Transition Density of a Reflected Symmetric Stable Lévy Process in an Orthant

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Abstract

Let $\{Z^{(s,x)}(t) : t \geq s\}$ denote the reflected symmetric α-stable Lévy process in an orthant D (with nonconstant reflection field), starting at (s, x). For $1 < \alpha < 2, 0 \leq s < t, x \in \overline{D}$ it is shown that $Z^{(s,x)}(t)$ has a probability density function which is continuous away from the boundary, and a representation given.

1 Introduction

Due to their applications in diverse fields, symmetric stable Lévy processes have been studied recently by several authors; see [4], [5] and the references therein. In the meantime reflected Lévy processes have been advocated as heavy traffic models for certain queueing/stochastic networks; see [14]. The natural way of defining a reflected/regulated Lévy process is via the Skorokhod problem as in [9], [3], [11], [1].

In this article we consider reflected/regulated symmetric α-stable Lévy process in an orthant, show that transition probability density function exists when $1 < \alpha < 2$ and is continuous away from the boundary; the reflection field can have fairly general time-space dependencies as in [11]. It may be emphasized that unlike the case of reflected diffusions (see [10]) powerful tools/methods of PDE theory are not available to us. To achieve our purpose we use an analogue of a representation for transition density (of a reflected diffusion) given in [2].

Section 2 concerns preliminary results on symmetric α-stable Lévy process in $\mathbb{R}^d$, its transition probability density function and the potential operator. In Section 3, corresponding reflected process with time-space dependent reflection field at the boundary is studied. A major effort goes into proving that the distribution of the reflected process at any given time $t > 0$ gives zero probability to the boundary.

2 Symmetric stable Lévy process

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a filtered probability space, $d \geq 2, 0 < \alpha < 2$. Let $\{B(t) : t \geq 0\}$ be an $\mathcal{F}_t$-adapted d-dimensional *symmetric α-stable Lévy process.* That is, $\{B(t)\}$ is an $\mathbb{R}^d$-valued homogeneous Lévy process (with independent increments) with r.c.l.l. sample paths; it is roation invariant and

$$E[\exp\{i\langle u, B(t) - x\rangle\}|B(0) = x] = \exp\{-t|u|^{\alpha}\} \tag{2.1}$$

for $t \geq 0, u \in I\!R^d, x \in I\!R^d$. It is a pure jump strong Markov process. Using Lévy-Ito theorem and Ito's formula, it can be shown that the (weak) infinitesimal generator of $B(\cdot)$ is given by the *fractional Laplacian*

$$\Delta^{\alpha/2} f(x) = \lim_{r \downarrow 0} C(d,\alpha) \int\limits_{|\xi|>r} \frac{f(x+\xi)-f(x)}{|\xi|^{d+\alpha}} d\xi \tag{2.2}$$

whenever the right side makes sense, where $C(d,\alpha) = \Gamma(\frac{d+\alpha}{2})/[2^{-\alpha}\pi^{d/2}|\Gamma(\frac{\alpha}{2})|]$; the measure $\nu(d\xi) = C(d,\alpha)\frac{1}{|\xi|^{d+\alpha}}d\xi$ is called the Lévy measure of $B(\cdot)$. Also, for any $t > 0$,

$$P(B(t) \neq B(t-)) = 0. \tag{2.3}$$

See [4], [5], [7], [8] for more information.

For a function g on $I\!R^d, g_i(x) = \partial g(x)/\partial x_i, g_{ij}(x) = \partial^2 g(x)/\partial x_i \partial x_j, 1 \leq i,j \leq d$.

Lemma 2.1. *If $f \in C_b^2(I\!R^d)$ then $\Delta^{\alpha/2} f \in C_b(I\!R^d)$.*

Proof: For $0 < r < s$, $\Delta_{r,s}^{\alpha/2}$ is defined by

$$\Delta_{r,s}^{\alpha/2}\psi(z) = C(d,\alpha) \int\limits_{r<|\xi|<s} \frac{\psi(z+\xi)-\psi(z)}{|\xi|^{d+\alpha}} d\xi. \tag{2.4}$$

Let $f \in C_b^2(I\!R^d)$. For any $x \in I\!R^d$ observe that

$$\frac{|f(x+\xi)-f(x)|}{|\xi|^{d+\alpha}} 1_{(1,\infty)}(|\xi|) \leq 2\|f\|_\infty \frac{1}{|\xi|^{d+\alpha}} 1_{(1,\infty)}(|\xi|) \tag{2.5}$$

and that as $\alpha > 0$

$$\int\limits_{|\xi|>1} \frac{1}{|\xi|^{d+\alpha}} d\xi = C \int\limits_1^\infty r^{-(\alpha+1)} dr < \infty. \tag{2.6}$$

So continuity of f and dominated convergence theorem imply that $\Delta_{1,\infty}^{\alpha/2} f$ is well defined, bounded and continuous. Next, Taylor expansion gives

$$f(x+\xi) - f(x) = \sum_{i=1}^d f_i(x)\xi_i + \frac{1}{2}\sum_{i,j=1}^d f_{ij}(y)\xi_i\xi_j \tag{2.7}$$

where y is point on the line segment joining x and $x+\xi$. Since $\xi \mapsto \xi_i$ is an odd function for each i

$$\int\limits_{r<|\xi|<1} \xi_i \frac{1}{|\xi|^{d+\alpha}} d\xi = 0. \tag{2.8}$$

Note that $\sum\limits_{i,j=1}^d f_{ij}(y)\xi_i\xi_j = O(|\xi|^2)$ and

$$\int\limits_{0<|\xi|<1} |\xi|^2 \frac{1}{|\xi|^{d+\alpha}} d\xi = C \int\limits_0^1 r^{-\alpha+1} dr < \infty \tag{2.9}$$

as $\alpha > 2$. Since $f_{ij}(\cdot) \in C_b(I\!R^d)$ it is now easily seen that $\lim_{r\downarrow 0} \Delta^{\alpha/2}_{r,1} f$ is well defined, bounded and continuous. Since

$$\Delta^{\alpha/2} f(x) = \Delta^{\alpha/2}_{1,\infty} f(x) + \lim_{r\downarrow 0} \Delta^{\alpha/2}_{r,1} f(x) \tag{2.10}$$

the lemma now follows. □

It is known that the process $B(\cdot)$ has a transition density function; we now give a representation for it.

Theorem 2.2. *The transition probability density function of $B(\cdot)$ is given by*

$$\begin{aligned} &p(s,x;t,z)\\ =&(4\pi)^{-d/2}(t-s)^{-d/\alpha} \int_0^\infty \frac{g(r)}{r^d} \exp\left\{-\frac{1}{4(t-s)^{2/\alpha}}\frac{1}{r^2}|z-x|^2\right\} dr \end{aligned} \tag{2.11}$$

for $0 \le s < t < \infty, x, z \in I\!R^d$, where $g(\cdot)$ is the density function of the square root of an $\frac{\alpha}{2}$-stable positive random variable.

Proof: By homogeneity enough to consider $s = 0, x = 0$. Let $t > 0$. By (2.1) and Proposition 2.5.5 (on pp. 79-80) of [13] it follows that $B(t) = (B_1(t), \dots, B_d(t))$ is sub-gaussian and that there exist independent one-dimensional random variables $S, U_1, \dots, U_d$ such that $U_i \sim N(0, 2t^{2/\alpha}), 1 \le i \le d, S$ is $\frac{\alpha}{2}$-stable positive random variable and $(B_1(t), \dots, B_d(t)) \sim (S^{\frac12}U_1, S^{\frac12}U_2, \dots, S^{\frac12}U_d)$. Denoting by $g(\cdot)$ the density of $S^{1/2}$, the joint density of $(U_1, \dots, U_d, S^{1/2})$ is given by

$$h(\xi_1, \dots, \xi_d, r) = \left(\frac{1}{4\pi}\right)^{d/2} \left(\frac{1}{t}\right)^{d/\alpha} g(r) \exp\left\{-\frac{1}{4t^{2/\alpha}} \sum_{i=1}^d \xi_i^2\right\}.$$

Using the invertible transformation $(\xi_1, \dots, \xi_d, r) \mapsto (r\xi_1, \dots, r\xi_d, r)$ on $I\!R^d \times (0,\infty)$ the joint density of $(B_1(t), \dots, B_d(t), S^{1/2})$ is given by

$$\begin{aligned} \tilde h(y_1, \dots, y_d, r) &= \frac{1}{r^d} h\left(\frac{1}{r}y_1, \dots, \frac{1}{r}y_d, r\right) \\ &= \left(\frac{1}{4\pi}\right)^{d/2} \left(\frac{1}{t}\right)^{d/\alpha} \frac{1}{r^d} g(r) \exp\left\{-\frac{1}{4t^{2/\alpha}}\frac{1}{r^2} \sum_{i=1}^d y_i^2\right\}. \end{aligned}$$

Now integrating w.r.t. r we get (2.11). □

Remark 2.3. From the preceding theorem it follows that $\int_0^\infty \frac{1}{r^k} g(r) dr < \infty$ for $k = 2, 3, \dots$ Indeed note that $g(\cdot)$ depends only on α; so if we consider k-dimensional symmetric α-stable Lévy process then the transition density will be given by (2.11) with d replaced by k; and as the density is well defined at $x = z$ the claim follows.

Proposition 2.4. *Denote $p_0(s,x;t,z) = \partial p(s,x;t,z)/\partial s$, $p_i(s,x;t,z) = \partial p(s,x;t,z)/\partial x_i$, $p_{ij}(s,x;t,z) = \partial^2 p(s,x;t,z)/\partial x_i \partial x_j, 1 \le i,j \le d$.*

(i) Fix $t > 0, z \in I\!R^d$. Let $t_0 < t$; then $p, p_0, p_i, p_{ij}, 1 \leq i,j \leq d$ are bounded continuous functions of (s,x) on $[0,t_0] \times I\!R^d$.

(ii) For any $t > 0, \delta > 0$

$$\sup\{|\nabla_x p(s,x;t,z)| : 0 \leq s < t, |z-x| \geq \delta\} \leq K(d,\delta) \tag{2.12}$$

where $K(d,\delta)$ is a constant depending only on d, δ and ∇_x denotes gradient w.r.t. x-variables.

Proof: (i) Since $ye^{-y^2}, y^2e^{-y^2}$ are bounded, using Remark 2.3 and dominated convergence theorem, the assertion can be proved by differentiating w.r.t. s, x under the integral in (2.11).

(ii) Since $y^{d+2}e^{-y^2}$ is bounded, differentiating under the integral in (2.11) we get for *all* $0 \leq s < t, |z-x| \geq \delta$

$$\begin{aligned}
&|\nabla_x p(s,x;t,z)| \\
\leq\ & K(d) \int_0^\infty g(r) \left(\frac{2}{|z-x|}\right)^{d+1} \left(\frac{|z-x|}{2r(t-s)^{1/\alpha}}\right)^{d+2} \exp\left\{-\frac{|z-x|^2}{4r^2(t-s)^{2/\alpha}}\right\} dr \\
\leq\ & \hat{K}(d) \left(\frac{2}{\delta}\right)^{d+1} \int_0^\infty g(r)dr = K(d,\delta).
\end{aligned}$$

□

The following result indicates a connection between the transition density and the generator; though it is not unexpected, a proof is given for the sake of completeness.

Theorem 2.5. *For fixed $t > 0, z \in I\!R^d$ the function $(s,x) \mapsto p(s,x;t,z)$ satisfies the Kolmogorov backward equation*

$$p_0(s,x;t,z) + \Delta_x^{\alpha/2} p(s,x;t,z) = 0, s < t, x \in I\!R^d \tag{2.13}$$

where p_0 is as in the preceding proposition and x in $\Delta_x^{\alpha/2}$ signifies that $\Delta^{\alpha/2}$ is applied to p as a function of x.

Proof: By the preceding proposition and Lemma 2.1 $\Delta_x^{\alpha/2}p(s,x;t,z)$ is a bounded continuous function. Put $u(s,x) = p(s,x;t,z), s < t, x \in I\!R^d$. Using Ito's formula (see [7]) for $0 \leq s < c < t, x \in I\!R^d$

$$E\{u(c,B(c)) - u(s,B(s)) - \int_s^c [u_0(r,B(r)) + \Delta^{\alpha/2}u(r,B(r))]dr | B(s) = x\} = 0.$$

That is

$$\begin{aligned}
&\int_{I\!R^d} p(c,y;t,z)p(s,x;c,y)dy - p(s,x;t,z) \\
=\ & \int_s^c \int_{I\!R^d} [p_0(r,y;t,z) + \Delta_y^{\alpha/2}p(r,y;t,z)]p(s,x;r,y)dy\, dr.
\end{aligned}$$

By Chapman-Kolmogorov equation, l.h.s. of the above is zero. As the above holds for all $c > s$ and the quantity within double brackets is bounded continuous in (r, y), by Feller continuity one can obtain (2.13) from the above letting $c \downarrow s$. $\square$

We next look at the 0-resolvent (or potential operator) associated with the process $B(\cdot)$. For a measurable function φ on $\mathbb{R}^d, x \in \mathbb{R}^d$ define

$$G\varphi(x) = \int_{\mathbb{R}^d} \varphi(z) \int_0^\infty p(0, x; t, z)dt\ dz = \int_0^\infty \int_{\mathbb{R}^d} \varphi(z)p(0, x; t, z)dz\ dt \quad (2.14)$$

whenever the r.h.s. makes sense. Since $0 < \alpha < 2 \leq d$, using (2.11) it is not difficult to see that

$$\int_0^\infty p(0, x; t, z) = C\frac{1}{|z - x|^{d-\alpha}}, z \neq x \quad (2.15)$$

which is the so called Riesz kernel.

Theorem 2.6. *Let $\varphi \in C_b^2(\mathbb{R}^d)$ and $\varphi, \varphi_i, \varphi_{ij}, 1 \leq i, j \leq d$ be integrable w.r.t. the d-dimensional Lebesgue measure. Then (a) $G\varphi \in C_b^2(\mathbb{R}^d)$, (b) $(G\varphi)_i(x) = G\varphi_i(x)$, $(G\varphi)_{ij}(x) = G\varphi_{ij}(x)$, $x \in \mathbb{R}^d, 1 \leq i, j \leq d$ (c) $\Delta^{\alpha/2}G\varphi(x) = -\varphi(x), x \in \mathbb{R}^d$.* $\square$

We need a lemma

Lemma 2.7. *If $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ then Gf is well defined, bounded and continuous.*

Proof: Let $\{T_t\}$ be the contraction semigroup associated with $B(\cdot)$. Observe that

$$Gf(x) = \int_0^1 T_t f(x)dt + \int_1^\infty \int_{\mathbb{R}^d} f(z)p(0, x; t, z)dz\ dt. \quad (2.16)$$

Since $T_t f$ is continuous for each $t > 0$ and $|T_t f(\cdot)| \leq \|f\|_\infty$ it is clear that the first term on r.h.s. is bounded and continuous. By (2.11)

$$|f(z)p(0, x; t, z)1_{(1,\infty)}(t)| \leq K\ t^{-d/\alpha}|f(z)|\ 1_{(1,\infty)}(t)$$

which is integrable as $0 < \alpha < 2 \leq d$. So continuity of p in x now implies that the second term on r.h.s. of (2.16) is bounded and continuous. $\square$

Proof of Theorem 2.6: By Lemma 2.6 we get $G\varphi, G\varphi_i, G\varphi_{ij}$ are bounded continuous. A simple change of variables yields

$$\begin{aligned} \frac{1}{h}[G\varphi(x + he_i) - G\varphi(x)] &= \int_0^\infty \int_{\mathbb{R}^d} \frac{\varphi(z + he_i) - \varphi(z)}{h} p(0, x; t, z)dz\ dt \\ &\to \int_0^\infty \int_{\mathbb{R}^d} \varphi_i(z)p(0, x; t, z)dz\ dt \end{aligned}$$

by dominated convergence theroem; thus $(G\varphi)_i(x) = G\varphi_i(x)$. An analogous argument gives $(G\varphi)_{ij}(x) = G\varphi_{ij}(x)$ for all x. By Lemma 2.1 note that $\Delta^{\alpha/2}G\varphi$ is well defined, bounded and continuous. To prove the last assertion, by Chapman-Kolmogorov equation we get

$$\Delta^{\alpha/2}G\varphi(x) = \lim_{t\downarrow 0}\frac{T_tG\varphi(x) - G\varphi(x)}{t}$$

$$= \lim_{t\downarrow 0}\frac{1}{t}\left[\int_0^\infty\int_{\mathbb{R}^d}\varphi(z)p(0,x;t+s,z)dz\,ds - \int_0^\infty\int_{\mathbb{R}^d}\varphi(z)p(0;x;s,z)dz\,ds\right]$$

$$= \lim_{t\downarrow 0}\frac{1}{t}\left[-\int_0^t\int_{\mathbb{R}^d}\varphi(z)p(0,x;s,z)dz\,ds\right] = -\varphi(x)$$

for each $x \in \mathbb{R}^d$, completing the proof. □

3 Reflected process

Let $D = \{x \in \mathbb{R}^d : x_i > 0, 1 \le i \le d\}$ be the d-dimensional positive orthant. The *reflection field* is a function $R : [0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{M}_d(\mathbb{R})$ where $\mathbb{M}_d(\mathbb{R})$ is the space of $(d \times d)$ matrices with real entries. We write $R(t,y,z) = (r_{ij}(t,y,z))$. We assume the following

Assumptions (A1) The function $(y,z) \mapsto r_{ij}(t,y,z)$ is Lipschitz continuous, uniformly in t, for $1 \le i,j \le d$.

(A2) For $i \ne j$, there exist v_{ij} such that $|r_{ij}(t,y,z)| \le v_{ij}$ for all t,y,z. Set $V = ((v_{ij}))$ with $v_{ii} = 0$. We assume spectral radius of $V = \sigma(V) < 1$.

(A3) Take $r_{ii}(\cdot,\cdot,\cdot) \equiv 1, 1 \le i \le d$.

(A2) is a uniform Harrison-Reiman condition that has proved useful in queueing networks; (A3) is just a suitable normalization.

Let $s \ge 0, x \in \bar{D}$. The Skorokhod problem in $\bar{D}$ corresponding to $\{B(t) : t \ge s\}$ and R consists in finding $\mathcal{F}_t$-adapted r.c.l.l. processes $Y^{(s,x)}(t), Z^{(s,x)}(t), t \ge s$ such that
(i) $Z^{(s,x)}(t) \in \bar{D}$ for all $t \ge s$;
(ii) $Y_i^{(s,x)}(s) = 0, Y_i^{(s,x)}(\cdot)$ is nondecreasing, $1 \le i \le d$;
(iii) $Y_i^{(s,x)}(\cdot)$ can increase only when $Z_i^{(s,x)}(\cdot) = 0$; that is, for $1 \le i \le d, t \ge s$,

$$Y_i^{(s,x)}(t) = \int_s^t 1_{\{0\}}(Z_i^{(s,x)}(r))dY_i^{(s,x)}(r), a.s. \tag{3.1}$$

(iv) Skorokhod equation holds, viz. for $1 \le i \le d, t \ge s$

$$\begin{aligned} Z_i^{(s,x)}(t) &= x_i + B_i(t) - B_i(s) + Y_i^{(s,x)}(t) \\ &\quad + \sum_{j \ne i} \int_s^t r_{ij}(u, Y^{(s,x)}(u-), Z^{(s,x)}(u-)) dY_j^{(s,x)}(u) \end{aligned} \tag{3.2}$$

or in vector notation

$$Z^{(s,x)}(t) = x + B(t) - B(s) + \int_s^t R(u, Y^{(s,x)}(u-), Z^{(s,x)}(u-)) dY^{(s,x)}(u). \tag{3.3}$$

Solving the deterministic Skorokhod problem path by path one can solve the above stochastic problem. Indeed the following result is given in [11].

Proposition 3.1. *Assume (A1) - (A3). For each $s \ge 0, x \in \bar{D}$ there is a unique pair $Z^{(s,x)}(\cdot), Y^{(s,x)}(\cdot)$ solving the above problem; also*

$$Y_i^{(s,x)}(t) \le ((I - V)^{-1} L^{(s,x)})_i(t), a.s. \tag{3.4}$$

for $t \ge s$ where $L^{(s,x)}(\cdot)$ is given by

$$L_i^{(s,x)}(t) = \sup_{s \le u \le t} \max\{0, -[x_i + B_i(t) - B_i(s)]\}.$$

Moreover $\{(Z^{(s,x)}(t), Y^{(s,x)}(t)) : t \ge s\}$ is an $\mathcal{F}_t$-adapted $\bar{D} \times \bar{D}$-valued Feller continuous strong Markov process. Any discontinuity of $Y^{(s,x)}(\cdot, \omega)$ or $Z^{(s,x)}(\cdot, \omega)$ has to be a discontinuity of $B(\cdot, \omega)$. If R is a function only of t, z then $\{Z^{(s,x)}(t) : t \ge s\}$ is a $\bar{D}$-valued Feller continuous strong Markov process. □

The z-part of the above viz. $\{Z^{(s,x)}(t) : t \ge s\}$ may be called the *reflected (or regulated) symmetric α-stable Lévy process.*

Proposition 3.2. *Assume (A1) - (A3) and let $1 < \alpha < 2$. Then $E[var\ (Y^{(s,x)}(\cdot); [s,t])] < \infty$ for all $t > s \ge 0, x \in \bar{D}$, where $var\ (g(\cdot); [a,b])$ denotes the total variation of g over $[a,b]$.*

Proof: As $Y_i^{(s,x)}(\cdot)$ is nondecreasing for each i it is enough to show that $E|Y_i^{(s,x)} t)|$
$< \infty$; also we may take $s = 0, x = 0$. Since $\alpha > 1$ note that $E|B_i(t)|^{\alpha'} < \infty$ for all $1 \le \alpha' < \alpha$. As $B(\cdot)$ is symmetric note that it is a martingale. (3.4) of the preceding proposition implies

$$E|Y_i^{(0,0)}(t)|^{\alpha'} \le C\ E\left[\sup_{0 \le r \le t} |B_i(r)|\right]^{\alpha'} \le \hat{C}\ E|B_i(t)|^{\alpha'} < \infty$$

by Doob's maximal inequality for any $1 < \alpha' < \alpha$. The required conclusion now follows. □

Note: In the context of reflected processes, the reflection terms are usually specified only for z on the boundary. However, no matter how the reflection field is extended to $\bar{D}$ or $\mathbb{R}^d$, only the values on the boundary determine the process; Theorem 4.5 of [12] and its proof can be easily adapted to our situation.

The next result concerns expected occupation time at the boundary.

Theorem 3.3. *Assume (A1) - (A3); let* $1 < \alpha < 2$. *Then for* $s \geq 0, x \in \bar{D}, t > s$

$$E\left[\int_s^t 1_{\partial D}(Z^{(s,x)}(r))dr\right] = 0. \tag{3.5}$$

Proof: We consider only $s=0$. Note that $\partial D = \{x \in \mathbb{R}^d : x_i = 0 \text{ for some } i\}$. Let $H = \{x \in \mathbb{R}^d : \min_i |x_i| \leq 1\}$. Let $\varphi \in C_b^2(\mathbb{R}^d)$ be such that (i) $0 \leq \varphi(\cdot) \leq 1$, (ii) $\partial D = \{\varphi = 1\}$, (iii) $\varphi(\cdot) = 0$ on H^c and (iv) $\varphi, \varphi_i, \varphi_{ij}$ are integrable.

For $0 < \epsilon \leq 1$ define φ_ϵ on $\mathbb{R}^d$ by $\varphi_\epsilon(z) = \varphi(z/\epsilon)$. Note that $\varphi_\epsilon, \varphi_{\epsilon,i}, \varphi_{\epsilon,ij}\ \epsilon C_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$; also they are supported on $\epsilon H \subseteq H$. Clearly

$$\lim_{\epsilon \downarrow 0} \varphi_\epsilon(z) = 1_{\partial D}(z), \text{ for all } z \in \mathbb{R}^d. \tag{3.6}$$

Next define g_ϵ on $\mathbb{R}^d$ by

$$g_\epsilon(x) = \int_{\mathbb{R}^d} -\frac{1}{\epsilon^\alpha}\varphi_\epsilon(x) \int_0^\infty p(0,x;t,z)dt\ dz. \tag{3.7}$$

By Theorem 2.6, $\Delta^{\alpha/2} g_\epsilon = \frac{1}{\epsilon^\alpha}\varphi_\epsilon, 0 < \epsilon \leq 1$. We now claim that

$$\sup_x \epsilon^\alpha |g_\epsilon(x)| \to 0 \text{ as } \epsilon \downarrow 0. \tag{3.8}$$

Putting $s = t/\epsilon^\alpha$ in (3.7) and as $|\varphi_\epsilon(\cdot)| \leq 1$ we get

$$\begin{aligned}
\epsilon^\alpha |g_\epsilon(x)| &\leq \epsilon^\alpha \int_0^1 \int_{\mathbb{R}^d} p(0,x;\epsilon^\alpha s, z)dz\ ds \\
&\quad + \epsilon^\alpha \int_{\mathbb{R}^d} |\varphi_\epsilon(z)| \int_1^\infty p(0,x;\epsilon^\alpha s, z)ds\ dz \\
&= I_1(x;\epsilon) + I_2(x;\epsilon).
\end{aligned}$$

As $p(0,x;\epsilon^\alpha s,\cdot)$ is a probability density $\sup_x |I_1(x;\epsilon)| \leq \epsilon^\alpha \to 0$. As φ is integrable, by (2.11)

$$\begin{aligned}
\sup_x |I_2(x;\epsilon)| &\leq \epsilon^\alpha \int_{\mathbb{R}^d} |\varphi_\epsilon(z)| \int_1^\infty C\left(\frac{1}{\epsilon^\alpha s}\right)^{d/\alpha} ds\ dz \\
&= C\ \epsilon^{\alpha-d} \int_{\mathbb{R}^d} \varphi(\frac{1}{\epsilon}z)dz = C\ \epsilon^\alpha \int_{\mathbb{R}^d} \varphi(z)dz \\
&= \hat{C}\epsilon^\alpha \to 0
\end{aligned}$$

whence (3.8) follows.

We next show that

$$\sup_x \epsilon^\alpha |\nabla g_\epsilon(x)| \to 0 \text{ as } \epsilon \downarrow 0. \tag{3.9}$$

By Theorem 2.6, and putting $s = t/\epsilon^\alpha$ gives

$$\begin{aligned}\frac{\partial}{\partial x_i} g_\epsilon(x) &= \int_{\mathbb{R}^d} -\varphi_{\epsilon,i}(z) \int_0^\infty p(0,x;\epsilon^\alpha s, z) ds\, dz \\ &= \int_{\mathbb{R}^d} -\varphi_i\left(\frac{z}{\epsilon}\right)\frac{1}{\epsilon} \int_0^\infty p(0,x;\epsilon^\alpha s, z) ds\, dz.\end{aligned}$$

Since φ_i is integrable for $1 \leq i \leq d$, an argument similar to the derivation of (3.8) gives

$$\sup_x \epsilon^\alpha |\nabla g_\epsilon(x)| \leq C\ \epsilon^{\alpha-1} \to 0 \text{ as } \epsilon \downarrow 0$$

because $\alpha > 1$; this proves (3.9).

Now applying Ito's formula to $\epsilon^\alpha g_\epsilon(Z^{(0,x)}(\cdot))$, denoting $Z^{(0,x)}(\cdot)$ by $Z(\cdot), Y^{(0,x)}(\cdot)$ by $Y(\cdot)$ and taking expectations we get

$$\begin{aligned}E[\epsilon^\alpha g_\epsilon(Z(t)) - \epsilon^\alpha g_\epsilon(x)] &= E\int_0^t \varphi_\epsilon(Z(r))dr \\ &+E\int_0^t \langle R(u,Y(u-),Z(u-))\epsilon^\alpha \nabla g_\epsilon(Z(u)), dY(u)\rangle.\end{aligned} \tag{3.10}$$

By (3.8) l.h.s. of (3.10) tends to zero as $\epsilon \to 0$. As R is bounded, Proposition 3.2 and (3.9) imply that the last term in (3.10) goes to zero as $\epsilon \to 0$. Finally, as $|\varphi_\epsilon(\cdot)| \leq 1$, (3.6) and (3.10) now imply (3.5). □

Remark 3.4. A function φ as indicated in the proof of the preceding theorem can, for example, be obtained as follows. Let H_1 be a closed set with smooth boundary such that $\partial D \subset \text{Int }(H_1) \subset H_1 \subset \text{Int }(H), \epsilon H_1 \subset H_1$ for $0 \leq \epsilon \leq 1, \lambda_d(H_1) < \infty$ where λ_d denotes the d-dimensional Lebesgue measure. Take $\varphi(z) = 0, z \notin H$ and

$$\varphi(z) = e\ \exp\left\{-\frac{1}{1-\exp\left[-\left(\frac{1}{z_1^2}+\ldots+\frac{1}{z_d^2}\right)\right]}\right\}, z \in H_1;$$

φ can be extended as required. □

Using Theorem 3.3 we now improve on it!

Theorem 3.5. *Assume (A1) - (A3),* $1 < \alpha < 2$. *Then for* $s \geq 0, x \in \bar{D}, t > s$

$$P(Z^{(s,x)}(t) \in \partial D) = 0. \tag{3.11}$$

Proof: Let $\zeta(z) = K \exp\left\{-\left(\frac{1}{z_1^2} + \ldots + \frac{1}{z_d^2}\right)\right\}$, where $K > 2$

$$h(r) = \begin{cases} e \ \exp\left\{-\frac{1}{1-r^2}\right\} & , \quad |r| \leq 1 \\ 0 & , \quad |r| \geq 1 \end{cases}$$

For $\epsilon > 0$ define $f_\epsilon(z) = h(\zeta(z/\epsilon)), z \in I\!R^d$. Clearly $f_\epsilon \in C_b^2(I\!R^d)$ and $\partial f_\epsilon(z)/\partial z_i = 0$ for any $z \in \partial D, 1 \leq i \leq d$. It is not difficult to see that

$$\lim_{\epsilon \downarrow 0} f_\epsilon(z) = 1_{\partial D}(z), z \in I\!R^d \tag{3.12}$$

(for $z \notin \partial D$ note that $z_i > c$ for all i for some $c > 0$; hence $\zeta(z/\epsilon) > 1$ for all small ϵ). Next, an argument as in Lemma 2.1 gives for $\epsilon > 0$

$$\sup_z |\Delta^{\alpha/2} f_\epsilon(z)| \leq \frac{C_1}{\alpha} + \frac{C_2}{(2-\alpha)} \frac{1}{\epsilon^2} \tag{3.13}$$

for suitable constants C_1, C_2.

Now we claim that for $z \in \bar{D} \backslash \partial D$,

$$\Delta^{\alpha/2} f_\epsilon(z) \to 0 \text{ as } \epsilon \downarrow 0. \tag{3.14}$$

Indeed let $z \notin \partial D$; there exist $r_0 > 0, c > 0$ such that $(z_i + \xi_i) > c, 1 \leq i \leq d$ for $|\xi| < r_0$. Choose $\epsilon_0 > 0$ so that for all $\epsilon < \epsilon_0, \zeta((z+\xi)/\epsilon) > K \exp\{-d\epsilon^2/c^2\} > 1$ for $|\xi| < r_0$. Therefore $f_\epsilon(z + \xi) = 0 = f_\epsilon(z)$ for all $|\xi| < r_0, \epsilon < \epsilon_0$ and hence

$$\Delta^{\alpha/2} f_\epsilon(z) = \int_{|\xi|>r_0} f_\epsilon(z+\xi) \frac{1}{|\xi|^{d+\alpha}} d\xi. \tag{3.15}$$

Since $\frac{1}{|\xi|^{d+\alpha}} 1_{(r_0,\infty)}(|\xi|)$ is integrable and $\lambda_d(\partial D) = 0$, by (3.12), (3.15) now the claim (3.14) follows.

To prove the theorem we consider only the case $s = 0$. Denote $Z^{(0,x)}(\cdot), Y^{(0,x)}(\cdot)$ by $Z(\cdot), Y(\cdot)$. We want to prove that for $x \in \bar{D}, t > 0$,

$$\lim_{\epsilon \downarrow 0} E \int_0^t \Delta^{\alpha/2} f_\epsilon(Z(r)) dr = 0. \tag{3.16}$$

By Theorem 3.3 and (3.13) for each $\epsilon > 0$,

$$E \int_0^t 1_{\partial D}(Z(r)) \Delta^{\alpha/2} f_\epsilon(Z(r)) dr = 0. \tag{3.17}$$

For $c > 0$, put $D_c = (2c, \infty)^d$. In view of (3.17), to prove (3.16) it is enough to prove that

$$\lim_{\epsilon \downarrow 0} E \int_0^t 1_{D_c}(Z(u)) \Delta^{\alpha/2} f_\epsilon(Z(u)) du = 0 \tag{3.18}$$

for any fixed $c > 0$. If $z \in D_c, |\xi| < c$ note that $z_i + \xi_i > c, 1 \le i \le d$. So one can choose $\epsilon_0 > 0$ such that $f_\epsilon(z+\xi) = 0$ for all $|\xi| < c, z \in D_c, \epsilon < \epsilon_0$. Hence for any $\epsilon < \epsilon_0$

$$|1_{D_c}(Z(u))\Delta^{\alpha/2} f_\epsilon(Z(u))| \le \int\limits_{|\xi|>c} \frac{1}{|\xi|^{d+\alpha}} d\xi \le C \frac{1}{\alpha c^\alpha}.$$

The required assertion (3.18) and hence (3.16) now follows by (3.14) and dominated convergence theorem.

Now to prove (3.11) (with $s = 0$), first consider the case $x \notin \partial D$. Since $\partial f_\epsilon(\cdot)/\partial z_i = 0$ on ∂D, and $Y(\cdot)$ can increase only when $Z(\cdot) \in \partial D$, by Ito's formula

$$E[f_\epsilon(Z(t))] - f_\epsilon(x) = E \int\limits_0^t \Delta^{\alpha/2} f_\epsilon(Z(r)) dr.$$

By (3.12), (3.16) letting $\epsilon \downarrow 0$ in the above we get (3.11).

Next let $x \in \partial D$; for $c > 0$ let $\eta \equiv \eta_c^{(x)} = \inf\{r \ge 0 : Z(r) \in \bar{D}_c\}$. By strong Markov property and the preceding case

$$E[1_{[0,t]}(\eta) 1_{\partial D}(Z(t))] = 0.$$

Note that $\{\eta_c^{(x)} \le t\} \uparrow \Omega$ (modulo null set) as $c \downarrow 0$; otherwise we will get a contradiction to Theorem 3.3. Letting $c \downarrow 0$ in the above we get the required conclusion. This completes the proof. □

Note: It may be interesting to compare the proofs of Theorems 3.3, 3.5 with those of their analogues for reflected Brownian motion given in [6].

In the following $\nabla_2 p(r, y; t, z) = \nabla_2 p(r, \cdot; t, z), \Delta_2^{\alpha/2} p(r, y; t, z) = \Delta_2^{\alpha/2} p(r, \cdot; t, z)$ denote respectively the operators $\nabla, \Delta^{\alpha/2}$ applied as function of y-variables. Our main result is

Theorem 3.6. *Assume (A1) - (A3); let $1 < \alpha < 2$. For $0 \le s < t < \infty, x \in \bar{D}, z \in D$ define*

$$p^R(s, x; t, z) = p(s, x; t, z) + E \int\limits_s^t \langle R(u, Y(u-), Z(u-)) \nabla_2 p(u, Z(u); t, z), dY(u) \rangle \quad (3.19)$$

where $Y(\cdot) = Y^{(s,x)}(\cdot), Z(\cdot) = Z^{(s,x)}(\cdot)$. For $0 \le s < t, x \in \bar{D}, z \in \partial D$ take $p^R(s, x; t, z) = 0$. Then (i) p^R is continuous on $\{0 \le s < t < \infty, x \in \bar{D}, z \in D\}$, it is also differntiable in (t, z); (ii) for any Borel set $A \subseteq \bar{D}, s < t, x \in \bar{D}$

$$P(Z^{(s,x)}(t) \in A) = \int\limits_A p^R(s, x; t, z) dz. \quad (3.20)$$

In case R is independent of y-variables, p^R is the transition probability density function of the Markov process $Z(\cdot)$. □

We need a lemma

Lemma 3.7. *Hypotheses and notation as in the Proposition 3.2. If $(s_n, x_n) \to (s, x)$ then for a.a. ω, for $T > s$*

$$\begin{aligned} var\ (Y^{(s_n,x_n)}(\cdot,\omega) - Y^{(s,x)}(\cdot,\omega); [s,T]) &\to 0 \\ \sup_{s \le t \le T} |Z^{(s_n,x_n)}(t,\omega) - Z^{(s,x)}(t,\omega)| &\to 0. \end{aligned}$$

Proof: Denote $Z^{(n)}(\cdot) = Z^{(s_n,x_n)}(\cdot), Y^{(n)}(\cdot) = Y^{(s_n,x_n)}(\cdot), Z(\cdot) = Z^{(s,x)}(\cdot)$, $Y(\cdot) = Y^{(s,x)}(\cdot)$. We first consider the case $s_n < s$ for all n. Clearly $Z^{(n)}(t,\omega)$, $Y^{(n)}(t,\omega), t \ge s$ is the solution to the Skorokhod problem corresponding to $Z^{(n)}(s,\omega) + B(\cdot,\omega) - B(s,\omega)$. For any $T > s$ note that

$$\begin{aligned} &\text{var } ([B(\cdot,\omega) - B(s,\omega) + Z^{(n)}(s,\omega)] - [B(\cdot,\omega) - B(s,\omega) + x]; [s,T]) \\ = &\ |Z^{(n)}(s,\omega) - x|. \end{aligned}$$

For any ω such that $B(\cdot,\omega)$ is continuous at s we have $x_n + B(s,\omega) - B(s_n,\omega) \to x$. Boundedness of R and (3.4) imply

$$\left| \int_{s_n}^{s} R(u, Y^{(n)}(u-), Z^{(n)}(u-)) dY^{(n)}(u,w) \right| \to 0 \text{ as } n \to \infty.$$

Thus $|Z^{(n)}(s,\omega) - x| \to 0$, and hence the result follows by Proposition 3.9 of [11]. □

Next let $s_n > s$ for all n. For any $n, Z(t,\omega), Y(t,\omega), t \ge s_n$ is the solution to the Skorokhod problem corresponding to $Z(s_n,\omega) + B(\cdot,\omega) - B(s_n,\omega)$. Clearly

$$\begin{aligned} &\text{var } ([x_n + B(\cdot,\omega) - B(s_n,\omega)] - [Z(s_n,\omega) + B(\cdot,\omega) - B(s_n,\omega)]); [s_n,T]) \\ = &\ |Z(s_n,\omega) - x_n|. \end{aligned}$$

So by the arguments as in [11]

$$\begin{aligned} \text{var } (Y^{(n)}(\cdot,\omega) - Y(\cdot,\omega); [s_n,T]) &\le C|Z(s_n,\omega) - x_n| \\ \sup_{s_n \le t \le T} |Z^{(n)}(t,\omega) - Z(t,\omega)| &\le C|Z(s_n,\omega) - x_n|. \end{aligned}$$

Note that for $s \le t \le s_n$ we may take $Z^{(n)}(t,\omega) = x_n, Y^{(n)}(t,\omega) = 0$. Clearly var $(Y(\cdot,\omega); [s,s_n])$, $\sup_{s \le t \le s_n} |x_n - Z(t,\omega)|, |Z(s_n,\omega) - x_n|$ all tend to 0 as $s_n \to s$ by right continuity. The required conclusion is now immediate.

Proof of Theorem 3.6: Since $dY^{(s,x)}(\cdot)$ can charge only when $Z^{(s,x)}(\cdot) \in \partial D$ and $d(z, \partial D) > 0$ for $z \notin \partial D$, well definedness of (3.19) follows from (2.12) and Proposition 3.2.

Assertion (i) now follows from properties of p (viz. (2.11), (2.12), Proposition 2.4), boundedness and continuity of R and Lemma 3.7.

To prove assertion (ii), in view of Theorem 3.5, it is enough to establish (3.20) when $A \subset D$.

Fix $t > s$; let $\epsilon > 0$. Apply Ito's formula to $p(r, Z^{(s,x)}(r); t, z), s \leq r \leq (t - \epsilon)$ corresponding to the semimartingale $Z^{(s,x)}(\cdot)$ and use Theorem 2.5 to get

$$\begin{aligned} p(t-\epsilon, Z(t-\epsilon); t, z) &= p(s, x; t, z) \\ &\quad + \int_s^{t-\epsilon} \langle R(r, Y(r-), Z(r-)) \nabla_2 p(r, Z(r); t, z), dY(r) \rangle \\ &\quad + \text{a stochastic integral.} \end{aligned} \tag{3.21}$$

Let f be a continuous function with compact support $K \subset D$. By (3.21) for any $\epsilon > 0$

$$E \int_D f(z) p(t-\epsilon, Z(t-\epsilon); t, z) dz = \int_D f(z) p(s, x; t, z) dz$$
$$+E \int_D f(z) \int_s^{t-\epsilon} \langle R(r, Y(r-), Z(r-)) \nabla_2 p(r, Z(r); t, z), dY(r) \rangle dz \tag{3.22}$$

For any ω, note that $p(t-\epsilon, Z(t-\epsilon, \omega); t, z) dz \Rightarrow \delta_{Z(t-,\omega)}(dz)$ as $\epsilon \downarrow 0$. And since $P(Z(t) \neq Z(t-)) = 0$ it now follows that

$$\lim_{\epsilon \downarrow 0} [\text{l.h.s. of (3.22)}] = E[f(Z^{(s,x)}(t))]. \tag{3.23}$$

As $d(K, \partial D) > 0$, by (2.12), Proposition 3.2 and boundedness of $f(\cdot), R(\cdot)$

$$\lim_{\epsilon \downarrow 0} [\text{r.h.s. of (3.22)}] = \int_D f(z) p^R(s, x; t, z) dz. \tag{3.24}$$

Thus

$$\int_D f(z) p^R(s, x; t, z) dz = E[f(Z^{(s,x)}(t))] \tag{3.25}$$

for any continuous function f with compact support in D.

Next for any open set $F \subset D$, let $\{f_n\}$ be a sequence of continuous functions with compact support in D such that $f_n \uparrow 1_F$ pointwise. Clearly

$$\lim_{n \to \infty} E[f_n(Z^{(s,x)}(t))] = E[1_F(Z^{(s,x)}(t))]. \tag{3.26}$$

Taking expectation in (3.21) and letting $\epsilon \downarrow 0$ we get

$$p^R(s, x; t, z) = \lim_{\epsilon \downarrow 0} E[p(t-\epsilon, Z(t-\epsilon); t, z)] \geq 0.$$

Therefore by monotone convergence theorem

$$\lim_{n \to \infty} \int_D f_n(z) p^R(s, x; t, z) dz = \int_D 1_F(z) p^R(s, x; t, z) dz. \tag{3.27}$$

Now (3.25), (3.26), (3.27) imply that (3.20) holds for any open $F \subset D$, and hence for any Borel set $A \subset D$.

Finally, the last assertion is immediate from (ii); this completes the proof. □

We conclude with the following questions.

1. Can $(x, z) \mapsto p^R(s, x; t, z)$ given by (3.19) be extended continuously to $\bar{D} \times \bar{D}$?

2. Is $p^R(s, x; t, z) > 0$ for $s < t, x, z \in D$?

3. When is p^R symmetric in x, z?

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