theta,\ i = 1, \ldots, N_{nt}\right\} = 0$$

Condition (E) is only violated in exceptional cases. It is meant to guarantee that if $\Lambda_n(\delta) = EQ_n(\delta)$ attains its minimum value at $\theta + O(k/|W_n|)$, this implies that $\delta = k\tau$ for ν-almost every r; otherwise (A.1) may fail. It is satisfied in trigonometric models where λ can be expressed by a Fourier series. Note that (E) implies that λ is not constant a.e. $[\nu]$. This latter condition ensures that τ is identifiable, a necessary condition for the existence of a consistent estimator.

Our main result is:

Theorem 1.1. *Suppose λ is cyclic (with period τ) and Lipschitz. In addition assume that condition* (E) *is satisfied. Let $k = k_n \sim |W_n|^c$, for some $0 \leq c < \frac{1}{3}$. Then we have for any $\gamma < \frac{1}{2}$*

$$|W_n|^\gamma |\hat{\tau}_{n,k} - \tau| \underset{p}{\rightarrow} 0 \tag{1.5}$$

as $n \to \infty$.

The requirement $c < \frac{1}{3}$ in Theorem 1.1 ensures that the deterministic part $\Lambda_n(\delta)$ of $Q_n(\delta)$ dominates its purely random part $\tilde{Q}_n(\delta) = Q_n(\delta) - \mathrm{E}\, Q_n(\delta)$, whenever $\delta \in \Theta_k$; otherwise (1.5) would fail. Throughout the relation $k_n \sim |W_n|^c$ will mean that $\lim_{n\to\infty} k_n/|W_n|^c = 1$.

Vere-Jones (1982) proposed a periodogram estimate and obtained an almost sure rate of order $o(n^{-1})$ where $(0, n)$ denotes the observation window, provided λ admits a Fourier series with coefficients which are monotone decreasing, a condition which seems rather restrictive in our nonparametric framework.

In Helmers, Mangku and Zitikis (2003a) kernel type estimation of a cyclic intensity function λ (with unknown period τ) at a given point $s \in W$, using only a single realization $X(\omega)$ of the cyclic Poisson process X, is investigated. Using *any* estimator $\hat{\tau}_n$ of τ the estimator

$$\hat{\lambda}_{n,k}(s) = \frac{\hat{\tau}_n}{|W_n|} \sum_{k=-\infty}^{\infty} \int_{W_n} K\left(\frac{x - (s + k\hat{\tau}_n)}{h_n}\right) X(dx) \tag{1.6}$$

of $\lambda(s)$ is shown to be weakly consistent for any Lebesgue point s of λ, provided K is a bounded probability density function with support being a subset of $[-1, 1]$, and having at most a finite number of discontinuities, while

$$\frac{|W_n|}{h_n} |\hat{\tau}_n - \tau| \underset{p}{\rightarrow} 0 \tag{1.7}$$

as $h_n \downarrow 0$ and $h_n |W_n| \to \infty$. Statistical properties such as the MSE of $\hat{\lambda}_{n,k}$ are obtained in Helmers, Mangku and Zitikis (2003b). Helmers and Zitikis (1999) consider a uniform kernel type estimator for $\lambda(s)$ in the case where λ is a parametric function of spatial location. In the present paper and also in Helmers and Zitikis (1999), Helmers, Mangku and Zitikis (2003a, b) and Mangku (2001) λ is assumed to be fixed, but the observation window W_n expands, that is $|W_n| \to \infty$. This approach appears to be an useful one, since the size of W_n is often under the investigator's control.

To conclude this section we note that by taking $\hat{\tau}_n$ in (1.6) and (1.7) to be our estimator $\hat{\tau}_{n,k}$ (cf. (1.4)) one can check that assumption (1.7) is satisfied, provided $\hat{\tau}_{n,k}$ is computed using the whole information about X in the window W_n, while to construct $\hat{\lambda}_{n,k}(s)$ (cf. (1.6)) we use only the information about X in a smaller window $W_{0,n} \subset W_n$ of a size determined by the rate at which $h_n \downarrow 0$ (cf. Mangku, 2001, p. 106). Bebbington and Zitikis (2002) recently proposed some modifications of our estimator of τ, which appear to work more satisfactory in practice. In any case better methods than ours for estimating τ with high accuracy are desirable. The authors hope to pursue this problem elsewhere.

2. Proof of Theorem 1.1

To prove (1.5) we have to show that for any $\gamma < \frac{1}{2}$ and each $\epsilon > 0$

$$P(|W_n|^\gamma |\hat{\tau}_{n,k} - \tau| > \epsilon) \to 0, \quad \text{as } n \to \infty. \tag{2.1}$$

For each k satisfying $k = k_n = o(|W_n|)$, define the auxiliary quantity $\hat{\tau}_{n,k,s}$ by

$$k\hat{\tau}_{n,k,s} = \arg \min_{\delta \in \Phi_{n,k}} Q_n(\delta) \tag{2.2}$$

where $\Phi_{n,k} = (k\tau - \epsilon_n, k\tau + \epsilon_n)$ and ϵ_n is such that $\epsilon_n \downarrow 0$ as $n \to \infty$. Then

$$P(|W_n|^\gamma |\hat{\tau}_{n,k} - \tau| > \epsilon) \le P(|W_n|^\gamma |\hat{\tau}_{n,k,s} - \tau| > \epsilon) + P(\hat{\tau}_{n,k,s} \neq \hat{\tau}_{n,k}) \tag{2.3}$$

where

$$P(\hat{\tau}_{n,k,s} \neq \hat{\tau}_{n,k}) = P(|k\hat{\tau}_{n,k} - k\tau| \ge \epsilon_n) \tag{2.4}$$

with $\epsilon_n \downarrow 0$ as in (2.2). To prove that (2.4) converges to zero, as $n \to \infty$, we first note that by a standard argument

$$\begin{aligned} P(|k\hat{\tau}_{n,k} - k\tau| \ge \epsilon_n) &\le P\Big(\inf_{\delta \in \Theta_k \setminus \Phi_{n,k}} Q_n(\delta) \le Q_n(k_n\tau)\Big) \\ &\le P\Big(\min_{i:\delta_i \in \Theta_k \setminus \Phi_{n,k}} Q_n(\delta_i) - Q_n(k_n\tau) < \alpha_n\Big) \\ &\quad + P\Big(\max_{1 \le i \le L_n} \sup_{\delta_{i-1} < \delta \le \delta_i} |Q_n(\delta) - Q_n(\delta_i)| \ge \alpha_n\Big) \end{aligned} \tag{2.5}$$

for any $\alpha_n > 0$ (cf., e.g., Guyon, 1995, pp. 119–120). Here and elsewhere $\{\delta_i\}_{i=0}^{L_n}$ denotes a grid of equally spaced points in Θ_k, so that for any $\delta \in \Theta_k$ there exists δ_i such that $|\delta - \delta_i| \le \eta = |\Theta_k|/L_n$, for all $i = 1, \ldots, L_n$. Take

$L_n = [|W_n|^\beta]$ for some $\beta > 2$ (cf. also the appendix). The first probability on the r.h.s. of (2.5) does not exceed

$$\begin{aligned}(2.6)\qquad & \sum_{i,\delta_i \in \Theta_k \setminus \Phi_{n,k}} P(Q_n(\delta_i) - Q_n(k_n\tau) < \alpha_n) \\ & \qquad = \sum_{i,\delta_i \in \Theta_k \setminus \Phi_{n,k}} P\Big(\tilde{Q}_n(\delta_i) - \tilde{Q}_n(k_n\tau) < \alpha_n - \big(\Lambda_n(\delta_i) - \Lambda_n(k_n\tau)\big)\Big)\end{aligned}$$

where $\Lambda_n(\delta) = \mathrm{E}\,Q_n(\delta)$ and $\tilde{Q}_n(\delta) = Q_n(\delta) - \mathrm{E}\,Q_n(\delta)$. Relation (A.1) of Lemma A.1 in the Appendix tells us that $\Lambda_n(\delta_i) - \Lambda_n(k_n\tau) > c_1(\delta_i - k_n\tau)^2/k_n + O(k/|W_n|)$ for all $\delta_i \in \Theta_k \setminus \Phi_{n,k}$ and some constant $c_1 > 0$. With $\delta_i \in \Theta_k \setminus \Phi_{n,k}$ and a proper choice of ϵ_n (cf. the argument given after (2.7)) this directly yields that $\Lambda_n(\delta_i) - \Lambda_n(k_n\tau) > c_1\epsilon_n^2|W_n|^{-c}\big(1 + o(1)\big)$, so that with $\alpha_n = \frac{1}{2}c_1\epsilon_n^2|W_n|^{-c}$ we find that $\alpha_n - \big(\Lambda_n(\delta_i) - \Lambda(k_n\tau)\big)$ is negative for n large enough and its absolute value is bigger than $\frac{1}{4}c_1\epsilon_n^2|W_n|^{-c}$. Hence, the sum in (2.6) does not exceed

$$\begin{aligned}(2.7)\qquad & \sum_{i,\delta_i \in \Theta_{k_n} \setminus \Phi_{n,k_n}} P(|\tilde{Q}_n(\delta_i) - \tilde{Q}_n(k_n\tau)| > \tfrac{1}{4}c_1\epsilon_n^2|W_n|^{-c}) \\ & \qquad \le \sum_{i=1}^{L_n} P(|\tilde{Q}_n(\delta_i)| > \tfrac{1}{8}c_1\epsilon_n^2|W_n|^{-c}) + L_n P(|\tilde{Q}_n(k_n\tau)| > \tfrac{1}{8}c_1\epsilon_n^2|W_n|^{-c}).\end{aligned}$$

By applying Lemma A.2 in the Appendix twice this directly yields that the r.h.s. of (2.7) converges to zero, as $n \to \infty$, provided $0 \le c < \frac{1}{3}$ and $\epsilon_n \downarrow 0$ such that $\epsilon_n|W_n|^{(1-3c)/4} \to \infty$. Hence the first probability on the r.h.s. of (2.5) $\to 0$, as $n \to \infty$.

From Lemma A.3 in the Appendix we know that the second probability on the r.h.s. of (2.5), with $\alpha_n = \frac{1}{2}c_1\epsilon_n^2|W_n|^{-c}$, converges to zero, as $n \to \infty$, provided $0 \le c < 1$ and $\epsilon_n \downarrow 0$ such that $\epsilon_n|W_n|^{(1-c)/4} \to \infty$, as $n \to \infty$. Since for any $\epsilon_n \downarrow 0$ satisfying $\epsilon_n|W_n|^{(1-3c)/4} \to \infty$ we have $\epsilon_n|W_n|^{(1-c)/4} \to \infty$, the exponential bounds appearing in Lemmas A.2 and A.3 both converge to zero, as $n \to \infty$, provided $0 \le c < \frac{1}{3}$ and $\epsilon_n \downarrow 0$ at a slower rate than $|W_n|^{(3c-1)/4}$. This completes this proof that (2.4) converges to zero, whenever $k \sim |W_n|^c$, for some $0 \le c < \frac{1}{3}$. Therefore, the estimator $\hat{\tau}_{n,k}$, obtained by global minimisation of $Q_n(\delta)$ on Θ_k, can for our purposes be replaced by $\hat{\tau}_{n,k,s}$, the auxiliary quantity (2.2), defined by local minimisation of $Q_n(\delta)$ on $\Phi_{n,k}$. In other words, it remains to show that the first term on the r.h.s. of (2.3) converges to zero, that is, for any $\gamma < \frac{1}{2}$ and each $\epsilon > 0$

$$(2.8)\qquad P(|W_n|^\gamma|\hat{\tau}_{n,k,s} - \tau| > \epsilon) \to 0, \quad \text{as } n \to \infty.$$

To prove (2.8) we need a stochastic expansion for the purely random part $\tilde{Q}_n(\delta) = Q_n(\delta) - \Lambda_n(\delta)$ of $Q_n(\delta)$: for any positive integer k satisfying $k = o(|W_n|)$,

$$
(2.9)\quad \tilde{Q}_n(\delta) = \frac{2(\delta - k\tau)}{|W_n|} \sum_{i=1}^{N_{n\delta}} \widetilde{X}(U_{\delta,i}) \Big(\lambda\big(a_n+r+(i-1)(\delta-k\tau)\big) - N_{n\delta}^{-1} \sum_{j=1}^{N_{n\delta}} \lambda(a_n+r+(j-1)(\delta-k\tau)) \Big) + \frac{1}{|W_n|} \sum_{i=1}^{N_{n\delta}} \widetilde{X^2(U_{\delta,i})} + O_p\Big(\frac{(\delta - k\tau)^2}{|W_n|^{1/2}} + \frac{k}{|W_n|} \Big)
$$

as $n \to \infty$ and $\delta - k\tau \to 0$. The Lipschitz condition on λ is needed to obtain (2.9). Here and elsewhere we write $\widetilde{X}(U_{\delta,i})$ to indicate $X(U_{\delta,i}) - \mathrm{E}\, X(U_{\delta,i})$; in particular $\widetilde{X^2(U_{\delta,i})}$ means $\big(\widetilde{X}(U_{\delta,i})\big)^2 - \mathrm{E}\big(\widetilde{X}(U_{\delta,i})\big)^2$. In addition, the variances of the first two terms on the r.h.s. of (2.9) are—up to a factor $(\delta - k\tau)^2/|W_n|$ respectively $1/|W_n|$—given by

$$
(2.10)\quad \mathrm{Var}\Big(\frac{2}{|W_n|^{1/2}} \sum_{i=1}^{N_{n\delta}} \widetilde{X}(U_{\delta,i}) \Big(\lambda\big(a_n+r+(i-1)(\delta-k\tau)\big) - N_{n\delta}^{-1} \sum_{j=1}^{N_{n\delta}} \lambda\big(a_n+r+(j-1)(\delta-k\tau)\big) \Big) \Big) = \frac{4\theta}{\tau} \int_0^\tau (\lambda(s) - \theta)^2\, ds + O\Big(\frac{k}{|W_n|}\Big) + O\Big(|\delta - k\tau| + \frac{k}{|W_n||\delta - k\tau|} \Big)
$$

and

$$
(2.11)\quad \mathrm{Var}\Big(\frac{1}{|W_n|^{1/2}} \sum_{i=1}^{N_{n\delta}} \widetilde{X^2(U_{\delta,i})} \Big) = 2\theta^2\delta + \theta + O\Big(\frac{k^2}{|W_n|} + (\delta - k\tau)^2 \Big)
$$

and the covariance satisfies the relation

$$
(2.12)\quad \mathrm{cov}\Big(\frac{2}{|W_n|^{1/2}} \sum_{i=1}^{N_{n\delta}} \widetilde{X}(U_{\delta,i}) \Big(\lambda\big(a_n+r+(i-1)(\delta-k\tau)\big) - N_{n\delta}^{-1} \sum_{j=1}^{N_{n\delta}} \lambda(a_n+r+(j-1)(\delta-k\tau)) \Big), \frac{1}{|W_n|^{1/2}} \sum_{i=1}^{N_{n\delta}} \widetilde{X}^2(U_{\delta,i}) \Big)
$$

$$
= \frac{2(\delta - k\tau)}{k\tau} \Big(\frac{1}{\tau} \int_0^\tau \lambda^2(s)\, ds - \theta^2 \Big) + O\Big(\frac{k}{|W_n|}\Big) + O\Big((\delta - k\tau)^2 + \frac{k}{|W_n||\delta - k\tau|} \Big)
$$

as $n \to \infty$ and $\delta - k\tau \to 0$. Note that the $O(k/|W_n|\,|\delta - k\tau|)$ term appearing in (2.10) and (2.12) is negligible for our purposes, whenever $\delta - k\tau \to 0$ at a slower rate than $\frac{k}{|W_n|^{1/2}}$. We refer to Mangku (2001, pp. 126–138) for detailed proofs of (2.9)–(2.12).

Next we approximate the r.v.'s appearing in (2.10) and (2.11)—up to remainder terms of lower order—by

$$(2.13)\quad \frac{2}{|W_n|^{1/2}} \sum_{j=1}^{M_{n,\delta,k\tau}} \sum_{i=(j-1)J_{\delta,k\tau}+1}^{jJ_{\delta,k\tau}} \widetilde{X}(U_{\delta,i})\big(\lambda(a_n + r + (i-1)(\delta - k\tau)) - \theta + O\Big(|\delta - k\tau| + \frac{k}{|W_n||\delta - k\tau|}\Big)\Big)$$

and

$$(2.14)\quad \frac{1}{|W_n|^{1/2}} \sum_{j=1}^{M_{n,\delta,k\tau}} \sum_{i=(j-1)J_{\delta,k\tau}+1}^{jJ_{\delta,k\tau}} \widetilde{\widetilde{X}^2(U_{\delta,i})}$$

respectively, where $J_{\delta,k\tau} = [\tau/|\delta - k\tau|]$ and $M_{n,\delta,k\tau} = [|W_n|\,|\delta - k\tau|/k\tau^2]$, as $n \to \infty$ and $\delta - k\tau \to 0$. Since λ is cyclic with period τ and Lipschitz, the collection $\widetilde{X}(U_{\delta,i})\lambda\big(a_n + r + (i-1)(\delta - k\tau)\big)$ respectively $\widetilde{\widetilde{X}^2(U_{\delta,i})}$, $i = 1, \ldots, N_{n\delta}$, is splitted into $M_{n,\delta,k\tau}$ blocks, such that within each block the sum of the $J_{\delta,k\tau}$ r.v.'s corresponding to this block has the same distribution for each of the $M_{n,\delta,k\tau}$ blocks, up to a remainder term, negligible for our purposes. Application of Theorem 9.3.1 in Dudley (1999) to (2.13) and (2.14) directly yields that on some probability space there exist, for any positive integer k, $M_{n,\delta,k\tau}$ $\mathbb{R}^2$-valued independent r.v.'s Z_{jk}, with the same law as

$$\left(\left(\frac{|\delta - k\tau|}{k}\right)^{1/2} \sum_{i=1}^{J_{\delta,k\tau}} \widetilde{X}(U_{\delta,i})\lambda\big(a_n + r + (i-1)(\delta - k\tau)\big),\ \left(\frac{|\delta - k\tau|}{k^2}\right)^{1/2} \sum_{i=1}^{J_{\delta,k\tau}} \widetilde{\widetilde{X}^2(U_{\delta,i})}\right)$$

and Y_{jk} independent mean zero r.v.'s with bivariate normal law with variances c_1^2 and c_2^2, where

$$c_1^2 = \theta\tau \int_0^\tau (\lambda(s) - \theta)^2 ds, \quad c_2^2 = 2\theta^2\tau^3$$

and covariance

$$c_{12k} = \frac{(\delta - k\tau)}{k^{3/2}}\left(\int_0^\tau \lambda^2(s)\, ds - \theta^2\tau\right),$$

such that

$$\left\| \sum_{j=1}^{M_{n,\delta,k\tau}} Z_{jk} - \sum_{j=1}^{M_{n,\delta,k\tau}} Y_{jk} \right\| = o_p\left(\left(\frac{|W_n|}{k} |\delta - k\tau| \right)^{1/2} \right), \tag{2.15}$$

as $n \to \infty$ and $|\delta - k\tau| \to 0$. Here $\| \ \|$ denotes the Euclidian norm in $\mathbb{R}^2$.

Define $\bar{\tau}_{n,k,s}$ by $k\bar{\tau}_{n,k,s} = \arg\min_{\delta \in \Phi_{n,k}} \bar{Q}_n(\delta)$, with

$$\begin{aligned} \bar{Q}_n(\delta) = {} & \frac{(\delta - k\tau)^2}{k\tau^2} \int_0^\tau (\lambda(s) - \theta)^2 \, ds \\ & + \frac{(\delta - k\tau)}{|W_n|^{1/2}} \left\{ \left(\frac{4\theta}{\tau} \int_0^\tau (\lambda(s) - \theta)^2 \, ds \right)^{1/2} T_1 + \frac{\theta^{3/2}}{(2\theta k\tau + 1)^{1/2}} T_2 \right\} \\ & + \frac{(2\theta^2 k\tau + \theta)^{1/2}}{|W_n|^{1/2}} T_2 + \theta \end{aligned} \tag{2.16}$$

where T_j, $j = 1, 2$ denote standard normal r.v.'s, with vanishing correlation as $|\delta - k\tau| \to 0$.

Note that $\bar{Q}_n(\delta)$ is nothing but the sum of the quadratic approximation to $\Lambda_n(\delta) = \mathrm{E}\, Q_n(\delta)$ (cf. Lemma A.1) and a "normal approximation" based on (2.15) (cf. also (2.10)–(2.12)) to $\tilde{Q}_n(\delta) = Q_n(\delta) - \mathrm{E}\, Q_n(\delta)$. Minimising the quadratic function $\bar{Q}_n(\delta)$ w.r.t. $(\delta - k\tau)$ yields a stochastic expansion for $\bar{\tau}_{n,k,s} - \tau$:

$$\begin{aligned} & |W_n|^{1/2} (\bar{\tau}_{n,k,s} - \tau) \\ & \quad = - \frac{\tau^{3/2} \theta^{1/2} T_1}{(\int_0^\tau (\lambda(s) - \theta)^2 \, ds)^{1/2}} - \frac{\tau^2 \theta^{3/2} T_2}{2(2\theta k\tau + 1)^{1/2} \int_0^\tau (\lambda(s) - \theta)^2 \, ds} \end{aligned} \tag{2.17}$$

so that $\bar{\tau}_{n,k,s} - \tau = O_p(|W_n|^{-1/2})$, which directly yields (1.5) with $\bar{\tau}_{n,k,s}$ instead of $\hat{\tau}_{n,k}$.

Note that the error of the normal approximation (2.15) will not affect the rate of convergence $|W_n|^{-1/2}$ of $\bar{\tau}_{n,k,s}$ to τ, because it amounts to replacing T_j by $T_j + o_p(1)$, $j = 1, 2$ in (2.16) and (2.17); let $\bar{Q}'_n$ denote $\bar{Q}_n$ (cf. (2.16)) with T_j replaced by $T_j + o_p(1)$, $j = 1, 2$.

It remains to verify that the transition from $\bar{\tau}_{n,k,s}$ to $\hat{\tau}_{n,k,s}$ (cf. (2.2)) is of negligible order, that is, for any $\gamma < \frac{1}{2}$,

$$\hat{\tau}_{n,k,s} - \bar{\tau}_{n,k,s} = o_p(|W_n|^{-\gamma}). \tag{2.18}$$

To see this note first that a simple calculation using (2.16) yields

$$\begin{aligned} & \bar{Q}'_n\left(\delta + o_p\left(\frac{k}{|W_n|^\gamma} \right) \right) \\ & \quad = \bar{Q}'_n(\delta) + o_p\left(\frac{k}{|W_n|^{\gamma + 1/2}} \right) + o_p\left(\frac{|\delta - k\tau|}{|W_n|^\gamma} \right) + o_p\left(\frac{k}{|W_n|^{2\gamma}} \right) \end{aligned} \tag{2.19}$$

as $n \to \infty$ and $|\delta - k\tau| \to 0$. Combining now (2.9) with (A.2) directly gives

$$Q_n(\delta) = \bar{Q}'_n(\delta) + R_n(\delta) \tag{2.20}$$

where

$$R_n(\delta) = O_p\left(\frac{|\delta - k\tau|k^{1/2}}{|W_n|} + \frac{(\delta - k\tau)^2}{|W_n|^{1/2}} + \frac{k}{|W_n|}\right) + O\left(\frac{|\delta - k\tau|^3}{k}\right). \tag{2.21}$$

The three random terms on the r.h.s. of (2.21) are easily seen to be of the order of the remainder term in (2.19), whenever $|\delta - k\tau| = O(\epsilon_n)$, whereas the deterministic error term $|\delta - k\tau|^3/k$ (cf. (A.2)) can be incorporated with impunity in the first term on the r.h.s. of (2.16). The latter change will not affect the order of magnitude $|W_n|^{-1/2}$ of $\bar{\tau}_{n,k,s} - \tau$. We can conclude that minimisation of $Q_n(\delta)$ on $\Theta_{n,k}$ behaves asymptotically like minimisation of $\bar{Q}'_n(\delta + o_p(k/|W_n|^\gamma))$ w.r.t. δ, that is (2.18) holds true. This completes the proof of Theorem 1.1. □

APPENDIX

In this appendix we provide some technical tools—in the form of three lemmas—which were needed in the proof of Theorem 1.1. In our first lemma we show that $\Lambda_n(\delta) - \Lambda_n(k\tau) = \mathrm{E}(Q_n(\delta) - Q_n(k\tau))$ behaves like a quadratic in a neighborhood of $k\tau$, whenever $k = o(|W_n|^{1/2})$.

Lemma A.1. *Suppose that λ is cyclic and Lipschitz. Moreover, assume that condition* (E) *is satisfied. Then, for ν-almost every r and any positive integer k satisfying $k = o(|W_n|^{1/2})$,*

$$\Lambda_n(\delta_i) - \Lambda_n(k\tau) > c_1(\delta_i - k\tau)^2/k + O\left(\frac{k}{|W_n|}\right) \tag{A.1}$$

for all $\delta_i \in \Theta_k \setminus \Phi_{n,k}$ and all $n \geq n_0$, for some positive integer n_0 and constant $c_1 > 0$. In addition, we have

$$\begin{aligned} &\Lambda_n(\delta) - \Lambda_n(k\tau) \\ &\quad = \frac{(\delta - k\tau)^2}{k\tau^2} \int_0^\tau (\lambda(s) - \theta)^2\, ds + O\left(\frac{k}{|W_n|}\right) + O\left(\frac{|\delta - k\tau|^3}{k}\right) \end{aligned} \tag{A.2}$$

as $n \to \infty$ and $\delta - k\tau \to 0$.

Proof. See Mangku (2001, pp. 153–157). □

In the next lemma we derive an exponential probability bound for the $\tilde{Q}_n(\delta_i)$'s, $i = 1, \ldots, L_n$. The proof uses the fact that the $\widetilde{X^2(U_{\delta,i})}$, $i = 1, \ldots, L_n$ are easily shown to be sub-Gaussian r.v.'s so that Lemma 8.2 of van de Geer (2000) can be applied.

Lemma A.2. *Let $L_n = [|W_n|^\beta]$ for some $\beta \geq 0$. Suppose that λ is cyclic and locally integrable, then, for each integer $k = k_n \sim |W_n|^c$ for some $0 \leq c < \frac{1}{3}$ and for each sequence $\epsilon_n \downarrow 0$, there exists a positive constant C_1 such that, as $n \to \infty$,*

$$\text{(A.3)} \qquad \sum_{i=1}^{L_n} P(|\tilde{Q}_n(\delta_i)| \geq \epsilon_n^2 |W_n|^{-c}) = O\big(\exp(-C_1 \epsilon_n^4 |W_n|^{1-3c})\big)$$

with $\delta_i \in \Theta_k$, $i = 1, \ldots, L_n$.

Proof. A simple calculation (cf. Mangku, 2001, p. 111) shows that

$$\text{(A.4)} \quad \tilde{Q}_n(\delta_i) = \frac{1}{|W_n|} \sum_{j=1}^{N_{n\delta_i}} \widetilde{X^2(U_{\delta_i,j})} - \frac{1}{|W_n| N_{n\delta_i}} \widetilde{X^2(W_{N_{n\delta_i}})} + O(|W_n|^{-1}) \widetilde{X}(W_{N_{n\delta_i}}),$$

where $W_{N_{n\delta_i}} = \cup_{j=1}^{N_{n\delta_i}} U_{\delta,j}$.

To obtain an exponential bound for the first term on the r.h.s. of (A.4) we apply Lemma 8.2 of van de Geer (2000) to find that

$$\text{(A.5)} \quad \sum_{i=1}^{L_n} P(|W_n|^{-1} | \sum_{j=1}^{N_{n\delta_i}} \widetilde{X^2(U_{\delta_i,j})}| \geq \frac{1}{3} \epsilon_n^2 |W_n|^{-c}) \leq 2\exp(-C_1 \epsilon_n^4 |W_n|^{1-3c})$$

for some constant $C_1 > 0$. This bound together with a similar bound—namely $2\exp(-C_2 \epsilon_n^2 |W_n|^{1-2c})$—for the other two terms in (A.4) (cf. Reiss, 1993, p. 222), completes the proof. □

In the third and final lemma we give an exponential probability bound for the modulus of continuity of the Q_n-process.

Lemma A.3. *Let $L_n = [|W_n|^\beta]$ for some $\beta > 2$. Suppose that λ is cyclic and bounded, then, for any integer $k = k_n \sim |W_n|^c$ for some $0 \leq c < 1$ and for each sequence $\epsilon_n \downarrow 0$, there exists a positive constant C_2 such that*

$$\text{(A.6)} \quad P\Big(\max_{1 \leq i \leq L_n} \sup_{\delta_{i-1} < \delta \leq \delta_i} |Q_n(\delta) - Q_n(\delta_i)| \geq \epsilon_n^2 |W_n|^{-c}\Big) = O\big(\exp(-C_2 \epsilon_n^2 |W_n|^{(1-c)/2})\big)$$

as $n \to \infty$.

Proof. See Mangku (2001, pp. 114–123, 125). □

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Roelof Helmers
CWI
P.O.Box 94079
1090 GB Amsterdam
The Netherlands
helmers@cwi.nl

I Wayan Mangku
Bogor Agricultural University
Jl. Raya Pajajaran Bogor
16144 Indonesia