TOETJES NA

THOMAS S. FERGUSON
University of California
at Los Angeles

CHRISTIAN GENEST
Université Laval

Toetjes is a constant-sum game of perfect information in which players guess in turn the outcome of a random variable X with known distribution. The contestants, who play in a predetermined order, know the values announced by their predecessors but must make a different choice. The winner is that participant whose guess is closest to the value eventually taken by X. Feder (*Amer. Math. Monthly*, 1990), who investigated Toetjes as a non-cooperative game when X is uniformly distributed, observed that it provides a clear advantage to the first guesser, while the last player is at a disadvantage. It is shown here that this result extends to more general densities, but that in contrast, Toetjes is most favorable to the last player when the contestants can form alliances. It is also shown that the last person who speaks has a strong advantage in a version of Toetjes related to the American television show *The Price is Right*, in which the winner is the contestant whose bid is closest to, but less than, the unknown retail price X of an important collection of consumer goods.

1. Introduction

In Dutch, "toetjes" means "afters," or dessert. In the work of Feder (1990), it also refers to a game that is played around many family dinner tables in the world, at times when a single leftover piece of cake needs to be allocated to one of the children. In the continuous version of the game considered by Feder, a parent secretly picks a number X at random in the interval $[0, 1]$ and one by one, in a predetermined order, the $n \geq 2$ children make their guesses known to all. Child 1 thus selects a number x_1 and, knowing $x_1, \ldots, x_{i-1}$, Child $i \geq 2$ chooses x_i distinct from all previous guesses. Once everybody has spoken, the parent announces whose guess is closest, and this child gets the treat. The others get nothing.

The question addressed by Feder (1990) is that of determining the guess that each child should make in order to maximize his/her probability of winning, knowing that all children do likewise. Feder introduces "tie-breaking" rules to require a specific action when several possible guesses give a child the same maximal probability of win. This paper revisits the issue—"toetjes na" means "after afters" in colloquial Dutch and can be interpreted either as "extra dessert" or "more (on) Toetjes"—in order to characterize the set of optimal strategies from a game-theoretic viewpoint and to explore what happens in situations where the parent's secret number is drawn from other continuous distributions than the uniform.

Toetjes with $n = 2$ players is briefly discussed in Section 2, where it is seen to be a game of perfect information that is essentially fair to rational contestants who engage non-cooperatively in it. Somewhat unexpectedly, however, it turns out that when there are $n \geq 3$ players, Toetjes often proves least beneficial to the person who announces his/her guess last. The case $n = 3$ is considered in detail in Section 3, where the set of equilibrium payoff vectors is characterized when X has a continuous density f that is either uniform on a bounded interval, strictly decreasing on a possibly unbounded interval, or unimodal and symmetric on the real line. More generally, it is seen in Section 4 that when f is strictly decreasing, the probability of winning is a non-increasing function of player number in the strategy leading to the unique, perfect equilibrium of the n-player version of the game.

In Section 5, Toetjes is also treated alternatively as a cooperative game with transferable utility. From the point of view of the Shapley value, the early players are seen to be at a disadvantage. In particular, the ability of the last contestant to make threats makes him/her very valuable in forming coalitions and confers to him/her a significant edge.

Closely related to Toetjes is a three-player version of "The Showcase Showdown" that concludes the American television show *The Price is Right.* In the latter, participants must guess the retail price X of a large collection of consumer durables. The winner is the person whose bid is closest to X without going over, and in the event that all bids (which must be different) exceed X, the process is repeated. A brief analysis of this game is mentioned in the last section, along with a few disjointed remarks regarding other possible extensions of Toetjes. So long as all players have access to the same information, *The Price is Right* turns out to be simpler than Toetjes in that the equilibrium payoff does not depend on the distribution of X, provided that it is continuous. In a three-player version of "The Showcase Showdown," the third contestant is clearly favored. Other interesting aspects of *The Price is Right* are discussed by Even (1966), Coe and Butterworth (1995), Berk, Hughson and Vandezande (1996), Grosjean (1998) and Biesterfeld (2001).

2. The two-person game

Since it is common knowledge that contestants share the same distribution for X and do not have access to any private data, Toetjes is a game of complete information. This is in contrast, e.g., with "second guessing" contests investigated by Steele and Zidek (1980), Pittenger (1980) and Hwang and Zidek (1982), in which two persons are challenged to guess the weight X of a given object. In the latter context, the participants have a common prior for X but also get to see the object, from which they derive private hunches, say G_1, G_2, about its weight. Assuming, as these authors do, that Player 1 announces G_1 as his/her guess and that Player 2 knows this, the latter can turn this extra information into a substantial advantage. When the G_i's are conditionally independent given X and have the same symmetric distribution centered at the true weight, for instance, Steele and Zidek (1980) show that Player 2 can achieve a success probability as large as $\frac{3}{4}$ by guessing a value infinitely close to, and either to the left or to the right of G_1, according as $G_2 < G_1$ or $G_2 > G_1$.

In the two-player version of Toetjes, neither player is at an advantage, because they share exactly the same information. When X is uniform on an interval $[A, B]$, or simply $[0, 1]$ without loss of generality, it is clear that the optimal choice for Player 1 is $x_1 = \frac{1}{2}$, since he/she can then guarantee a payoff of $\frac{1}{2}$ for him/herself. Indeed if $x_1 < \frac{1}{2}$, say, Player 2 could then play $x_1 + \epsilon < \frac{1}{2}$ for some suitably small $\epsilon > 0$, and Player 1's probability of winning would be $x_1 + \epsilon/2 < \frac{1}{2}$. Of course, the smaller ϵ, the higher the payoff for Player 2.

This simple example brings out a difficulty with the game, namely the fact that there may not always exist an optimal move. To overcome this problem, Feder (1990) defines "limiting plays" which are such that all players can obtain payoffs arbitrarily close to optimal by picking numbers sufficiently close to the limit. To keep things compact, the notations x^- and x^+ are used below to refer to a point infinitesimally smaller or greater than x. Expressions such as $(x^-)^-$, $(x^+)^+$ and the like may also be defined unambiguously when needed.

With these conventions, the only optimal moves in the two-player version of Toetjes with X uniform are $x_1 = (A+B)/2$ and x_2 equals x_1^- or x_1^+. Similarly if X is assumed to have a continuous distribution F, so that ties occur with zero probability, Player 1 can guarantee a return of $\frac{1}{2}$ by choosing x_1 to be a median of F. Player 2 again chooses x_2 to be x_1^- or x_1^+. The limiting payoffs are thus $p_1 = p_2 = \frac{1}{2}$ in all cases, as Hotelling (1929) had already pointed out a long time ago in an economic context where two competitive vendors selling the same commodity seek optimal business locations on an idealized linear town where customers distributed according to F are equally interested in their product and would always buy it from the closest outlet.

3. Three-person games

When Toetjes is played with $n \geq 3$ children, the game's sets of payoffs and equilibrium strategies depend strongly on the density f of the parent's secret random number X. Three cases are considered in turn, namely those where f is (i) uniform on a bounded interval; (ii) strictly decreasing on a possibly unbounded interval; and (iii) both symmetric and unimodal on the reals.

3.1. Uniform density on a bounded interval

Suppose that X is uniform on a bounded interval $[A, B]$. It may again be assumed without loss of generality that $A = 0$ and $B = 1$. Since each contestant's strategy may depend on previous guesses, the pure strategies for Players 1, 2 and 3 are most conveniently denoted by $x_1 = x$, $x_2 = y(x)$ and $x_3 = z(x, y)$, respectively.

Of prime interest is the set of perfect equilibrium points, $(x, y(\cdot), z(\cdot,\cdot))$ and associated payoffs (p_1, p_2, p_3). For the equilibrium to be perfect, $z(x, y)$ must be a best response to $(x, y(x))$, and $y(x)$ must be a best response to x, knowing Player 3 will use $z(x, y)$. The optimal payoff to Player 3 is symmetric in x and y, so in finding his/her best guess, attention may be restricted to $x \le y$, as is done below.

Observe that given values of x and y, Player 3 is indifferent between all points in $[x, y]$, since his/her payoff is $|y - x|/2$ no matter which point he/she picks in that interval. This choice can affect the payoff of the previous players, however, and hence their strategy. Accordingly, Feder (1990) speaks of the need for a criterion that all participants would use to choose among equally desirable moves. Under the "play closest to Player 1" tie-breaking rule, he shows (this is his Corollary 1) that the optimal limiting plays and associated payoffs are

$$(x_1, x_2, x_3) = (\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{4}+) \quad \text{and} \quad (p_1, p_2, p_3) = (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4}),$$

while under the "play rightmost" rule, he finds (Corollary 2)

$$(x_1, x_2, x_3) = (\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{3}{4}-) \quad \text{and} \quad (p_1, p_2, p_3) = (\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}).$$

As a complement to Feder's work, note that the set of equilibrium payoffs is of the form

$$(p_1, p_2, p_3) = \left(\frac{1+\alpha}{4}, \frac{2-\alpha}{4}, \frac{1}{4}\right) \tag{3.1}$$

with α running in the interval $[0, 1]$. As can be seen, therefore, Player 3 is always at a disadvantage.

To derive Equation (3.1), first observe that for given $x < y$, Player 3 need consider only three possibilities: $z = x^-$, $z = y^+$, and z at some arbitrary point $t \in [x^+, y^-]$. The (limiting) expected returns for these possibilities are, x, $1 - y$, and $(y - x)/2$, respectively. Player 3 will thus choose that response with the maximum payoff, viz.

$$z(x,y) = \begin{cases} x^- & \text{if } x > 1 - y \text{ and } 3x > y, \\ y^+ & \text{if } 1 - y > x \text{ and } 3(1 - y) > (1 - x), \\ t & \text{if } 3x < y \text{ and } 3(1 - y) < (1 - x). \end{cases}$$

The vector (p_1, p_2, p_3) of payoffs in these three regions is given by

$$\begin{cases} \left(\frac{y-x}{2}, 1-\frac{x+y}{2}, x\right) & \text{if } x > 1-y \text{ and } 3x > y, \\ \left(\frac{x+y}{2}, \frac{y-x}{2}, 1-y\right) & \text{if } 1-y > x \text{ and } 3(1-y) > (1-x), \\ \left(\frac{x+t}{2}, 1-\frac{y+t}{2}, \frac{y-x}{2}\right) & \text{if } 3x < y \text{ and } 3(1-y) < (1-x). \end{cases}$$

On the boundaries of the regions, either of the strategies, optimal on either side of the boundary, is a best response. In particular, the point $(x, y) = (\frac{1}{4}, \frac{3}{4})$ is on the boundary of all three regions, so x^-, y^+ and t are all best responses.

Consider first the response by Player 3 that is "fair" in the sense that it treats the other two players equally. For $x < y$, this strategy satisfies

$$z_*(x,y) = \begin{cases} x^- & \text{if } x > 1-y \text{ and } 3x > y, \\ y^+ & \text{if } 1-y > x \text{ and } 3(1-y) > (1-x), \\ t(x,y) & \text{if } 3x \le y \text{ and } 3(1-y) \le (1-x), \\ \delta & \text{if } x = 1-y \text{ and } \frac{1}{4} < x \le \frac{1}{2}, \end{cases}$$

where $t(x, y) = (x + y)/2$, and δ is the randomized strategy that chooses x^- and y^+ with probability $\frac{1}{2}$ each.

If Player 2 knows that Player 3 will proceed in this fashion, his/her best response to $x < \frac{1}{2}$ from Player 1 is

$$y_*(x) = \begin{cases} \frac{x+2}{3} & \text{if } x \le \frac{1}{4}, \text{ giving value } \left(\frac{1+5x}{6}, \frac{1-x}{2}, \frac{1-x}{3}\right), \\ (1-x)^+ & \text{if } \frac{1}{4} < x < \frac{1}{2}, \text{ giving value } \left(\frac{1}{2}-x, \frac{1}{2}, x\right). \end{cases}$$

As a function of x, the first component of this value is maximized at $x = \frac{1}{4}$. Therefore, $\left(\frac{1}{4}, y_*(\cdot), z_*(\cdot, \cdot)\right)$ is a perfect equilibrium with value $(\frac{3}{8}, \frac{3}{8}, \frac{1}{4})$.

More generally, the perfect equilibrium payoff (3.1) obtains by changing $t(x, y)$ to

$$t(x, y) = (1 - \alpha)x + \alpha y$$

for some $0 \le \alpha \le 1$ in the definition of z_*. Feder's tie-breaking rules correspond to the choices $\alpha = 0$ ("closest to Player 1") and $\alpha = 1$ ("rightmost"), respectively.

3.2. Strictly decreasing density on a possibly unbounded interval

Suppose that X has an absolutely continuous distribution F with strictly decreasing density on $[A, B)$, where B can be finite or infinite. Continue to denote by x and y the guesses of Players 1 and 2, and assume for the sake of argument that $A < x < y$. Should Player 3 decide to put $x_3 = z$ between x and y, then it would be best for him/her to choose it as close to x as possible. Thus the three choices for Player 3 would be $z = x^-$, $z = x^+$ and $z = y^+$. This argument shows that, of the first two players, he/she who picks the smaller number is at a disadvantage because if Player 3 plays between them, he/she is obliged by perfectness to play close to the lower number.

In view of the above, Player 1 will choose x large, say around the third quartile. Then Player 2 will choose y small, typically around the first quartile, to make Player 3 indifferent between choosing $z = y^-$ and $z = y^+$. The value of y will thus be chosen so that

$$F(y) = F\left(\frac{x+y}{2}\right) - F(y). \tag{3.2}$$

It is easy to check that only one number $y_x \in [A, x]$ meets these conditions for given $x > 0$. Consequently, the smallest choice of x that Player 1 can get away with is such that

$$F(y_x) = 1 - F(x). \tag{3.3}$$

Because y_x is a monotone increasing function of x, the solution to this equation may also be seen to be unique. There is, therefore, a single Nash equilibrium point.

Figure 1 illustrates the solution in the case where X is exponentially distributed with unit mean. In this case, numerical work leads to $x = 1.2859$ and $y = .3235$, which correspond to the 72.36th and the 27.64th percentile, respectively. When expressed in the latter terms, these moves are actually optimal whatever the mean of the exponential distribution. The corresponding perfect equilibrium payoff vector is

$$(p_1, p_2, p_3) = (.4472, .2764, .2764).$$

Thus, Player 1 enjoys a substantial advantage here, while Players 2 and 3 are ex aequo.

As a second example, consider the case where X follows a Pareto distribution with parameter $\alpha \geq 1$, i.e., $F(x) = 1 - 1/x^\alpha$ for $x \in [1, \infty)$. A simple calculation shows that when $\alpha = 1$, say, the only solution to (3.2) is

$$y_x = \frac{-x + \sqrt{x^2 + 8x}}{2},$$

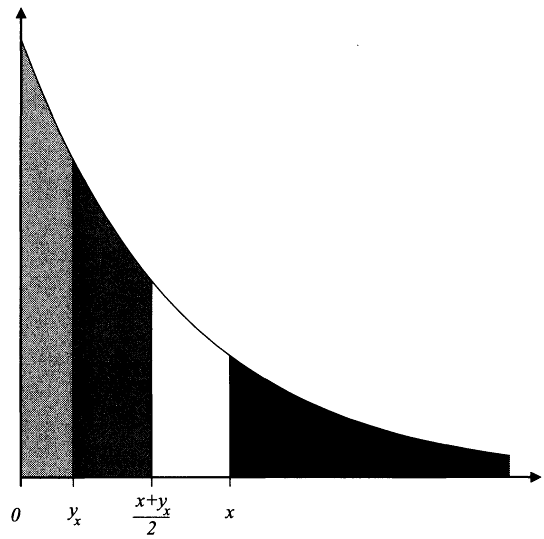

Figure 1. Representation of the unique perfect equilibrium payoff for Toetjes with $n = 3$ players when the density of X is strictly decreasing on a possibly unbounded interval. The limiting optimal moves for Players 1 and 2, respectively denoted by x and $y = y_x$, are such that the area to the right of x is equal to the area to the left of y_x and to the area between y_x and $(x + y_x)/2$.

which is indeed monotone increasing in x on the entire support. Substitution into (3.3) then yields $x = 2 + \sqrt{2}$ and $y_x = \sqrt{2}$, so that in this case, Players 1 and 2 would play approximately at the 70.71th and at the 29.29th percentile, respectively. The perfect equilibrium payoff vector is then

$$(p_1, p_2, p_3) = (.4142, .2929, .2929).$$

Numerical solutions for other integer values of α are given in Table 1, where Player 1 is seen to increase his/her advantage as α gets larger. When expressed in terms of quantiles, of course, the moves given in the table continue to be optimal for the more general Pareto distribution $F(x) = 1 - (\beta/x)^\alpha$ with scale parameter $\beta > 0$ and support $[\beta, \infty)$.

3.3. Symmetric and unimodal density on the real line

Suppose that the density of X may be written as $f(x) = f_0(x - m)$ in terms of a function f_0 that is symmetric about 0 and strictly decreasing on $[0, \infty)$. One may as well assume, without loss of generality, that $m = 0$ is the mode, so that $f(x) = f(-x) > 0$ for all possible values of x. In this case, it is easy to convince oneself that the best option for the first two players is to choose points that are symmetric about 0, say $x_1 = x < 0$ and $x_2 = y = -x > 0$. Then the best response for Player 3 is either at one of the end points, namely $z = x^-$ or $z = y^+$, or, when it is between x and y, it will be at 0.

Table 1. Player 1's optimal move, x, and corresponding percentile of the distribution of X when the latter follows a Pareto distribution $F(x) = 1 - 1/x^\alpha$ for $x \geq 1$ with $\alpha = 1, \ldots, 10$ and $\alpha = 100$; Player 2's optimal move is $y_x = (1 - 1/x^\alpha)^{-1/\alpha}$, and his/her probability of win is half the complement of Player 1's equilibrium payoff, displayed in the right-most column.

Parameter value	Player 1's optimal guess x	Corresponding percentile	Player 1's equilibrium payoff
1	3.4142	70.71	.4142
2	1.8712	71.44	.4288
3	1.5235	71.72	.4344
4	1.3731	71.87	.4374
5	1.2896	71.97	.4393
6	1.2366	72.03	.4407
7	1.1999	72.07	.4415
8	1.1731	72.11	.4424
9	1.1526	72.14	.4429
10	1.1364	72.16	.4432
100	1.0129	72.34	.4449

Thus, to make Player 3 indifferent between these three choices, Player 1's guess must be such that the events $X < x$ and $x/2 < X < -x/2$ have the same probability. This leads to the equation,

$$F(x) = F(-x/2) - F(x/2),$$

whose solution is unique. Refer to Figure 2 for an illustration.

In the special case where F is the Laplace (or double exponential) distribution, for example, one finds

$$x = 2\log(\sqrt{3} - 1) = -0.6238,$$

which means that Player 1 should guess the 26.79th percentile, approximately. The perfect equilibrium payoff is then

$$(p_1, p_2, p_3) = (.36605, .36605, .2679).$$

Additional illustrations are provided in Table 2 for the normal and the Student distribution with various degrees of freedom. In all these examples, Players 1 and 2 have identical expected returns and as in the uniform case, Player 3 is at a disadvantage.

4. The n-person game with a strictly decreasing density

The search for optimal strategies in a game of Toetjes with $n > 3$ players is quite complicated in general. Subject to the same specific "tie-breaking" rules mentioned in Section 3.1, Feder (1990) gave a solution in the case where the secret number X is selected uniformly in a bounded interval $[A, B]$.

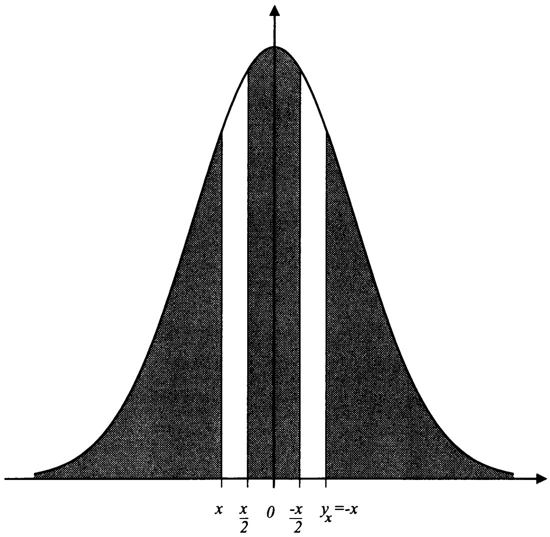

Figure 2. Representation of the unique perfect equilibrium payoff for Toetjes with $n = 3$ players when the density of X is symmetric and unimodal on the real line and Player 1 chooses to play at $x < 0$ while Player 2 picks $y_x = -x$. These moves are optimal whenever the area to the left of $x < 0$ is equal to the area between $-x/2$ and $x/2$.

The discussion for three players given in Sections 3.2 and 3.3 provides indications as to what the general form of the solution is for $n > 3$ players when the density of X is strictly decreasing on a possibly unbounded interval, or unimodal and symmetric on the real line. Only the former case is considered here.

Assuming that f is strictly decreasing on $[A, B)$ with B possibly infinite, the first $n-1$ contestants are expected to choose a pattern of points to make Player n indifferent as to which region he/she plays in. If he/she plays in any region other than the left most, he/she will play at the extreme left of the region to obtain the highest coverage probability.

Suppose for example that $n = 4$ and that f is the standard exponential distribution. The particular configuration sought by the first three players is a set a, b, c with $a < b < c$, such that

$$1 - e^{-a} = e^{-a} - e^{-(a+b)/2} = e^{-b} - e^{-(b+c)/2} = e^{-c}.$$

These four quantities are the four coverage probabilities if Player 4 chooses a^-, a^+, b^+ and c^+, respectively. The solution of these equations is easily found to be $(a, b, c) = (.2052, .7722, 1.6846)$, and the common coverage probability is $e^{-c} = .1855$. This breaks the positive axis into six regions, viz.

$$[0, a), \quad \left(a, \frac{a+b}{2}\right), \quad \left(\frac{a+b}{2}, b\right), \quad \left(b, \frac{b+c}{2}\right), \quad \left(\frac{b+c}{2}, c\right), \quad (c, \infty)$$

Table 2. Player 1's optimal move x and corresponding percentile of the distribution of X when the latter follows a Student distribution with $d = 1, 2, 3, 4, 5$, and 10 degrees of freedom; $d = \infty$ corresponds to the standard normal distribution. Player 1 can also play optimally at $-x$, in which case Player 2 would choose x; both strategies yield the same equilibrium payoff for these two players, as displayed in the right-most column.

Degrees of freedom	Player 1's optimal guess x	Corresponding percentile	Player 1's equilibrium payoff
1	−0.8944	26.77	.3661
2	−0.7668	26.17	.3692
3	−0.7274	25.98	.3701
4	−0.7084	25.89	.3705
5	−0.6971	25.84	.3708
10	−0.6753	25.74	.3713
∞	−0.6543	25.64	.3718

whose corresponding probabilities are .1855, .1855, .1433, .1855, .1147 and .1855.

Player 1 would like to choose $x = b$ to obtain a coverage probability $.1855 + .1433 = .3288$. Then Player 2 would like to choose $y = c$ to obtain a coverage probability $.1855 + .1147 = .3002$. Then, hopefully, Player 3 would choose $z = a$, and Player 4 would choose $w = a^-$ or $w = a^+$. To make sure these last two choices are as hoped, Player 1 would have to choose $x = b + \epsilon$ for some very small $\epsilon > 0$, and Player 2 would choose $y = c$, say. Then Player 3 can achieve a coverage probability slightly higher than .1855 by choosing $z = a + \delta$ for some very small $\delta > 0$, and Player 4 would choose $w = z^-$ or $w = z^+$. This would guarantee something very close to the equilibrium payoff $(.3288, .3002, .1855, .1855)$.

For the special Pareto distribution $F(x) = 1 - 1/x$ for $x \geq 1$, the three corresponding cutoff points are $a = 1.249$, $b = 2.077$, and $c = 5.015$. The equilibrium payoff is $(.3192, .2829, .1994, .1994)$.

By the method used by Feder (1990), one can show that the same analysis holds for any number $n \geq 3$ of players for all distributions that have a (strictly) decreasing density on $[A, B)$. Suppose for instance that the distribution function F is concave on $[0, \infty)$ with $F(0) = 0$. Let the sequence $0 < a_1 < \cdots < a_{n-1} < \infty$ be defined in such a way that

$$F(a_1) = F\left(\frac{a_1 + a_2}{2}\right) - F(a_1) = \cdots$$
$$= F\left(\frac{a_{n-2} + a_{n-1}}{2}\right) - F(a_{n-2}) = 1 - F(a_{n-1}).$$

One can check easily that such a set of $n - 1$ numbers exists. Let

$$q_k = F(a_{k+1}) - F\left(\frac{a_k + a_{k+1}}{2}\right), \quad k = 1, \ldots, n - 2$$

so that $q_1 + \cdots + q_{n-2} + nF(a_1) = 1$. Then Player 1 would choose a number slightly larger than that a_{k+1} for which q_k is largest, then Player 2 would choose a point close to the next largest, and so on down to Player $n-2$, and then Player $n-1$ would choose a number slightly bigger than a_1. This leaves Player n, who would choose something very close to the choice of Player $n-1$, either slightly above or below. In this fashion, which is optimal in the limit, Player 1 receives the largest expected payoff, Player 2 the next largest, and so on down to Players $n-1$ and n who receive the same least amount, something slightly more than $F(a_1)$.

5. Toetjes in coalitional form

This section explores briefly the features of Toetjes when it is played as a cooperative game, i.e., when there are no restrictions on the agreements that may be reached between the children to coordinate their actions. For simplicity, the discussion is limited to the case $n = 3$.

Clearly, Player 3's strength in making threats and forming alliances becomes important in a cooperative version of Toetjes. One way to illustrate this is to compute the characteristic function v of the game, which assigns a worth to every possible coalition, i.e., every subset S of the grand coalition $N = \{1, 2, 3\}$. By definition, one has $v(\varnothing) = 0$ and $v(S) = 1 - v(N \setminus S)$ for any S, because the game has constant sum equal to 1.

As an example, suppose that X is uniformly distributed on [0,1]. Player 1, acting alone, can achieve nothing because Players 2 and 3 can play infinitesimally below and above him/her. Consequently, $v(\{1\}) = 0$ and $v(\{23\}) = 1$.

In this context, Player 2, acting alone can achieve at most $\frac{1}{6}$, because Player 1 can choose $x_1 = \frac{1}{2}$, forcing Player 2 to play at $\frac{1}{6}$ or $\frac{5}{6}$ in order to obtain at least $\frac{1}{6}$. But if Player 1 does not play at $\frac{1}{2}$, Player 2 can obtain at least $\frac{1}{6}$ by picking an appropriate point on the opposite side of $\frac{1}{2}$. Thus, $v(\{2\}) = \frac{1}{6}$ and $v(\{13\}) = \frac{5}{6}$.

Finally, as shown in Section 3.1, Player 3 acting alone can achieve $\frac{1}{4}$, but no more than $\frac{1}{4}$. So $v(\{3\}) = \frac{1}{4}$ and $v(\{12\}) = \frac{3}{4}$.

It has been suggested by Shapley (1953) that given a characteristic function v, a measure of the value or power of the ith player in a game can be obtained by computing

$$\phi_i = \sum_{\substack{S \subset N \\ i \in S}} \frac{(|S| - 1)!\,(n - |S|)!}{n!} [v(S) - v(S \setminus \{i\})],$$

where $|S|$ denotes the cardinality of S and $n = |N| = 3$ in the present case. Since Toetjes is a constant-sum game, the Shapley value $\phi = (\phi_1, \phi_2, \phi_3)$ is also the nucleolus, an alternative notion of compromise payoff vector for the players due to Schmeidler (1969).

When X is uniform, a simple calulation shows that

$$\phi = \left(\frac{7}{36}, \frac{13}{36}, \frac{16}{36}\right) = (.1944, .3611, .4444).$$

As one can see, therefore, Player 3 is more than twice as strong as Player 1 at the bargaining table. This remains true when the distribution of X is either exponential or normal. In the first case, one has

$$v(\{1\}) = 0, \quad v(\{2\}) = .1464, \quad v(\{3\}) = .2764$$

and $\phi = (.1924, .3388, .4688)$. In the second case,

$$v(\{1\}) = 0, \quad v(\{2\}) = .1779, \quad v(\{3\}) = .2564$$

and $\phi = (.1886, .3664, .4450)$.

Generally speaking, it is interesting how the strengths of the players are reversed in going from a non-cooperative to a cooperative version of Toetjes.

As a side remark, observe that in its coalitional form, Toetjes is unstable in that whatever division $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ of $v(N) = 1$ might be envisaged among the players, there would be a tendency for some coalition $S \subset N$ to form and upset the proposed imputation, because such an alliance could guarantee each of its members more than they would receive from φ. In other words, Toetjes has an empty core, viz.

$$C = \left\{(\varphi_1, \varphi_2, \varphi_3) : \sum_{i \in N} \varphi_i = 1 \text{ and } \sum_{i \in S} \varphi_i \geq v(S) \text{ for all } S \subset N\right\} = \varnothing,$$

as prevails for all n-person constant-sum essential (i.e., non-trivial) games.

6. Closing remarks

The analysis of Toetjes with a parent's secret number X other than uniform has shown that the last child to announce his/her guess is often at a disadvantage in the game. A natural question is whether by letting the last player choose the distribution F from which X will be drawn, equity could be reinstated. The answer is yes. In the case $n = 3$, for example, Player 3 can achieve his/her maximal payoff of $\frac{1}{3}$ by choosing F to be trimodal, with mass $\frac{1}{3}$ within distance $\frac{1}{3}$ of each of -1, 0 and 1, and distance at least 1 between the nodes. This way, Player 3 is then guaranteed at least $\frac{1}{3}$, by playing at 0, unless at least one player plays within $\frac{2}{3}$ of 0, in which case he/she can capture all of one of the end modes. The equilibrium payoff for the game is thus $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

At the opposite, the best distribution for Player 1 seems to be one close to a uniform distribution but whose density has a small positive (say) slope.

It would be good to formalize this observation which, in the case $n = 3$, implies that the optimal equilibrium payoff from Player 1's point of view is $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$.

Another question of interest suggested by Feder (1990) is that of identifying optimal strategies for a multivariate version of Toetjes in which X is drawn from a p-variate distribution F. Taking $n = 2$ for simplicity and denoting by x and y the moves of Players 1 and 2, their respective chances of winning correspond to the probabilities under F of the two half-planes H and $\overline{H} = \mathbb{R}^p \backslash H$ determined by the $(p-1)$-dimensional plane passing through the point $(x + y)/2$ and perpendicular to the segment joining x and y. In order to optimize his/her payoff, Player 2 will obviously choose $y \approx x$ in such a way that the half-space H that this determines as his/her winning set has the largest probability possible under F. Consequently, an optimal strategy for Player 1 consists of choosing a value x for which, under F, the infimum of $\min\{\mathrm{P}(H), \mathrm{P}(\overline{H})\}$ over all possible pairs of half-spaces going through x is as close to $\frac{1}{2}$ as possible.

The above argument shows that x should be a point of maximal depth in the sense of Tukey (1975). Alternatively, x should be a half-space median of F. As in the univariate version of Toetjes, the solution to this problem is not necessarily unique, but Small (1987) identifies various technical conditions under which it is. It would be worth exploring further connections of this sort between multivariate notions of location and solutions to multi-player versions of Toetjes in several dimensions.

Finally, consider a three-player version of the final auction in the popular American television show *The Price is Right*, hosted by Bob Barker. In the so-called "Showcase Showdown," contestants are challenged to guess the total retail price X of a large collection of consumer goods. All bids must be different, and the participant whose answer is closest to, but less than, X becomes the winner. This process is repeated as many times as necessary when all players overbid. In practice, participants in this game would typically be asymmetrically informed and have different guessing abilities, much as in the framework considered by Steele and Zidek (1980). Thus they would rarely hold consensual views on X. In the spirit of Toetjes, however, assume that it is common knowledge that contestants are identically informed, so that they share the same continuous distribution for X. It may then be assumed without loss of generality that this distribution is uniform on the interval $[0, 1]$, because in this version of Toetjes, the probability of win does not actually depend on the distance metric used.

If the players play in order, x_1, x_2, x_3, then with probability $\min x_i$ the game is played again. If the same strategy is repeated indefinitely, the payoff to the players will be the conditional probability of winning given that $X > \min x_i$. For example, if the first two players play at a and b in some order, with $a < b$, then it is clear that the third player will play only at 0, or just above a or just above b. Under the assumption that X is uniform on $[0,1]$, the payoff to Player 3 for these three cases is a, $(b-a)/(1-a)$, and $(1-b)/(1-a)$, respectively, the last two being conditional probabilities given that the game ends. It is clear from this that the game is strongly in Player 3's favor, since one of the last two numbers is at least $\frac{1}{2}$.

By a method similar to that of Section 3, one finds that the equilibrium payoff is $(z, \sqrt{z}-z, 1-\sqrt{z}) \approx (.1850, .2451, .5699)$, where z is the real root of $z^3+2z^2+5z-1=0$. This payoff is obtained as closely as desired if Player 1 takes x_1 to be slightly above $1-z \approx .8150$, Player 2 chooses x_2 slightly above $1-\sqrt{1-x_1}$, and Player 3 picks x_3 at 0. This result should be compared with Proposition 3 of Berk, Hughson and Vandezande (1996), where the solution to the four-player version of this game of perfect information is given. As shown by these authors, the unique Nash equilibrium solution obtains when contestants choose in order, $\frac{7}{9}$, $\frac{5}{9}$, $\frac{3}{9}$, and 0, yielding a payoff vector of $(\frac{2}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{3})$. It is interesting to note that when asked to participate in this game, Player 1 would prefer to have three opponents rather than just two!

Acknowledgements. Part of this work was done while the authors were visiting the Institut de statistique et de recherche opérationnelle at the Université libre de Bruxelles; the warm hospitality of the personnel and their first-rate research facilities were greatly appreciated. Special thanks are due to Professor F. Thomas Bruss, who organized the first author's visit to Belgium and made several useful comments and suggestions in early stages of this project. The second author is indebted to the Natural Sciences and Engineering Research Council of Canada and to the Fonds québécois de la recherche sur la nature et les technologies for their generous financial support.

REFERENCES

Berk, J.B., Hughson, E. and Vandezande, K. (1996). The price is right, but are the bids? An investigation of rational decision theory. *The American Economic Review* 86, 954–970.

Biesterfeld, A. (2001). The price (or probability) is right. *Journal of Statistics Education*, 9 (1). `http://www.amstat.org/publications/jse/v9n3/biesterfeld.html`

Coe, P.R. and Butterworth, W. (1995). Optimal stopping in "The Showcase Showdown." *Amer. Statist.* 49, 271–275.

Even, S. (1966). "The Price is Right" game. *Amer. Math. Monthly* 73, 180–182.

Feder, T. (1990). Toetjes. *Amer. Math. Monthly* 97, 785–794.

Grosjean, J.H. (1998). Beating the showcase showdown. *Chance* 11, 14–19.

Hotelling, H. (1929). Stability in competition. *The Economic Journal* 39, 41–57.

Hwang, J.T. and Zidek, J.V. (1982). Limit theorems for out-guesses with mean-guided second guessing. *J. Appl. Probab.* 19, 321–331.

Pittenger, A.O. (1980). Success probabilities for second guessers. *J. Appl. Probab.* 17, 1133–1137.

Schmeidler, D. (1969). The nucleolus of a characteristic function game. *SIAM J. Appl. Math.* 17, 1163–1170.

Shapley, L.S. (1953). A value for n-person games. In *Contributions to the Theory of Games*, vol. 2, pp. 307–317. Ann. of Math. Stud., vol. 28. Princeton Univ. Press, Princeton, NJ.

Small, C.G. (1987). Measures of centrality for multivariate and directional distributions. *Canad. J. Statist.* 15, 31–39.

Steele, J.M. and Zidek, J.V. (1980). Optimal strategies for second guessers. *J. Amer. Statist. Assoc.* 75, 596–601.

Tukey, J.W. (1975). Mathematics and the picturing of data. In *Proceedings of the International Congress of Mathematicians* (Vancouver, Canada, 1974), vol. 2, pp. 523–531. Canadian Mathematical Congress, Montréal, QC.

Thomas S. Ferguson
Dept. of Mathematics
University of California at Los Angeles
Box 951555
Los Angeles, CA 90095–1555
USA
tom@math.ucla.edu

Christian Genest
Dép. de mathématiques et de statistique
Université Laval
Québec QC G1K 7P4
Canada
genest@mat.ulaval.ca