θ-expansions and the generalized Gauss map

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Abstract

Motivated by problems in random continued fraction expansions, we study θ-expansions of numbers in $[0, \theta)$ where $0 < \theta < 1$. For such a number θ, we study the generalized Gauss transformation defined on $[0, \theta)$ as follows:

$$T(x) = \begin{cases} \frac{1}{x} - \theta[\frac{1}{\theta x}] & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0 \end{cases}$$

One of the problems that concerns us is the symbolic dynamics of this map and existence of absolutely continuous invariant probability.

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1 Introduction

Suppose that μ is a probability on the real line. Consider the following law of motion: If you are at x pick a number Z according to the law μ and move to $Z + x$. Continue the motion with independent choices at each stage. This is nothing but the familiar random walk. Suppose that by an error the law is transcribed as : move to $Z + \frac{1}{x}$, then what happens? To make sense of the problem, from now on we consider the state space to be $(0, \infty)$. Let μ be a probability on $[0, \infty)$ which drives the motion. If you are at x move to $Z + \frac{1}{x}$ where Z is chosen independent of the past and has law μ. This leads us to the Markov process

$$X_0 = x > 0; \qquad X_{n+1} = Z_{n+1} + \frac{1}{X_n} \quad \text{for } n \geq 0$$

where $(Z_n; n \geq 1)$ is an i.i.d sequence of random variables, each having law μ. The purpose of the paper is to discuss this process.

2 Generalities

If μ is δ_0, the point mass at zero, then $X_n = x$ or $1/x$ according as n is even or odd. Unless $x = 1$ the process does not converge in distribution. For each $x > 0$, $\frac{1}{2}(\delta_x + \delta_{1/x})$ is an invariant distribution for the process. In fact any invariant probability is a mixture of these. If $\mu = \delta_a$ where $a > 0$, then the process starting at x is deterministic and is the sequence – in the usual notation of continued fractions – $[x;]$, $[a; x]$, $[a; a, x]$, $\cdots$ which converges to the number given by the continued fraction $[a; a, a, \cdots]$. We leave the easy calculation involving convergents to the interested reader. The point mass at this point is the unique

invariant distribution for the process. From now on we assume that μ is not a degenerate probability, on $[0, \infty)$. It may however have some mass at zero. Then $X_n = [Z_n; Z_{n-1}, \cdots, Z_1, x]$ has the same law as $[Z_1; Z_2, \cdots, Z_n, x]$ and consequently X_n converges in distribution to

$$X_\infty = Z_1 + \cfrac{1}{Z_2 + \cfrac{1}{Z_3 + \cdots}}$$

simply denoted by $[Z_1; Z_2, Z_3, \cdots]$. The almost sure convergence of the expression on the right side is argued as follows. Since Z_i are i.i.d with strictly positive mean, we have $\sum Z_n = \infty$ a.e and for nonnegative numbers (a_n) the continued fraction $[a_1; a_2, a_3, \cdots]$ is convergent iff $\sum a_n = \infty$ (Khinchin [9], Th 10,p. 10). Since X_n converges to X_∞ in distribution irrespective of the initial point x we have the following:

Theorem 1 (Bhattacharya and Goswami [1]):

(1) The Markov process X_n has a unique invariant distribution Π, and X_n converges in distribution to Π.

(2) Π is the unique probability on $(0, \infty)$ characterized by $\Pi = \mu * \frac{1}{\Pi}$ in the sense that whenever X, Z are independent random variables with X strictly positive, $Z \sim \mu$ and $X \sim Z + \frac{1}{X}$, then $X \sim \Pi$.

In view of the last part of the theorem, each explicit evaluation of Π leads to a characterization of Π as the unique distribution satisfying the convolution equation above. It is in this context the problem was first discussed by Letac and Seshadri [11],[12]. They observed that when μ is exponential then Π is inverse Gaussian, thereby obtaining a characterization of the inverse Gaussian distribution. A systematic study of the markov process was initiated in Bhattacharya and Goswami [1] motivated by problems in random number generation. They showed , among other things, that Π is always non-atomic. An excellent review is in Goswami [8].

3   Positive integer driver

One problem that concerns us here is the nature of the invariant probability – whether it is absolutely continuous or singular. Since the invariant probability Π is nothing but the distribution of $X_\infty = [Z_1; Z_2, Z_3, \cdots]$, the problem reduces to studying the nature of the distribution of X_∞.. Let us assume that the driving probability μ is concentrated on the set of strictly positive integers. In this case note that the representation $[Z_1(\omega); Z_2(\omega), Z_3(\omega), \cdots]$ is already the usual continued fraction expansion of the number $X_\infty(\omega)$. Well known results about usual continued fraction expansions lead to an interesting consequence. The range of $1/X_\infty$ is contained in $(0, 1)$. Under the distribution of $1/X_\infty$, the digits in the continued fraction expansion are i.i.d so that it is an invariant and ergodic measure for the Gauss transformation. So it must be same as the Gauss measure or must be singular to the Gauss measure and hence singular. But as one knows the digits are not independent under the Gauss measure. So the distribution of $1/X_\infty$ is singular. Consequently the distribution of X_∞ is singular too. Thus,

Theorem 2: *Suppose μ is concentrated on strictly positive integers. Then X_∞ has singular distribution. This is perhaps known, but we have not found in the literature. Thus we here have a naturally arising family of singular distributions.*

4 Bernoulli driver

The arguments used above fail when μ has mass at zero. Due to the presence of zeros, $[Z_1(\omega); Z_2(\omega), Z_3(\omega), \cdots]$ is no longer the usual continued fraction expansion of the number $X_\infty(\omega)$. Let us assume that the driving probability μ puts mass α at 0 and $1 - \alpha$ at 1. Since each Z_i takes only two values 0 and 1, it is not difficult to discover the continued fraction expansion of $X_\infty(\omega)$ This is what we obtain now. Let us assume that $Z_1(\omega) = 1$ or equivalently, consider the set $\Omega_1 = \{\omega : Z_1(\omega) = 1\}$. Define the stopping times for the process $(Z_i)_{i \geq 1}$ as follows :

$$\tau_0(\omega) = \text{First even integer } i \text{ such that } Z_i(\omega) \neq 0.$$

$$\tau_1(\omega) = \text{First odd integer } i > \tau_0 \text{such that} Z_i(\omega) \neq 0.$$

$$\tau_2(\omega) = \text{First even integer } i > \tau_1 \text{ such that } Z_i(\omega) \neq 0 \quad \&c.$$

Let us now define,

$$a_0(\omega) = \sum_{1 \leq i < \tau_0(\omega)} Z_i(\omega); \quad a_1(\omega) = \sum_{\tau_0(\omega) \leq i < \tau_1(\omega)} Z_i(\omega) \quad \&c.$$

Then, we have for a.e. $\omega \in \Omega_1$, $[a_0(\omega); a_1(\omega), a_2(\omega), \cdots]$. is the usual continued fraction expansion of $X_\infty(\omega)$. If $S_k = \sum_{1 \leq i \leq k} Z_i$, then $a_0 = S_{\tau_0 - 1}$; $a_1 = S_{\tau_1 - 1} - S_{\tau_0 - 1}$; $\cdots$ so that,

$$\frac{1}{k+1} \sum_{i=0}^{k} a_i(\omega) = \frac{S_{\tau_k - 1}}{k+1} = \frac{S_{\tau_k - 1}}{\tau_k - 1} \cdot \frac{\tau_k - 1}{k+1}$$

By the SLLN, we conclude that, $\lim_{k \to \infty} \dfrac{S_{\tau_k - 1}}{\tau_k - 1}$ converges to $1 - \alpha$ a.e. Further, $\tau_0 - 1, \tau_1 - \tau_0, \tau_2 - \tau_1, \cdots$ are i.i.d. random variables taking values $1, 3, 5, \cdots$ with probabilities $1 - \alpha$; $\alpha(1 - \alpha)$; $\alpha^2(1 - \alpha), \cdots$. So, again by SLLN $\lim_{k \to \infty} \dfrac{\tau_k - 1}{k+1} = \dfrac{1 + \alpha}{1 - \alpha}$. Thus

$$\lim_{k \to \infty} \frac{1}{k+1} \sum_{i=0}^{k} a_i(\omega) = 1 + \alpha \quad \text{for a.e. } \omega \in \Omega_1.$$

Thus for almost every $\omega \in \Omega_1$, the average of the digits in the (continued fraction) expansion of $X_\infty(\omega)$ has a finite limit. This argument can be repeated from the first nonzero Z_i to get the same conclusion a.e. This shows that the range of X_∞ is Lebesgue null ([9], Th. 30, p. 63 or * below). In fact the proof shows more.

Theorem 3:

   (i) *Let Π_α be the distribution of X_∞ when μ takes values 0 and 1 with probabilities α and $1 - \alpha$ respectively. Then each Π_α is singular. The family $(\Pi_\alpha : 0 < \alpha < 1)$ is a uniformly singular family.*

(ii) *If μ is concentrated on the set of nonnegative integers and has finite mean then X_∞ is singular.*

Singularity of Π_α in (i) above was shown by Bhattacharya and Goswami [1], using again the properties of the Gauss transformation, by different methods. The case when the driving probability μ puts mass α at 0 and mass $1 - \alpha$ at θ where $0 < \theta < 1$, leads to the interesting concept of θ expansions and a generalization of the Gauss transformation.

5 θ - expansions

Throughout our discussion we fix a θ with $0 < \theta < 1$. We start with a discussion of continued fraction expansion w.r.t θ, analogous to the usual expansion which corresponds to the case $\theta = 1$.

Let $x > 0$. Let $a_0 = \max\{n \geq 0 : n\theta \leq x\}$. If x already equals $a_0\theta$, we write $x = [a_0\theta]$. Otherwise, define r_1 by $x = a_0\theta + \dfrac{1}{r_1}$ where $0 < \dfrac{1}{r_1} < \theta$. Then $r_1 > \dfrac{1}{\theta} \geq \theta$ and let $a_1 = \max\{n \geq 0 : n\theta \leq r_1\}$. If $r_1 = a_1\theta$, then we write $x = [a_0\theta, a_1\theta]$, i.e., $x = a_0\theta + \dfrac{1}{a_1\theta}$. If $a_1\theta < r_1$, define r_2 by $r_1 = a_1\theta + \dfrac{1}{r_2}$ where $0 < \dfrac{1}{r_2} < \theta$. So, $r_2 > \dfrac{1}{\theta} \geq \theta$ and let $a_2 = \max\{n \geq 0 : n\theta \leq r_2\}$. Proceeding in this way, either the process terminates at, say, n steps or it continues indefinitely. In the former case, we write $x = [a_0\theta; a_1\theta, \cdots, a_n\theta]$ and we call this the continued fraction expansion of x with respect to θ terminating at the n-th stage. In the latter case, we write $x = [a_0\theta; a_1\theta, a_2\theta, \cdots]$ and it is called the infinite or non-terminating continued fraction expansion of x with respect to θ. From now on, unless otherwise mentioned, we refer to this expansion as the continued fraction expansion of a number in $(0, \infty)$. Since during any discussion a particular value of θ is fixed, we shall omit the phrase 'w.r.t θ'. We shall now briefly argue that the infinite expansion does converge to x. To do this, as with usual expansion, we define the n-th convergent of a number $x \in (0, \infty)$ as

$$\frac{p_n}{q_n} = [a_0\theta; a_1\theta, \cdots, a_n\theta], \quad n \geq 0.$$

In case x has terminating expansion, say, $x = [a_0\theta; a_1\theta, \cdots, a_k\theta]$, then clearly $\dfrac{p_k}{q_k} = x$. We make the usual convention that in this case

$$\frac{p_n}{q_n} = x \quad \text{for} \quad n \geq k.$$

When $x < \theta$, we have $a_0 = 0$ and instead of writing $x = [0; a_1\theta, a_2\theta, \cdots]$, we write $x = [a_1\theta, a_2\theta, \cdots]$ which is same as writing, in the usual notation,

$$x = \cfrac{1}{a_1\theta + \cfrac{1}{a_2\theta + \cdots}}.$$

Let $0 < x < \theta$, $x = [a_1\theta, a_2\theta, \cdots]$. In what follows a_n, p_n and q_n depend on x. The following are routine to verify (The stated identities hold for all n

in case x has non-terminating expansion and they hold for $n \leq k$ in case x has expansion terminating at the k-th stage):
For $n \geq 1$,

$$p_n = a_n \theta p_{n-1} + p_{n-2} \qquad 5.1(i)$$

and

$$q_n = a_n \theta q_{n-1} + q_{n-2} \qquad (5.1)$$

Following the convention of the usual continued fraction expansions, namely, $p_{-1} = 1, p_0 = 0, q_{-1} = 0, q_0 = 1$, we arrive at

$$p_{n-1} q_n - p_n q_{n-1} = (-1)^n \quad \text{for} \quad n \geq 0 \qquad (5.2)$$

Let $n \geq 1$, and $x = [a_1\theta, a_2\theta, \cdots, a_n\theta + \dfrac{1}{r_n}]$. Then a little algebra shows that

$$x = \frac{p_n + \dfrac{1}{r_n} p_{n-1}}{q_n + \dfrac{1}{r_n} q_{n-1}} \qquad (5.3)$$

$$\left| x - \frac{p_n}{q_n} \right| = \frac{1}{q_n(r_n q_n + q_{n-1})} \qquad (5.4)$$

Now, $a_{n+1}\theta \leq r_n \leq (a_{n+1} + 1)\theta$. Using these estimates in (5.4) and noting (5.1(i)),(5.1(ii)), we get,

$$\frac{1}{q_n(q_{n+1} + \theta q_n)} \leq \left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n q_{n+1}} \qquad (5.5)$$

Observing that $q_0 = 1 \geq \theta$ and $q_1 = a_1\theta \geq \theta$, using (5.1(ii)), we obtain that $q_n \geq \theta, \quad \forall n \geq 1$. Further, (5.1(ii)) also gives

$$q_n = a_n \theta q_{n-1} + q_{n-2} \geq \theta^2 + q_{n-2} \quad \forall n \geq 2$$

Using this inequality, we have, by induction on n,

$$q_n \geq [\frac{n}{2}] \theta^2. \qquad (5.6)$$

As a consequence $q_n \to \infty$. So, from (5.5), $\mid x - \dfrac{p_n}{q_n} \mid \to 0$ as $n \to \infty$. This was already observed in Bhattacharya and Goswami [1]. We shall now improve upon the estimate (5.6) which will be used later. To do this note that from (5.1(ii)), $q_n \geq (\theta^2 + 1)q_{n-2}$ for $n \geq 1$. Now using induction on n, we get,

$$q_n \geq (\theta^2 + 1)^{[\frac{n}{2}]} \theta \quad \forall n. \qquad (5.7)$$

$[a_1\theta, a_2\theta, \cdots]$ arises as the continued fraction expansion of a number x smaller than θ. It is easy to see that a sequence $[a_0\theta; a_1\theta, a_2\theta, \cdots]$ arises as the continued fraction expansion of a number if and only if $[a_1\theta, a_2\theta, \cdots]$ arises as the expansion of a number smaller than θ. To understand the idea, note that, for

the usual continued fraction expansion (case $\theta = 1$), $[0; 2, 1]$ does not arise as the expansion of any number – the correct one being $[0; 3]$. In fact, in the usual case this is the only restriction. More precisely $[a_1, a_2, \cdots]$ arises as the usual expansion of a number smaller than one iff (i) each $a_i \geq 1$; (ii) in case it is terminating the last a_k is strictly larger than one.

Let us start with the most simple case, namely, $1/\theta$ is an integer w.r.t θ – i.e. $1/\theta = n\theta$ for some integer $n \geq 1$.

Theorem 4: *Let $1/\theta = n\theta$ for some integer, say, $n \geq 1$. Then $[a_1\theta, a_2\theta, \cdots]$ arises as the expansion of a number smaller than θ iff*

(i) *each $a_i \geq n$;*

(ii) *in case it is terminating the last a_k is strictly larger than n.*

Proof: Suppose $[a_1\theta, a_2\theta, \cdots, a_k\theta]$ is the continued fraction expansion of a number $x < \theta$. Then $n\theta = \dfrac{1}{\theta} < \dfrac{1}{x}$ shows that $a_1 \geq n$. Now $\dfrac{1}{x} = a_1\theta + \dfrac{1}{r_2}$ where $0 < \dfrac{1}{r_2} < \theta$ implying as earlier $a_2 \geq n$. Proceeding this way, we get that $a_i \geq n$ for all $i < k$. Since $r_k = a_k\theta > \dfrac{1}{\theta} = n\theta$, we get $a_k > n$ as claimed. Conversely, suppose we have integers a_i for $1 \leq i \leq k$ so that $a_i \geq n$ for $i < k$ and $a_k > n$. Then, since $a_k > n$, we have $a_k\theta > n\theta$ or $\dfrac{1}{a_k\theta} < \theta$. Also, $a_{k-1} \geq n$ implies that $a_{k-1}\theta + \dfrac{1}{a_k\theta} > n\theta$ or $[a_{k-1}\theta, a_k\theta] < \theta$. Proceeding this way, we can show $[a_i\theta, \cdots, a_k\theta] < \theta$ for $1 \leq i \leq k$. Thus if we define $x = [a_1\theta, \cdots, a_k\theta]$ then $x < \theta$ and indeed $[a_1\theta, \cdots, a_k\theta]$ is the continued fraction expansion of x. Similar but simpler argument applies to show that in the non-terminating case, it is necessary and sufficient to have each $a_i \geq n$. To consider a slightly more general case suppose that $\dfrac{1}{\theta} = [n_1\theta; n_2\theta]$. Thus we have

$$\frac{1}{(n_1 + 1)\theta} < \theta < \frac{1}{n_1\theta}$$

and

$$\theta = \cfrac{1}{n_1\theta + \cfrac{1}{n_2\theta}}$$

It should be observed that in such a case $n_2 > (n_1 + 1)$. In this case $[a_1\theta, a_2\theta, \cdots]$ arises as the expansion of a number smaller than θ iff the following conditions hold:

(i) Each $a_i \geq n_1$;

(ii) In case an $a_i = n_1$ then $a_{i+1} < n_2$;

(iii) In the terminating case the last a_k must satisfy $a_k > n_1$.

This can be seen as follows. Suppose $[a_1\theta, \cdots, a_k\theta]$ is the continued fraction expansion of a number $x < \theta$. If $k = 1$, it trivially follows that $a_1 > n_1$ as claimed.

Let us assume that $k \geq 2$. Arguing as in the previous case, we obtain that $a_i \geq n_1$ for each i and $a_k > n_1$. Suppose that for some $i < k$, $a_i = n_1$. Then $[a_i\theta, a_{i+1}\theta, \cdots, a_k\theta] < \theta$ implies that $a_i\theta + [a_{i+1}\theta, \cdots, a_k\theta] > n_1\theta + \dfrac{1}{n_2\theta}$. But since $a_i = n_1$, this immediately implies $[a_{i+1}\theta, \cdots, a_k\theta] > \dfrac{1}{n_2\theta}$ so that $a_{i+1}\theta + [a_{i+2}\theta, \cdots, a_k\theta] < n_2\theta$ and consequently, $a_{i+1} < n_2$. Conversely, suppose that $a_1, \cdots, a_k$ are integers satisfying the conditions of the claim. As in the previous case, $a_k > n_1$ implies $[a_k\theta] < \theta$. Now if $a_{k-1} > n_1$, $a_{k-1}\theta + \dfrac{1}{a_k\theta} > n_1\theta + \dfrac{1}{n_2\theta}$ implying that $[a_{k-1}\theta, a_k\theta] < \theta$. On the other hand, if $a_{k-1} = n_1$, then using the hypothesis that $a_k < n_2$, we get $a_{k-1}\theta + \dfrac{1}{a_k\theta} = n_1\theta + \dfrac{1}{a_k\theta} > n_1\theta + \dfrac{1}{n_2\theta}$. So, $[a_{k-1}\theta, a_k\theta] < \theta$. One can now proceed as in the earlier case and show that $[a_1\theta, \cdots, a_k\theta]$ is indeed the continued fraction expansion of a number $x < \theta$. The non-terminating case is dealt with in an analogous manner. $\square$

The ideas used above, executed with a little care, will lead to a proof of the following theorem which considers the case when $\dfrac{1}{\theta}$ is rational w.r.t θ – *i.e.* it is a ratio of two polynomials in θ, with integer coefficients. We shall not go into the details.

Theorem 5: *Suppose that $\dfrac{1}{\theta} = [n_1\theta; n_2\theta, \cdots, n_m\theta]$. Then $[a_1\theta, a_2\theta, \cdots]$ arises as the expansion of a number smaller than θ iff the following conditions hold:*

(i) *Each $a_i \geq n_1$.*

(ii) *In case for some $i \geq 1$ and $p < m$, $\langle a_{i+1}, \cdots, a_{i+p} \rangle = \langle n_1, \cdots, n_p \rangle$ then we should have $a_{i+p+1} \leq n_{p+1}$ if $p+1$ is even while $a_{i+p+1} \geq n_{p+1}$ if $p+1$ is odd. Moreover if m is even and $p+1$ equals m, then $a_{i+p+1} < n_{p+1}$.*

(iii) *In the terminating case the last a_k must satisfy $a_k > n_1$ and further if for some even $p < m$, (i.e, $p+1$ is odd) $\langle a_{k-p}, \cdots, a_{k-1} \rangle = \langle n_1, \cdots, n_p \rangle$, then $a_k > n_{p+1}$.*

The case when $\dfrac{1}{\theta}$ is irrational w.r.t. θ – *i.e* has a non-terminating expansion can also be discussed in a similar fashion leading to the following theorem.

Theorem 6: *Let $\dfrac{1}{\theta} = [n_1\theta; n_2\theta, \cdots]$. Then $[a_1\theta, a_2\theta, \cdots]$ arises as the expansion of a number smaller than θ iff*

(i) *each $a_i \geq n_1$.*

(ii) *In case for some $i \geq 1$ and $p \geq 1$, $\langle a_{i+1}, \cdots, a_{i+p} \rangle = \langle n_1, \cdots, n_p \rangle$ then $a_{i+p+1} \leq n_{p+1}$ if $p+1$ is even while $a_{i+p+1} \geq n_{p+1}$ if $p+1$ is odd;*

(iii) *In the terminating case the last a_k must satisfy $a_k > n_1$ and further if for some even $p \geq 1$, (i.e, $p+1$ is odd) $\langle a_{k-p}, \cdots, a_{k-1} \rangle = \langle n_1, \cdots, n_p \rangle$ then, $a_k > n_{p+1}$.*

The above discussion gives necessary and sufficient criteria for an expression $[a_0\theta; a_1\theta, \cdots]$, to be actually the continued fraction expansion of a number. How-

ever these conditions depend on the expansion of $\frac{1}{\theta}$. More precisely the conditions depended on the sequence of integers $n_1, n_2, \cdots$ where $\frac{1}{\theta} = [n_1\theta; n_2\theta, \cdots]$, It is natural to enquire as to how the expansion of $\frac{1}{\theta}$ itself looks like. This is what we do in the next section.

6   Expansion of $1/\theta$

We shall now discuss the following problem: Given a finite sequence of integers $n_1, n_2, \cdots, n_k$, find conditions so that there is a number θ, $0 < \theta \le 1$, such that $\frac{1}{\theta} = [n_1\theta; n_2\theta, \cdots, n_k\theta]$. In case such a number exists, is it unique? Unfortunately we do not have the complete solution. One can easily observe that when such a θ exists each n_i must be at least as large as n_1 and $n_k > n_1$. However this is not a sufficient condition. For example we can not have $\frac{1}{\theta} = [2\theta; 3\theta]$, a simple algebra shows that the correct expansion is $\frac{1}{\theta} = [3\theta;]$.

The situation $k = 1$ is very simple. In this case for any integer $n_1 \ge 1$ there is indeed a unique θ and it is given by $\theta = 1/\sqrt{n_1}$.

The situation $k = 2$ is a little more involved. Note that the quadratic in θ, namely, $\frac{1}{\theta} = n_1\theta + \frac{1}{n_2\theta}$ can always be solved for θ; but it does not ensure that the resulting $\frac{1}{\theta}$ has the required expansion. In fact we must necessarily have $n_2 > n_1 + 1$. To see this, observe that if $n_1\theta + \frac{1}{n_2\theta}$ is the expansion of $\frac{1}{\theta}$, then we have $\frac{1}{\theta} > n_1\theta$ and $\frac{1}{n_2\theta} < \theta$. so that $n_2 > n_1$. Further if $n_2 = n_1 + 1$, then

$$\frac{1}{\theta} = n_1\theta + \frac{1}{(n_1+1)\theta}$$

reduces to $\frac{1}{\theta} = (n_1 + 1)\theta$, which is not the required expansion. Thus we must have $n_2 > n_1 + 1$. Moreover when this condition holds there is such a θ and it is given by $\theta = \sqrt{(n_2 - 1)/n_1 n_2}$. Indeed, for this θ

$$\frac{1}{\theta} = \frac{n_1 n_2}{n_2 - 1}\theta = n_1\theta + \frac{n_1}{n_2 - 1}\theta = n_1\theta + \frac{1}{n_2\theta}$$

Further such a θ is unique, because the quadratic equation to be satisfied by θ has only one positive root.

The situation $k = 3$ is more complicated. It is necessary to have $n_2 > n_1$ and also $n_3 > n_1$. Further when this condition holds there is such a θ and it is given by

$$\theta = \sqrt{\frac{\sqrt{(n_1 + n_3 - n_2 n_3)^2 + 4n_1 n_2 n_3} - (n_1 + n_3 - n_2 n_3)}{2n_1 n_2 n_3}}$$

In fact the equation to be satisfied by θ is of fourth degree having two nonreal complex roots, one positive and one negative root. Thus such a θ is unique as well. With this choice, $1/\theta$ has the required θ-expansion. Thus we have proved

Theorem 7:

(i) *For the existence of a number $0 < \theta < 1$ such that $1/\theta$ has the θ-expansion $[n_1\theta; n_2\theta]$ it is necessary and sufficient that $n_2 > n_1 + 1$. When this holds, such a θ is unique.*

(ii) *For the existence of a number $0 < \theta < 1$ such that $1/\theta$ has the θ-expansion $[n_1\theta; n_2\theta, n_3\theta]$ it is necessary and sufficient that $n_2 > n_1$ and $n_3 > n_1$. When this holds, such a θ is unique.*

However the situation for values of k larger than 3, eludes us. For the existence of a number θ, $0 < \theta < 1$ such that $1/\theta$ has the θ-expansion $[n_1\theta; n_2\theta, \cdots, n_m\theta]$ the following conditions appear to be necessary and sufficient:

(i) $n_i > n_1$ for $i = 2, m$ where as $n_i \geq n_1$ for $2 < i < m$.

(ii) If for some i and p with $i + p < m$, $\langle n_{i+1}, \cdots, n_{i+p} \rangle = \langle n_1, \cdots, n_p \rangle$, then we should have $n_{i+p+1} \leq n_{p+1}$ if $p + 1$ is even where as $n_{i+p+1} \geq n_{p+1}$ if $p + 1$ is odd.

Before proceeding further, we mention that in the literature, there exist several generalizations of the usual continued fraction expansions. See, Bissinger [3], Everett [7] and Renyi [14] Kraikamp and Nakada [10] and the references therein.

7 Generalized Gauss Transformation

Recall that the Gauss transformation on the interval $[0, 1)$ associated with the usual continued fraction expansions is defined by

$$U(x) = \begin{cases} \frac{1}{x} - [\frac{1}{x}] & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

The Gauss measure μ defined by $d\mu(x) = \dfrac{1}{\log 2} \dfrac{1}{1+x} \, dx$ on $[0, 1)$ is ergodic and invariant for U [2]. Further

$(*)$ $$\frac{a_1 + \cdots + a_n}{n} \longrightarrow \infty \quad \text{a.e.}$$

$(**)$ $$\lim_{n \to \infty} \frac{1}{n} \log q_n = \gamma \quad \text{a.e. for some finite number} \quad \gamma,$$

As in section 5, a_n are the digits in the continued fraction expansion and $p_n \,/\, q_n$ is the n-th convergent. And a.e. refers to μ, or equivalently to Lebesgue measure. These properties play a crucial role in [1].

The analogue of this transformation for the θ expansion is the transformation T – referred to as the generalized Gauss transformation – defined on $[0,\theta)$ as follows :

$$T(x) = \begin{cases} \frac{1}{x} - \theta[\frac{1}{\theta x}] & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0 \end{cases}$$

For several values of $\theta < 1$ it was shown in [5] – by using the concept of Markov maps – that T has an ergodic invriant measure equivalent to Lebesgue measure and moreover $(*)$ and $(**)$ hold. We shall not go into the details for two reasons. First, there may be simpler argument. Second, even after establishing these properties , which are no doubt interesting, we have not been able to draw conclusions about the distribution of X_∞. Theorems 4, 5, 6, and 7 are nothing but a description of the symbolic dynamics of this transformation. As remarked to us by Professor R.F Williams, this transformation is piecewise C^2 and is expanding – with derivative (in modulus) bounded below by $1/\theta^2$. By using well -known results (see [6] or perhaps implicit in [14]) we get.

Theorem 8: *The generalized Gauss transformation T on $[0,\theta)$ defined above admits an absolutely continuous invariant measure. A tractable special case of this transformation will be discussed in the next section.*

Before proceeding further, we remark the following. One can define a map on $[0,\theta)$ to itself by putting $U_1(x) = \theta(\frac{x}{\theta} - [\frac{x}{\theta}])$ and one can also define a map on $[0,\frac{1}{\theta})$ to itself by putting $U_2(x) = \frac{1}{\theta}(\frac{1}{\theta x} - [\frac{1}{\theta x}])$. Obviously, these maps are conjugate to the Gauss map U on $[0,1)$. However, the map T that we defined above is different from U_1 and U_2 and this map T is relevent for our discussion. We could not see if this is conjugate to the Gauss map U. Professor Y.Guivarch informed us that for several values of θ, T and U have different entropies and hence can not be conjugate.

8   Invariant Measure for T when $\dfrac{1}{\theta^2} \in I\!N$

In this section we assume that $\dfrac{1}{\theta^2} \in I\!N$. Thus for some integer, say l, $\dfrac{1}{\theta} = l\theta$. Thus $\dfrac{1}{\theta}$ has continued fraction expansion terminating at the first stage itself, $\dfrac{1}{\theta} = [l\theta]$. Throughout this section θ and hence the integer l is fixed.

We shall now extend the usual argument (see Billingsley [2]) to get an absolutely continuous invariant measure for the above transformation.
In fact, we claim that

$$dP(x) \quad = \quad \frac{1}{\log\frac{l+1}{l}} \, \frac{1}{\sqrt{l+x}} dx$$

which is same as saying

$$dP(x) \quad = \quad \frac{1}{\log(1+\theta^2)} \, \frac{\theta}{1+\theta x} dx$$

is the required invariant measure for T. In the present case, we are lucky enough to explicitly write down the invariant measure which is perhaps not possible in general.

Since we could not see any direct way of connecting the transformation T with the usual Gauss transformation U, we shall verify the above claim by carrying the same steps as in Billingsley referred to above. In order to show that T preserves P, it is enough to show that $P[0, \theta u) = P(T^{-1}[0, \theta u))$ for all $u \in [0, 1)$.

Since $T^{-1}[0, \theta u) = \bigcup_{k=l}^{\infty} (\dfrac{1}{(k+u)\theta}, \dfrac{1}{k\theta})$ (equality is upto a set of Lebesgue measure zero), it is enough to verify the following :

$$\int_0^{\theta u} \frac{\theta}{1+\theta x} dx = \sum_{k=l}^{\infty} \int_{\frac{1}{(k+u)\theta}}^{\frac{1}{k\theta}} \frac{\theta}{1+\theta x} dx.$$

The sum on the right side, after evaluation of the integrals, is a telescopic sum which equals $\log (\dfrac{l+u}{l})$ same as the left side.

We now show that P is ergodic too.

As in Billingsley[2], we introduce the sets $\Delta_{a_1, a_2, \cdots, a_n}$ and the maps $\psi_{a_1, a_2, \cdots, a_n} : [0, \theta) \to [0, \theta)$ as follows. $\Delta_{a_1, a_2, \cdots, a_n}$ is the set of all x such that $a_i(x) = a_i$ for $i = 1, 2, \cdots, n$. In view of the discussion in section 5, $\Delta_{a_1, a_2, \cdots, a_n}$ may be empty for some n-tuples $(a_1, a_2, \cdots, a_n)$. In what follows we assume that $\Delta_{a_1, a_2, \cdots, a_n}$ is non-empty for the n-tuple $(a_1, a_2, \cdots, a_n)$ under consideration. $\psi_{a_1, a_2, \cdots, a_n}$ is given by,

$$\psi_{a_1, a_2, \cdots, a_n}(t) = \cfrac{1}{a_1\theta + \cfrac{1}{a_2\theta + \cfrac{1}{\cdots + \cfrac{1}{a_n\theta + t}}}}, \quad t \in [0, \theta).$$

Then $\Delta_{a_1, a_2, \cdots, a_n}$ is the image of $[0, \theta)$ under $\psi_{a_1, a_2, \cdots, a_n}$. One can show that $\psi_{a_1, a_2, \cdots, a_n}(t) = \dfrac{p_n + tp_{n-1}}{q_n + tq_{n-1}}$ for $t \in [0, \theta)$ just like in (5.5). Recall that $\dfrac{p_n}{q_n} = [a_1\theta, a_2\theta, \cdots, a_n\theta]$. Also $\psi_{a_1, a_2, \cdots, a_n}(t)$ is decreasing for odd n and increasing for even n. So,

$$\Delta_{a_1, a_2, \cdots, a_n} = \begin{cases} \left[\dfrac{p_n}{q_n}, \ \dfrac{p_n + \theta p_{n-1}}{q_n + \theta q_{n-1}} \right] & \text{if} \quad n \quad \text{even,} \\[4mm] \left[\dfrac{p_n + \theta p_{n-1}}{q_n + \theta q_{n-1}}, \ \dfrac{p_n}{q_n} \right] & \text{if} \quad n \quad \text{odd.} \end{cases}$$

Using (5.2), we see,

$$\lambda(\Delta_{a_1, a_2, \cdots, a_n}) = \frac{\theta}{q_n(q_n + \theta q_{n-1})} \tag{8.1}$$

where λ, as usual, denotes Lebesgue measure.

Let us denote $\Delta_{a_1,a_2,\cdots,a_n}$ and $\psi_{a_1,a_2,\cdots,a_n}$ by Δ_n and ψ_n respectively. Here we fix $a_1, a_2, \cdots, a_n$. Then Δ_n has length $\mid \psi_n(\theta) - \psi_n(0) \mid$. Also, for $0 \leq x < y \leq \theta$, the interval $\{\omega : x \leq T^n(\omega) < y\} \cap \Delta_n$ has length $\mid \psi_n(y) - \psi_n(x) \mid$.

So, using the notation, $\lambda(A|B) = \lambda(A \cap B) / \lambda(B)$, we have,

$$\lambda(T^{-n}[x,y]|\Delta_n) = \frac{\psi_n(y) - \psi_n(x)}{\psi_n(\theta) - \psi_n(0)}.$$

In absolute value the numerator equals $\dfrac{y - x}{(q_n + xq_{n-1})(q_n + yq_{n-1})}$ and the denominator equals $\dfrac{\theta}{q_n(q_n + \theta q_{n-1})}$.

After some algebra,

$$\frac{\psi_n(y) - \psi_n(x)}{\psi_n(\theta) - \psi_n(0)} = \frac{y - x}{\theta} \frac{1}{1 + y\frac{q_{n-1}}{q_n}} \frac{1}{1 - \frac{(\theta - x)q_{n-1}}{q_n + \theta q_{n-1}}}. \tag{8.2}$$

Now $\dfrac{q_n}{q_{n-1}} \geq \theta$ and hence, the right hand side of (8.2) $\geq \dfrac{y - x}{2\theta}$.

Again, $\dfrac{q_{n-1}}{q_n + \theta q_{n-1}} \leq \dfrac{1}{2\theta}$ so that $1 - \dfrac{(\theta - x)q_{n-1}}{q_n + \theta q_{n-1}} \geq \dfrac{1}{2}$ and hence the right hand side of (5.2) is $\leq \dfrac{2(y - x)}{\theta}$. Thus,

$$\frac{y - x}{2\theta} \leq \lambda(T^{-n}[x,y]|\Delta_n) \leq \frac{2(y - x)}{\theta}.$$

Hence, for any Borel set A also, we have,

$$\frac{\lambda(A)}{2\theta} \leq \lambda(T^{-n}(A)|\Delta_n) \leq \frac{2\lambda(A)}{\theta}. \tag{8.3}$$

Now, since $0 \leq x < \theta$,

$$\frac{1}{\log(1 + \theta^2)} \frac{\theta}{1 + \theta^2} \leq \frac{1}{\log(1 + \theta^2)} \frac{\theta}{1 + \theta x} \leq \frac{\theta}{\log(1 + \theta^2)}.$$

Hence, for any Borel set M, we have,

$$\frac{1}{\log(1 + \theta^2)} \frac{\theta}{1 + \theta^2} \lambda(M) \leq P(M) \leq \frac{\theta}{\log(1 + \theta^2)} \lambda(M). \tag{8.4}$$

So, $\lambda(M) \leq \dfrac{1 + \theta^2}{\theta} \log(1 + \theta^2) P(M)$ and $\lambda(M) \geq \dfrac{\log(1 + \theta^2)}{\theta} P(M)$.

Therefore, using these inequalities together with (8.3) and (8.4), we get the following :

$$C_1(\theta)P(A) \leq P(T^{-n}(A)|\Delta_n) \leq C_2(\theta)P(A)$$

where C_1, C_2 are constants depending on θ only. Now if A is invariant, the above inequality becomes

$$C_1(\theta)P(A) \leq P(A|\Delta_n) \leq C_2(\theta)P(A).$$

Assuming $P(A) > 0$, we get,

$$C_1(\theta)P(\Delta_n) \leq P(\Delta_n|A) \leq C_2(\theta)P(\Delta_n).$$

Hence, for any Borel set E,

$$C_1(\theta)P(E) \leq P(E|A) \leq C_2(\theta)P(E).$$

Taking $E = A^c$, one gets $P(A^c) = 0$ so that $P(A) = 1$. Therefore, T is ergodic under P, as claimed.

We now prove $(*)$ and $(**)$ also hold – again following Billingsley closely.

By ergodic theorem, if f is any non-negative function on $[0, \theta]$, integrable or not, we have,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(\omega)) = \frac{1}{\log(1+\theta^2)} \int_0^\theta \frac{\theta f(x)}{1+\theta x} dx \quad \text{a.e.} \quad [P].$$

Taking $f = a_1$, the right hand side becomes,

$$\frac{1}{\log(1+\theta^2)} \int_0^\theta \frac{\theta a_1(x)}{1+\theta x} dx = \sum_{k=l}^\infty \frac{1}{\log(1+\theta^2)} \int_{\frac{1}{(k+1)\theta}}^{\frac{1}{k\theta}} \frac{k\theta}{1+\theta x} dx$$

$$= \frac{1}{\log(1+\theta^2)} \sum_{k=l}^\infty k \log(1 + \frac{1}{k^2+2k}) = \infty.$$

Thus,

$$\lim_{n \to \infty} \sum_{k=0}^{n-1} a_1(T^k(\omega)) = \lim_{n \to \infty} \sum_{k=1}^n a_k(\omega) = \infty \quad \text{a.e.} \quad [P].$$

This proves $(*)$. Towards $(**)$, first notice that,

$$\frac{1}{q_n(\omega)} = \prod_{k=1}^n \frac{p_{n+1-k}(T^{k-1}(\omega))}{q_{n+1-k}(T^{k-1}(\omega))}. \tag{8.5}$$

Also, from (5.5),

$$\mid x - \frac{p_n}{q_n} \mid \leq \frac{1}{q_n(x)q_{n+1}(x)}$$

Or,

$$\mid \frac{x}{\frac{p_n}{q_n}} - 1 \mid \leq \frac{1}{p_n(x)q_{n+1}(x)} \leq \frac{1}{(1+\theta^2)^n}$$

Or,

$$\mid \log(\frac{x}{\frac{p_n}{q_n}}) \mid \leq \log(1 + \frac{1}{(1+\theta^2)^n}) \leq \frac{1}{(1+\theta^2)^n} \tag{8.6}$$

So, using (8.5) and (8.6),

$\mid \log[a_k(\omega)\theta, a_{k+1}(\omega)\theta, \cdots] - \log[a_k(\omega)\theta, a_{k+1}(\omega)\theta, \cdots, a_n(\omega)\theta] \mid$

$= \mid \log(T^{k-1}(\omega)) - \log[a_1(T^{k-1}(\omega))\theta, a_2(T^{k-1}(\omega))\theta, \cdots, a_{n-k+1}(T^{k-1}(\omega))\theta] \mid$

$= \mid \log(T^{k-1}(\omega)) - \log \frac{p_{n+1-k}(T^{k-1}(\omega))}{q_{n+1-k}(T^{k-1}(\omega))} \mid \leq \frac{1}{(1+\theta^2)^{n-k+1}}.$

Using this in (8.5) we get

$$\frac{1}{n}\log\frac{1}{q_n(\omega)} = \frac{1}{n}\log\prod_{k=1}^{n}\frac{p_{n+1-k}(T^{k-1}(\omega))}{q_{n+1-k}(T^{k-1}(\omega))}$$

$$= \frac{1}{n}\sum_{k=1}^{n}\log(T^{k-1}(\omega)) + \frac{1}{n}\sum_{k=1}^{n}\frac{\zeta_{n,k}}{(1+\theta^2)^{n-k+1}}$$

for some numbers $\zeta_{n,k}$ which are smaller than one in modulus. Moreover since $\sum_{i=1}^{\infty}\frac{1}{(1+\theta^2)^i}$ is finite, the second term on the right side converges to zero. The ergodic theorem implies that the first term on the right side converges to $\frac{1}{1+\theta^2}\int_0^{\theta}\frac{\theta\log x}{1+\theta x}dx$ which is finite, say, γ. This proves (∗∗). Thus we have the following:

Theorem 9: *Let $\theta = \dfrac{1}{\sqrt{l}}$, $l \in I\!N$. Then μ given by*

$$d\mu(x) = \frac{1}{\log(1+\theta^2)}\frac{\theta}{1+\theta x}dx$$

is invariant and ergodic for the generalised Gauss map T on $[0,\theta)$ defined by

$$T(x) = \begin{cases} \frac{1}{x} - \theta[\frac{1}{\theta x}] & if \quad x \neq 0, \\ 0 & if \quad x = 0 \end{cases}.$$

Moreover, for a.e. $x = [a_1\theta, a_2\theta, \cdots]$ we have,

$$\lim_{n\to\infty}\frac{a_1 + \cdots + a_n}{n} = \infty.$$

Further, there is a finite number γ, such that if $\dfrac{p_n}{q_n}$ denotes the n-th convergent of x, then for a.e. x, we have,

$$\lim_{n\to\infty}\frac{1}{n}\log q_n = \gamma.$$

9 Remarks

We conclude with few remarks. The entire investigation above is directed to finding the nature of the distribution of the random continued fraction

$$X_\infty = [Z_1; Z_2, Z_3, \cdots]$$

where (Z_i) is a sequence of nondegenerate iid random variables with values in $[0,\infty)$. It makes sense to talk about the expression even when the variables are not identically distributed.

Thus assume that (Z_i) are independent random variables with values in $[0,\infty)$. By the zero - one law, we know that $\sum Z_n$ is either almost surely

finite or almost surely infinite. In the first case the continued fraction does not converge (K, Theorem 10, page 10). So let us consider the case when it is infinity almost surely. Then X_∞ is defined a.e. The question is whether one can establish results similar to sums of independent random variables.

Towards this end, suppose that each Z_i is discrete. One can show that the purity law of Jessen - Wintner holds. Here is the argument which is an adaptation of a well known argument – see for instance [4]. Let S denote the range of all the variables Z_i. Note that S is a countable set. Put

$$D = S \cup (-S) \cup \{0\} = \{x : x \in S \text{ or } -x \in S \text{ or } x = 0\}$$

For any two sets A and B of R , let $A + B = \{x + y : x \in A, y \in B\}$, $B^+ = \{x : x \in B, x \geq 0\}$ and $1/B = \{x : x = 0 \text{ or } x = 1/y \text{ for some } y \in B\}$. For any Borel set $B \subset [0, \infty)$ define $B_0 = (B + D)^+ \cup (1/B)$ and in general for $n \geq 0$, let $B_{n+1} = (B_n + D)^+ \cup (1/B_n)$. Finally, $B_\infty = \cup_{n \geq 0} B_n$. Note that D being countable, these are all again Borel sets. Moreover if B is countable then so is B_∞ and if B is Lebesgue null then so is B_∞. We claim that for every Borel set B the event $(X_\infty \in B_\infty)$ is a tail event for the sequence (Z_i). Indeed if $[Z_1; Z_2, Z_3, \cdots] \in B_n$, then $[0; Z_2, Z_3, \cdots] \in B_{n+1}$ and $[Z_2; Z_3, \cdots] \in B_{n+2}$. Conversely if $[Z_2; Z_3, \cdots] \in B_n$ then $[0; Z_2, Z_3, \cdots] \in B_{n+1}$ and $[Z_1; Z_2, Z_3, \cdots] \in B_{n+2}$. Thus for any Borel B, the event $(X_\infty \in B_\infty)$ has probability one or zero. If for some countable B this has probability one then X_∞ is discrete, otherwise it has continuous distribution. Suppose it is continuous. If for some Lebesgue null B, this event has probability one then X_∞ is singular, otherwise it has an absolutely continuous distribution. Thus we have

Theorem 10: *If Z_i are independent discrete nonnegative random variables with $\sum Z_i = \infty$, then the law of the continued fraction $X_\infty = [Z_1; Z_2, Z_3, \cdots]$ is pure.*

Suppose that the sequence (Z_i) is equivalent to a constant sequence in the sense of Khinchin – that is, there is a sequence of numbers (z_i) such that $\sum P(Z_i \neq z_i) < \infty$. Then it is clear that X_∞ is discrete. Perhaps the converse is also true as in the case of sums (see Levy [13]).

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