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STATISTICAL INFERENCE FOR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

Stochastic partial differential equations (SPDE) are used for stochastic modelling , for instance, in the study of neuronal behaviour in neurophysiology, in modelling sea surface temparature and sea surface height in physical oceanography , and in building stochastic models for turbulence. Probabilistic theory underlying the subject of SPDE is discussed in Ito (1984) and more recently in Kallianpur and Xiong (1995) among others. The study of statistical inference for the parameters involved in SPDE is more recent. Asymptotic theory of maximum likelihood estimators for a class of SPDE is discussed in Huebner, Khasminskii and Rozovskii (1993) and Huebner and Rozovskii(1995) following methods in Ibragimov and Khasminskii (1981). Bayes estimation problems for such a class of SPDE are investigated in Prakasa Rao (1998, 2000) following the techniques developed in Borwanker et al.(1971). An analogue of the Bernstein-von Mises theorem for parabolic stochastic partial differential equations is proved in Prakasa Rao (1998). As a consequence, the asymptotic properties of the Bayes estimators of the parameters are investigated using the asymptotic properties of maximum likelihood estimators proved in Huebner and Rozovskii (1995). Nonparametric estimation of a linear multiplier for some classes of SPDE are studied in Prakasa Rao(2000a,b) by the kernel method of density estimation following the techniques in Kutoyants(1994).

Key words: Bernstein-von Mises theorem, Stochastic partial differential equation, Maximum likelihood estimation, Bayes estimation, Nonparametric inference, Linear multiplier.

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1 Introduction

Stochastic partial differential equations(SPDE) are used for stochastic modelling, for instance, in the study of neuronal behviour in neurophysiology, in modelling sea surface temperature and sea surface height in physical oceanography and in building stochastic models for the behaviour of turbulence (cf. Kallianpur and Xiong (1995)). The probabilistic theory of SPDE is investigated in Ito (1984), Rozovskii (1990), Kallianpur and Xiong (1995) and De Prato and Zabczyk (1992) among others. Huebner et al.(1993) started the investigation of maximum likelihood estimation of parameters for a class of SPDE and extended their results to parabolic SPDE in Huebner and Rozovskii (1995) following the approach of Ibragimov and Khadsminskii(1981). Bernstein -von Mises type theorems were developed for such SPDE in Prakasa Rao (1998,2000) following the techniques in Borwanker et al.(1971) and Prakasa Rao (1981). Asymptotic properties of the Bayes estimators of parameters for SPDE were discussed in Prakasa Rao (1998,2000). Statistical inference for diffusion type processes and semimartingales in general is studied in Prakasa Rao (1999a,b).

Our aim in this paper is to review some recent work of us on the Bernstein-von Mises type theorems for parabolic SPDE and to present some new results on the problem of estimation of a linear multiplier for a class of SPDE using the methods of nonparametric inference following the approach of Kutoyants (1994).

2 Main Results

2.1 Bernstein-von Mises theorem

Let $(\Omega, \mathcal{F}, P)$ be a probability space and consider a stochastic partial differential equation (SPDE) of the form

$$du^\theta(t,x) = A^\theta u^\theta(t,x)dt + dW(t,x), 0 \leq t \leq T, x \in G \tag{2.1}$$

where $A^\theta = \theta A_1 + A_0$, A_1 and A_0 being partial differential operators, $\theta \in \Theta \subset R$ and $W(t,x)$ is a cylindrical Brownian motion in $L_2(G), G$ being a bounded domain in R^d with the boundary ∂G as a C^∞-manifold of dimension $(d-1)$ and locally G is totally on one side of ∂G. For the definition of cylindrical Brownian motion, see, Kallianpur and Xiong (1995), p.93.

The order $Ord(A)$ of a partial differential operator A is defined to be the order of the highest partial derivative in A. Let m_0 and m_1 be the orders of the operators A_0 and A_1 respectively. We assume that the operators A_0 and A_1 commute, m_1 is even and

$$\text{(C0)} \qquad m_1 \geq \frac{1}{2}(Ord(A^\theta) - d)$$

in the following discussion.

Suppose the solution $u^\theta(t,x)$ of (2.1) has to satisfy the boundary conditions

$$u^\theta(0,x) = u_0(x) \tag{2.2}$$

and

$$D^\gamma u^\theta(t,x)|_{\partial G} = 0 \tag{2.3}$$

for all multiindices γ such that $|\gamma| = m-1$ where $2m = \max(m_1, m_0)$. Here

$$D^\gamma f(\boldsymbol{x}) = \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \cdots \partial x_d^{\gamma_d}} f(\boldsymbol{x}) \tag{2.4}$$

with $|\gamma| = \gamma_1 + \cdots + \gamma_d$. Suppose that

$$A_i(\boldsymbol{x})u = - \sum_{|\alpha|,|\beta| \leq m_i} (-1)^{|\alpha|} D^\alpha (a_i^{\alpha\beta}(\boldsymbol{x}) D^\beta u) \tag{2.5}$$

where

$$a_i^{\alpha\beta}(x) \in C^\infty(\bar{G}). \tag{2.6}$$

Let

$$a^{\alpha\beta}(\theta, x) = \theta a_1^{\alpha\beta}(x) + a_0^{\alpha\beta}(x). \tag{2.7}$$

Suppose θ_0 is the true parameter.

We follow the notation introduced in Huebner and Rozovskii (1995). Assume that the following conditions hold.

(H1) The operators A_0 and A_1 satisfy the condition

$$\int_G A_i uv dx = \int_G uA_i v dx, u, v \in C_0^\infty(G), i = 0, 1.$$

(H2) There is a compact neighbourghood Θ of θ_0 so that $\{A_\theta, \theta \in \Theta\}$ is a family of uniformly strongly elliptic operators of order $2m = max(m_1, m_0)$.

For $s > 0$, denote the closure of $C_0^\infty(G)$ in the Sobolev space $W^{s,2}(G)$ by $W_0^{s,2}$.

The operator A^θ with boundary conditions defined by (2.3) can be extended to a closed self-adjoint operator $\mathcal{L}_\theta$ on $L_2(G)$ (Shimakura(1992)). In view of the condition (H2) , the operator $\mathcal{L}_\theta$ is lower semibounded, that is there exists a constant $k(\theta)$ such that $-\mathcal{L}_\theta + k(\theta)I > 0$ and the resolvent $(k(\theta)I - \mathcal{L}_\theta)^{-1}$ is compact. Let $\Lambda_\theta = (k(\theta)I - \mathcal{L}_\theta)^{\frac{1}{2m}}$. Let $h_i(\theta)$ be an orthonormal system of eigen functions of Λ_θ. We assume that the following condition holds.

(H3) There exists a complete orthonormal system $\{h_i, i \geq 1\}$ *independent* of θ such that

$$\Lambda_\theta h_i = \lambda_i(\theta) h_i, \theta \in \Theta.$$

The elements of the basis $\{h_i, i \geq 1\}$ are also eigen functions for the operator $\mathcal{L}_\theta$, that is

$$\mathcal{L}_\theta h_i = \mu_i^\theta h_i$$

where

$$\mu_i^\theta = -\lambda_i^{2m}(\theta) + k(\theta).$$

For $s \geq 0$, define H_θ^s to be the set of all $u \in L_2(G)$ such that

$$||u||_{s,\theta} = (\sum_{j=1}^{\infty} \lambda_j^{2s}(\theta)|(u,h_j)_{L_2(G)}|^2)^{1/2} < \infty.$$

For $s < 0$, H_θ^s is defined to be the closure of $L_2(G)$ in the norm $||u||_{s,\theta}$ given above. Then H_θ^s is a Hilbert space with respect to the inner product $(.,.)_{s,\theta}$ associated with the norm $||.||_{s,\theta}$ and the functions $h_{i,\theta}^s = \lambda_i^{-s}(\theta)h_i, i \geq 1$ form an orthonormal basis in H_θ^s. Condition (H2) imples that for every s ,the spaces H_θ^s are equivalent for all θ. We identify the spaces H_θ^s (denoted by H^s) and the norms $||.||_{s,\theta}$ for different $\theta \in \Theta$.

In addition to the conditions (H1)-(H3), we assume that
(H4)$u_0 \in H^{-\alpha}$ where $\alpha > \frac{d}{2}$. Note that $u_0 \in L_2(G)$,
(H5) the operator A_1 is uniformly strongly elliptic of even order m_1 and has the same system of eigen functions $\{h_i, i \geq 1\}$ as $\mathcal{L}_\theta$.

The conditons (H1)-(H5) described above are the same as those in Huebner and Rozovskii (1995).

Note that $u_0 \in H^{-\alpha}$. For $\theta \in \Theta$, define

$$u_{0i}^\theta = (u_o, h_{i\theta}^{-\alpha})_{-\alpha}. \tag{2.8}$$

Then the random field

$$u^\theta(t,x) = \sum_{i=1}^{\infty} u_i^\theta(t) h_{i\theta}^{-\alpha}(x) \tag{2.9}$$

is the solution of (2.1) subject to the boundary conditions (2.2) and (2.3) where $u_i^\theta(t)$ is the unique solution of the stochastic differential equation

$$du_i^\theta(t) = \mu_i^\theta u_i^\theta(t)dt + \lambda_i^{-\alpha}(\theta)dW_i(t), 0 \leq t \leq T, \tag{2.10}$$

$$u_i^{(\theta)}(0) = u_{0i}^\theta. \tag{2.11}$$

Let π^N be the orthogonal projection operator of $H^{-\alpha}$ onto the subspace spanned by $\{h_{i\theta}^{-\alpha}, 1 \leq i \leq N\}$. Let

$$\begin{aligned} u^{N,\theta}(t,x) &= \pi^N u^\theta(t,x) \\ &= \sum_{i=1}^{N} u_i^\theta(t) h_{i\theta}^{-\alpha}(x) \end{aligned} \tag{2.12}$$

where $u_i^\theta(t)$ is the solution of (2.10) subject to (2.11). Note that

$$du^{N,\theta}(t,x) = A^\theta u^{N,\theta}(t,x)dt + dW^N(t,x), 0 \leq t \leq T, x \in G \tag{2.13}$$

with

$$u^{N,\theta}(0,x) = \pi^N u_0(x) \tag{2.14}$$

and

$$W^N(t,x) = \sum_{i=1}^{N} \lambda_i^{-\alpha} W_i(t) h_{i\theta}^{-\alpha}(x). \tag{2.15}$$

Here $\{W_i(t), t \geq 0\}, i \geq 1$ are independent standard Wiener processes.

Let P_θ^N be the probability measure generated by $u^{N,\theta}$ on $C([0,T];R^N)$. Let $h_i^{-\alpha}$ denote $h_{i,\theta_0}^{-\alpha}, u^N$ denote u^{N,θ_0} and u denote u^{θ_0} when θ_0 is the true parameter. It is known that, for any $\theta \in \Theta$, the measures P_θ^N and $P_{\theta_0}^N$ are absolutely continuous with respect to each other and

$$\begin{aligned} \log \frac{dP_\theta^N}{dP_{\theta_0}^N}(u^N) &= (\theta-\theta_0)\int_0^T (A_1 u^N(s), du^N(s))_0 - \frac{(\theta^2-\theta_0^2)}{2}\int_0^T \|A_1 u^N(s)\|_0^2 ds \\ & -(\theta-\theta_0)\int_0^T (A_1 u^N(s), A_0 u^N(s))_0 ds. \end{aligned} \tag{2.16}$$

It is easy check that (cf. Huebner and Rozovskii (1995))

$$\hat{\theta}_N - \theta_0 = \frac{\int_0^T (A_1 u^N(s), dW^N(s))_0}{\int_0^T \|A_1 u^N(s)\|_0^2 ds} \tag{2.17}$$

where $\hat{\theta}_N$ is the maximum likelihood estimator of θ_0. Huebner and Rozovskii (1995) studied the asymptotic properties of this estimator under the conditions (H1)-(H5). Further more the Fisher information is given by

$$I_N = E\int_0^T \|A_1 u^{N,\theta_0}(s)\|_0^2 ds. \tag{2.18}$$

Note that $I_N \to \infty$ as $N \to \infty$ from the Lemma 2.1 of Huebner and Rozovskii (1995).

Suppose that Λ is a prior probability measure on $(\Theta, \mathcal{B})$ where $\mathcal{B}$ is the σ-algebra of Borel subsets of set $\Theta \subset R$. We assume that the true parameter $\theta_0 \in \Theta^0$, the interior of Θ. Further suppose that Λ has the density $\lambda(\cdot)$ with respect to the Lebesgue measure and the density $\lambda(\cdot)$ is continuous and positive in an open neighbourhood of θ_0, the true parameter.

Let

$$\tau = I_N^{1/2}(\theta - \hat{\theta}_N) \tag{2.19}$$

and

$$p^*(\tau|u^N) = I_N^{-1/2} p(\hat{\theta}_N + \tau I_N^{-1/2}|u^N) \tag{2.20}$$

where $p(\theta|u^N)$ is the posterior density of θ given u^N. Note that

$$p(\theta|u^N) = \frac{\frac{dP_\theta^N}{dP_{\theta_0}^N}(u^N)\lambda(\theta)}{\int_\Theta \frac{dP_{\theta_0}^N}{dP_{\theta_0}^N}(u^N)\lambda(\theta)d\theta} \tag{2.21}$$

and let $p^*(\tau|u^N)$ denote the posterior density of $I_N^{1/2}(\theta - \hat{\theta}_N)$. Let

$$\begin{aligned} \nu_N(\tau) &= \frac{dP^N_{\hat{\theta}_N + \tau I_N^{-1/2}}}{dP^N_{\theta_0}} \Big/ \frac{dP^N_{\hat{\theta}_N}}{dP^N_{\theta_0}} \\ &= \frac{dP^N_{\hat{\theta}_N + \tau I_N^{-1/2}}}{dP^N_{\hat{\theta}_N}} \text{ a.s.} \end{aligned} \tag{2.22}$$

In view of (2.16), it follows that

$$\log \nu_N(\tau) = -\frac{1}{2}\tau^2 I_N^{-1} \int_0^T \|A_1 u^N(s)\|_0^2 ds \tag{2.23}$$

since

$$\hat{\theta}_N = \frac{\int_0^T (A_1 u^N(s), du^N(s) - A_0(s)u^N(s)ds)_0}{\int_0^T \|A_1 u^N(s)\|_0^2 ds}. \tag{2.24}$$

Let

$$C_N = \int_{-\infty}^{\infty} \nu_N(\tau)\lambda(\hat{\theta}_N + \tau I_N^{-1/2})d\tau. \tag{2.25}$$

It can be checked that

$$p^*(\tau|u^N) = C_N^{-1}\nu_N(\tau)\lambda(\hat{\theta}_N + \tau I_N^{-1/2}). \tag{2.26}$$

Note that (C1) $\beta_N = I_N^{-1}\int_0^T \|A_1 u^N(s)\|_0^2 ds \to 1$ a.s. $[P_{\theta_0}]$ as $N \to \infty$. from the Lemma 2.2 of Huebner and Rozovskii (1995). Then the following relations hold:

(i) $\lim_{N\to\infty} \nu_N(\tau) = \exp(-\frac{1}{2}\tau^2)$ a.s. $[P_{\theta_0}]$,

(ii) for any $0 < \gamma < 1$,

$$\log \nu_N(\tau) \leq -\frac{1}{2}\tau^2(1-\gamma)$$

for every τ for sufficiently large N, and

(iii) for every $\delta > 0$, there exists $\gamma' > 0$ such that

$$\sup_{|\tau|>\delta I_N^{1/2}} \nu_N(\tau) \leq \exp\{-\frac{1}{4}\gamma' I_N^{-1}\}$$

as $N \to \infty$.

Further more (C2) the maximum likelihood estimator $\hat{\theta}_N$ is strongly consistent, that is

$$\hat{\theta}_N \to \theta_0 \text{ a.s. } [P_{\theta_0}] \text{ as } N \to \infty.$$

from the Lemmas 2.1 and 2.2 in Huebner and Rozovskii (1995). Suppose that (C3) $K(\cdot)$ is a nonnegative function such that, for some $0 < \gamma < 1$,

$$\int_{-\infty}^{\infty} K(\tau)e^{-\frac{1}{2}\tau^2(1-\gamma)}d\tau < \infty.$$

(C4) For every $\eta > 0$ and $\delta > 0$,

$$e^{-\eta I_N^{-1}} \int_{|\tau|>\delta} K(\tau I_N^{-1/2})\lambda(\hat{\theta}_N + \tau)d\tau \to 0 \text{ a.s } [P_{\theta_0}]$$

as $N \to \infty$.

We now have the following main theorem which is an analogue of the Bernstein-von Mises theorem (cf.Prakasa Rao (1981, 1984)) for diffusion processes and diffusion fields. A special case of this result for some classes of SPDE's was recently proved in Prakasa Rao (2000).

Theorem 2.1: Suppose the conditions (C3) and (C4) hold in addition to the conditions (H1)-(H5) stated earlier where $\lambda(\cdot)$ is a prior density which is continuous and positive in an open neighbourhood of θ_0, the true parameter. Then

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} K(\tau)|p^*(\tau|u^N) - \left(\frac{1}{2\pi}\right)^{1/2} e^{-\frac{1}{2}\tau^2}|d\tau = 0 \text{ a.s. } [P_{\theta_0}]. \tag{2.27}$$

As a consequence of Theorem 2.1, it is easy to get the following result.

Theorem 2.2: Suppose the conditions (H1)-(H5) hold. In addition suppose that:

(D1) $\lambda(\cdot)$ is a prior density which is continuous and positive in an open neighbourhood of θ_0, the true parameter; and

(D2) $\int_{-\infty}^{\infty} |\theta|^m \lambda(\theta) d\theta < \infty$ for some integer $m \geq 0$.

Then

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} |\tau|^m |p^*(\tau|u^N) - \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\tau^2}|d\tau = 0 \text{ a.s. } [P_{\theta_0}]. \tag{2.28}$$

Remarks: It is obvious that the condition (D2) holds for $m = 0$. Suppose the condition (D1) holds. Then it follows that

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} |p^*(\tau|u^N) - \left(\frac{1}{2\pi}\right)^{1/2} e^{-\frac{1}{2}\tau^2}|d\tau = 0 \text{ a.s. } [P_{\theta_0}]. \tag{2.29}$$

This is the analogue of the Bernstein-von Mises theorem in the classical statistical inference. As a particular case of Theorem 2.2, we obtain that

$$E_{\theta_0}[I_N^{1/2}(\hat{\theta}_N - \theta_0)]^m \to E[Z]^m \text{ as } N \to \infty \tag{2.30}$$

where Z is $N(0,1)$.

For proofs of Theorems 2.1 and 2.2, see Prakasa Rao(1998).

3 Bayes Estimation

We define an estimator $\tilde{\theta}_N$ for θ to be a Bayes estimator based on the path u^N corresponding to the loss function $\tilde{L}(\theta,\varphi)$ and the prior density $\lambda(\theta)$ if it is an estimator which minimizes the function

$$B_N(\varphi) = \int \tilde{L}(\theta,\varphi)p(\theta|u^N)d\theta, \varphi \in \Theta$$

where $\tilde{L}(\theta,\varphi)$ is defined on $\Theta \times \Theta$. Suppose there exist a Bayes estimator $\tilde{\theta}_N$. Further suppose that the loss function $\tilde{L}(\theta,\varphi)$ satisfies the following conditions:

(E1) $\tilde{L}(\theta,\varphi) = L(|\theta - \varphi|) \geq 0$;

(E2) $L(t)$ is nondecreasing for $t \geq 0$;

(E3) there exists nonnegative functions $R_N, K(\tau)$ and $G(\tau)$ such that

(a) $R_N L(\tau I_N^{-1/2}) \leq G(\tau)$ for all $N \geq 1$;

(b) $R_N L(\tau I_N^{-1/2}) \to K(\tau)$ as $N \to \infty$ uniformly on bounded intervals of τ;

(c) the function

$$\int_{-\infty}^{\infty} K(\tau + m)e^{-\frac{1}{2}\tau^2}d\tau$$

achieves its minimum at $m = 0$, and

(d) $G(\tau)$ satisfies the conditions similar to (C3) and (C4).

The following result can be proved by arguments similar to those given in Borwanker et al. (1971). We omit the proof.

Theorem 3.1: Suppose the conditions (D1)-(D2) of Theorem 2.2 hold in addition to (H1)-(H5) stated earlier. In addition , suppose that the loss function $\tilde{L}(\theta, \varphi)$ satisfies the conditions (E1) - (E3) stated above. Then

$$I_N^{1/2}(\hat{\theta}_N - \tilde{\theta}_N) \to 0 \text{ a.s. } [P_{\theta_0}] \text{ as } N \to \infty \tag{3.1}$$

and

$$\begin{aligned} \lim_{N\to\infty} R_N B_N(\tilde{\theta}_N) &= \lim_{N\to\infty} R_N B_N(\hat{\theta}_N) \\ &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} K(\tau) e^{-\frac{1}{2}\tau^2} d\tau. \end{aligned} \tag{3.2}$$

Huebner and Rozovskii (1995) proved that

$$\hat{\theta}_N \to \theta_0 \text{ a.s. } [P_{\theta_0}] \text{ as } N \to \infty \tag{3.3}$$

and

$$I_N^{1/2}(\hat{\theta}_N - \theta_0) \xrightarrow{\mathcal{L}} N(0,1) \text{ as } N \to \infty \tag{3.4}$$

under the conditions (H1)-(H5). As a consequence of Theorem 3.1, it follows that

$$\tilde{\theta}_N \to \theta_0 \text{ a.s } [P_{\theta_0}] \text{ as } N \to \infty \tag{3.5}$$

and

$$I_N^{1/2}(\tilde{\theta}_N - \theta_0) \xrightarrow{\mathcal{L}} N(0,1) \text{ as } N \to \infty. \tag{3.6}$$

In other words the Bayes estimator $\tilde{\theta}_N$ of the paramaeter θ in the parabolic SPDE given by (2.1) is strongly consistent, asymptotically normal and asymptotically efficient as $N \to \infty$ under the conditions (H1)-(H5) of Huebner and Rozovskii (1995) and the conditions stated in Theorem 3.1.

Remarks: A general approach for the study of asymptotic properties of maximum likelihood estimators and Bayes estimators is by proving the local asymptotic normality of the loglikelihood ratio process as was done in

Prakasa Rao (1968), Ibragimov and Khasminskii (1981) in the classical i.i.d. cases and by Huebner and Rozovskii (1993) for some classes of SPDE. Our approach for Bayes estimation, via the comparison of the rates of convergence of the difference between the maximum likelihood estimator and the Bayes estimator, is a consequence of the the Bernstein - Von Mises type theorem .

We now consider a nonparametric version of the problem discussed earlier for a class of SPDE.

4 Stochastic PDE with Linear Multiplier

Let $(\Omega, \mathcal{F}, P)$ be a probability space and consider the process $u_\varepsilon(t,x), 0 \leq x \leq 1, 0 \leq t \leq T$ governed by the stochastic partial differential equation

$$du_\varepsilon(t,x) = (\triangle u_\varepsilon(t,x) + \theta(t)u_\varepsilon(t,x))dt + \varepsilon dW_Q(t,x) \tag{4.1}$$

where $\triangle = \frac{\partial^2}{\partial x^2}$. Suppose that $\varepsilon \to 0$ and $\theta \in \Theta$ where Θ is a class of real valued functions $\theta(t), 0 \leq t \leq T$ uniformly bounded , k times continuously differentiable and suppose that the k-th derivative $\theta^{(k)}(.)$ satisfies the Lipschitz condition of order $\alpha \in (0,1]$, that is,

$$|\theta^{(k)}(t) - \theta^{(k)}(s)| \leq |t-s|^\alpha, \beta = k + \alpha. \tag{4.2}$$

Further suppose the initial and the boundary conditions are given by

$$\begin{cases} u_\varepsilon(0,x) = f(x), f \in L_2[0,1] \\ u_\varepsilon(t,0) = u_\varepsilon(t,1) = 0, 0 \leq t \leq T \end{cases} \tag{4.3}$$

and Q is the nuclear covariance operator for the Wiener process $W_Q(t,x)$ taking values in $L_2[0,1]$ so that

$$W_Q(t,x) = Q^{1/2}W(t,x)$$

and $W(t,x)$ is a cylindrical Brownian motion on $L_2[0,1]$. Then, it is known that (cf. Rozovskii (1990), Kallianpur and Xiong (1995))

$$W_Q(t,x) = \sum_{i=1}^{\infty} q_i^{1/2} e_i(x) W_i(t) \text{ a.s.} \tag{4.4}$$

where $\{W_i(t), 0 \leq t \leq T\}, i \geq 1$ are independent one - dimensional standard Wiener processes and $\{e_i\}$ is a complete orthonormal system in $L_2[0,1]$ consisting of eigen vectors of Q and $\{q_i\}$ eigen values of Q.

We assume that the operator Q is a special covariance operator Q with $e_k = sin(k\pi x), k \geq 1$ and $\lambda_k = (\pi k)^2, k \geq 1$. Then $\{e_k\}$ is a complete orthonormal system with eigen values $q_i = (1+\lambda_i)^{-1}, i \geq 1$ for the operator Q and $Q = (I - \triangle)^{-1}$. Note that

$$dW_Q = Q^{1/2} dW. \tag{4.5}$$

We define a solution $u_\varepsilon(t,x)$ of (4.1) as a formal sum

$$u_\varepsilon(t,x) = \sum_{i=1}^{\infty} u_{i\varepsilon}(t) e_i(x) \tag{4.6}$$

(cf. Rozovskii (1990)). It can be checked that the Fourier coefficient $u_{i\varepsilon}(t)$ satisfies the stochastic differential equation

$$du_{i\varepsilon}(t) = (\theta(t) - \lambda_i) u_{i\varepsilon}(t) dt + \frac{\varepsilon}{\sqrt{\lambda_i + 1}} dW_i(t),\ 0 \leq t \leq T \tag{4.7}$$

with the initial condition

$$u_{i\varepsilon}(0) = v_i,\ v_i = \int_0^1 f(x) e_i(x) dx. \tag{4.8}$$

We assume that the initial function f in (4.3) is such that

$$v_i = \int_0^1 f(x) e_i(x) dx > 0,\ \ i \geq 1.$$

Estimation of linear multiplier

We now consider the problem of estimation of the function $\theta(t), 0 \leq t \leq T$ based on the observation of the Fourier coefficients $u_{i\varepsilon}(t), 1 \leq i \leq N$ over $[0,T]$ or equivalently the projection $u_\varepsilon^{(N)}(t,x)$ of the process $u_\varepsilon(t,x)$ onto the subspace spanned by $\{e_1, \ldots, e_N\}$ in $L_2[0,1]$.

We will at first construct an estimator of $\theta(.)$ based on the path $\{u_{i\varepsilon}(t), 0 \leq t \leq T\}$. Our technique follows the methods in Kutoyants (1994), p.155.

Let us suppose that

$$\sup_{\theta\in\Theta}\sup_{0\leq t\leq T}|\theta(t)|\leq L_0. \tag{4.9}$$

Consider the differential equation

$$\frac{du_i(t)}{dt}=(\theta(t)-\lambda_i))u_i(t),u_i(0)=v_i,0\leq t\leq T. \tag{4.10}$$

It is easy to see that

$$u_i(t)=v_ie^{\int_0^t(\theta(s)-\lambda_i)ds},0\leq t\leq T$$

and hence

$$u_i(t)\geq v_ie^{-M_it},0\leq t\leq T \tag{4.11}$$

where

$$M_i=L_0+\lambda_i. \tag{4.12}$$

¿From the Lemma 1.13 of Kutoyants (1994), it follows that

$$\sup_{0\leq s\leq T}|u_{i\varepsilon}(s)-u_i(s)|\leq\frac{\varepsilon}{\sqrt{\lambda_i+1}}e^{M_it}\sup_{0\leq s\leq T}|W_i(s)| \tag{4.13}$$

almost surely. Let

$$A_t^{(i)}=\{\omega:\inf_{0\leq s\leq t}u_{i\varepsilon}(s)\geq\frac{1}{2}v_ie^{-M_it}\} \tag{4.14}$$

and $A_i=A_T^{(i)}$. Note that $A_t^{(i)}$ contains the set A_i for $0\leq t\leq T$.

Define the process $\{Y_{i\varepsilon}(t),0\leq t\leq T\}$ by the stochastic differential equation

$$\begin{aligned}dY_{i\varepsilon}(t) &= -\frac{\varepsilon^2}{2(\lambda_i+1)}u_{i\varepsilon}^{-2}(t)\chi(A_t^{(i)})dt\\ &\quad +u_{i\varepsilon}^{-1}(t)\chi(A_t^{(i)})du_{i\varepsilon}(t),0\leq t\leq T\end{aligned} \tag{4.15}$$

where $\chi(E)$ denotes the indicator function of a set E. Let $\phi_\varepsilon\to 0$ as $\varepsilon\to 0$ and define

$$\hat{\theta}_{i\varepsilon}(t) = \lambda_i + \chi(A_i)\phi_\varepsilon^{-1} \int_0^T G(\frac{t-s}{\phi_\varepsilon}) dY_{i\varepsilon}(s) \tag{4.16}$$

where $G(.)$ is a bounded kernel with finite support , that is , there exists constants a and b such that

$$\int_a^b G(u)du = 1, G(u) = 0 \text{ for } u < a \text{ and } u > b. \tag{4.17}$$

We suppose that $a < 0$ and $b > 0$. Further suppose that the kernel $G(.)$ satisfies the additional condition

$$\int_{-\infty}^{\infty} G(u)u^j du = 0, j = 1, \ldots, k. \tag{4.18}$$

Note that

$$\begin{aligned} \hat{\theta}_{i\varepsilon}(t) - \lambda_i &= \chi(A_i)\phi_\varepsilon^{-1} \int_0^T G(\frac{t-s}{\phi_\varepsilon}) dY_{i\varepsilon}(s) \qquad (4.19) \\ &= \chi(A_i)\phi_\varepsilon^{-1} \int_0^T G(\frac{t-s}{\phi_\varepsilon})[\theta(s) - \lambda_i - \frac{\varepsilon^2}{2(\lambda_i+1)} u_{i\varepsilon}^{-2}(s)]\chi(A_s^{(i)})ds \\ &\quad + \chi(A_i)\phi_\varepsilon^{-1} \int_0^T G(\frac{t-s}{\phi_\varepsilon}) \frac{\varepsilon}{\sqrt{\lambda_i+1}} u_{i\varepsilon}^{-1}(s)\chi(A_s^{(i)})dW_i(s). \end{aligned}$$

Hence

$$\begin{aligned} E[\hat{\theta}_{i\varepsilon}(t) - \lambda_i] &= E[\chi(A_i)\phi_\varepsilon^{-1} \int_0^T G(\frac{t-s}{\phi_\varepsilon})(\theta(s) - \lambda_i)ds] \\ &\quad - E[\chi(A_i)\phi_\varepsilon^{-1} \int_0^T G(\frac{t-s}{\phi_\varepsilon}) \frac{\varepsilon^2}{2(\lambda_i+1)} u_{i\varepsilon}^{-2}(s)\chi(A_s^{(i)})ds] \\ &\quad + E[\chi(A_i)\phi_\varepsilon^{-1} \int_0^T G(\frac{t-s}{\phi_\varepsilon}) \frac{\varepsilon}{\sqrt{\lambda_i+1}} u_{i\varepsilon}^{-1}(s)\chi(A_s^{(i)})dW_i(s)]. \end{aligned}$$

Note that

$$\begin{aligned} &(E[\chi(A_i)\phi_\varepsilon^{-1} \int_0^T G(\frac{t-s}{\phi_\varepsilon}) \frac{\varepsilon}{\sqrt{\lambda_i+1}} u_{i\varepsilon}^{-1}(s)\chi(A_s^{(i)})dW_i(s)])^2 \\ &\leq E[\phi_\varepsilon^{-2}\{\int_0^T G(\frac{t-s}{\phi_\varepsilon}) \frac{\varepsilon}{\sqrt{\lambda_i+1}} u_{i\varepsilon}^{-1}(s)\chi(A_s^{(i)})dW_i(s)\}^2] \\ &\leq \phi_\varepsilon^{-2} \int_0^T G^2(\frac{t-s}{\phi_\varepsilon}) \frac{\varepsilon^2}{\lambda_i+1} E[u_{i\varepsilon}^{-2}(s)\chi(A_s^{(i)})]ds \\ &= J_{1i\varepsilon}^2(\text{say}). \end{aligned}$$

Therefore, for sufficiently small $\varepsilon > 0$,

$$\begin{aligned} E[\hat{\theta}_{i\varepsilon}(t) - \theta(t)] &= E[\chi(A_i)\phi_\varepsilon^{-1}\int_0^T G(\frac{t-s}{\phi_\varepsilon})(\theta(s) - \theta(t))ds] \\ &\quad -E[\chi(A_i^c)(\theta(t) - \lambda_i)] \\ &\quad -E[\chi(A_i)\phi_\varepsilon^{-1}\int_0^T G(\frac{t-s}{\phi_\varepsilon})\frac{\varepsilon^2}{2(\lambda_i+1)}u_{i\varepsilon}^{-2}(s)\chi(A_s^{(i)})ds] \\ &\quad +O(J_{1i\varepsilon}) \end{aligned} \tag{4.20}$$

since, for $0 < t < T$,

$$\phi_\varepsilon^{-1}\int_0^T G(\frac{t-s}{\phi_\varepsilon})ds = 1 \tag{4.21}$$

for sufficiently small $\varepsilon > 0$ by the conditions imposed on the kernel G. Therefore, for $0 < t < T$, for sufficiently small $\varepsilon > 0$,

$$\begin{aligned} |E[\hat{\theta}_{i\varepsilon}(t) - \theta(t)]|^2 &\leq 4\{\phi_\varepsilon^{-1}\int_0^T G(\frac{t-s}{\phi_\varepsilon})(\theta(s) - \theta(t))ds\}^2 \\ &\quad +4[P(A_i^c)]^2(\theta(t) - \lambda_i)^2 \\ &\quad +4(E[\chi(A_i)\phi_\varepsilon^{-1}\int_0^T G(\frac{t-s}{\phi_\varepsilon})\frac{\varepsilon^2}{2(\lambda_i+1)}u_{i\varepsilon}^{-2}(s)\chi(A_s^{(i)})ds])^2 \\ &\quad +O(J_{1i\varepsilon}^2). \end{aligned} \tag{4.22}$$

Applying the Taylor series expansion and properties of the kernel $G(.)$ and the function $\theta(.)$, it is easy to see that the first term is bounded by

$$C_1\phi_\varepsilon^{2\beta}\{\int_{-\infty}^{\infty}|G(u)u^\beta|du\}^2 \tag{4.23}$$

where C_1 is a constant depending only on the the constants L_0 in (2.9) and the smoothness parameter k of $\theta(.)$. Note that

$$\begin{aligned} P(A_i^c) &= P\{\inf_{0\leq t\leq T} u_{i\varepsilon}(t) < \frac{1}{2}v_i e^{-M_iT}\}\} \\ &\leq P\{\inf_{0\leq t\leq T}[u_{i\varepsilon}(t) - u_i(t)] + \inf_{0\leq t\leq T} u_i(t) < \frac{1}{2}v_i e^{-M_iT}\} \\ &\leq P\{\inf_{0\leq t\leq T}[u_{i\varepsilon}(t) - u_i(t)] < -\frac{1}{2}v_i e^{-M_iT}\} \text{ (from (4.11))} \end{aligned}$$

$$
\begin{aligned}
&\le P\{\sup_{0\le t\le T}|u_{i\varepsilon}(t)-u_i(t)|>\frac{1}{2}v_ie^{-M_iT}\}\\
&\le 2\exp\{-\frac{v_i^2e^{-4M_iT}(\lambda_i+1)}{8\varepsilon^2T}\}
\end{aligned}
$$

since

$$
P(\sup_{0\le t\le T}|W_i(t)|>\alpha)\le \min(2,\frac{4}{\alpha}\sqrt{\frac{T}{2\pi}})e^{-\frac{\alpha^2}{2T}}
$$

from Kutoyants (1994), p.28. The third term is bounded by

$$
C_2\frac{\varepsilon^4}{(\lambda_i+1)^2}v_i^{-2}e^{2M_iT}\{\int_{-\infty}^{\infty}|G(u)|du\}^2 \tag{4.24}
$$

where C_2 is an absolute constant. Relations (4.22) -(4.24) show that

$$
\begin{aligned}
|E[\hat{\theta}_{i\varepsilon}(t)-\theta(t)]|^2 \le\ & C_3[\phi_\varepsilon^{2\beta}\{\int_{-\infty}^{\infty}|G(u)u^\beta|du\}^2\\
&+\exp\{-\frac{2v_i^2e^{-4M_iT}(\lambda_i+1)}{8\varepsilon^2T}\}(\theta(t)-\lambda_i)^2\\
&+\frac{\varepsilon^4}{(\lambda_i+1)^2}v_i^{-2}e^{2M_iT}\{\int_{-\infty}^{\infty}|G(u)|du\}^2]\\
&+O(J_{1i\varepsilon}^2)
\end{aligned} \tag{4.25}
$$

where C_3 is an absolute constant. Hence

$$
|E[\hat{\theta}_{i\varepsilon}(t)-\theta(t)]|^2\le C_4[\phi_\varepsilon^{2\beta}+e^{-d_i\varepsilon^{-2}}(\theta(t)-\lambda_i)^2+\varepsilon^4k_i+J_{1i\varepsilon}^2] \tag{4.26}
$$

where C_4 is a constant depending on the kernel $G(.)$ and the Lipschitz constant L_0,

$$
d_i=\frac{v_i^2e^{-4M_iT}(\lambda_i+1)}{4T} \tag{4.27}
$$

and

$$
k_i=\frac{e^{2M_iT}}{(\lambda_i+1)^2v_i^2}. \tag{4.28}
$$

Following computations as given above(cf. Kutoyants (1994), p.157), we can show that

$$
E[\hat{\theta}_{i\varepsilon}(t)-\theta(t)]^2\le C_5[\phi_\varepsilon^{2\beta}+e^{-d_i\varepsilon^{-2}}(\theta(t)-\lambda_i)^2+\varepsilon^4k_i+\varepsilon^2\phi_\varepsilon^{-1}k_i+J_{1i\varepsilon}^2] \tag{4.29}
$$

where C_5 is a constant depending on the kernel $G(.)$ and the Lipschitz constant L_0. Choosing ϕ_ε such that

$$\phi_\varepsilon^{2\beta} = \varepsilon^2 \phi_\varepsilon^{-1},$$

we obtain that $\phi_\varepsilon = \varepsilon^{\frac{2}{(2\beta+1)}}$ and we have

$$E[\hat{\theta}_{i\varepsilon}(t) - \theta(t)]^2 \leq C_5[\varepsilon^{\frac{4\beta}{2\beta+1}} + e^{-d_i\varepsilon^{-2}}(\theta(t) - \lambda_i)^2 + \varepsilon^4 k_i + \varepsilon^{\frac{4\beta}{2\beta+1}} k_i + J_{1i\varepsilon}^2] \tag{4.30}$$

and

$$(E[\hat{\theta}_{i\varepsilon}(t) - \theta(t)])^2 \leq C_4[\varepsilon^{\frac{4\beta}{2\beta+1}} + e^{-d_i\varepsilon^{-2}}(\theta(t) - \lambda_i)^2 + \varepsilon^4 k_i + J_{1i\varepsilon}^2]. \tag{4.31}$$

Note that $\theta_{i\varepsilon}(t), 1 \leq i \leq N$ are independent estimators of $\theta(t)$ since the processes $W_i, 1 \leq i \leq N$ are independent Wiener processes. The above inequalities imply that

$$\begin{aligned} \sup_{1\leq i\leq N} E[\hat{\theta}_{i\varepsilon}(t) - \theta(t)]^2 \quad \leq \quad & C_6[\varepsilon^{\frac{4\beta}{2\beta+1}} + e^{-(\inf_{1\leq i\leq N} d_i)\varepsilon^{-2}} \sup_{1\leq i\leq N} (\theta(t) - \lambda_i)^2 \\ & + \varepsilon^4 \sup_{1\leq i\leq N} k_i + \varepsilon^{\frac{4\beta}{2\beta+1}} \sup_{1\leq i\leq N} k_i \\ & + \sup_{1\leq i\leq N} J_{1i\varepsilon}^2] \end{aligned} \tag{4.32}$$

Note that

$$\sup_{1\leq i\leq N} k_i \leq \frac{e^{2(L_0+N^2\pi^2)T}}{\inf_{1\leq i\leq N} v_i^2} = \beta_N \text{ (say)} \tag{4.33}$$

and

$$\inf_{1\leq i\leq N} d_i \geq (\inf_{1\leq i\leq N} v_i^2)\frac{e^{-4L_0T}}{4T} = \gamma_N \text{ (say)}. \tag{4.34}$$

Therefore

$$\begin{aligned} \sup_{1\leq i\leq N} E[\hat{\theta}_{i\varepsilon}(t) - \theta(t)]^2 \quad \leq \quad & C_7[\varepsilon^{\frac{4\beta}{2\beta+1}} + e^{-\gamma_N\varepsilon^{-2}}(|\theta(t)| + N^2\pi^2)^2 \\ & + \varepsilon^4\beta_N + \varepsilon^{\frac{4\beta}{2\beta+1}}\beta_N \\ & + \sup_{1\leq i\leq N} J_{1i\varepsilon}^2]. \end{aligned} \tag{4.35}$$

In particular

$$\begin{aligned} \sup_{1\leq i\leq N} Var(\hat{\theta}_{i\varepsilon}(t)) \quad \leq \quad & C_7[\varepsilon^{\frac{4\beta}{2\beta+1}} + e^{-\gamma_N\varepsilon^{-2}}(|\theta(t)| + N^2\pi^2)^2 \\ & + \varepsilon^4\beta_N + \varepsilon^{\frac{4\beta}{2\beta+1}}\beta_N \\ & + \sup_{1\leq i\leq N} J_{1i\varepsilon}^2]. \end{aligned} \tag{4.36}$$

We assume that the following conditions hold for $1 \leq i \leq N$. Let $\gamma_\varepsilon = \varepsilon^{-\frac{2\beta}{2\beta+1}}$. Suppose that

$$(C_1)\gamma_\varepsilon^2 J_{1i\varepsilon}^2 \to \frac{1}{u_i^2(t)(\lambda_i+1)} \int_{-\infty}^{\infty} G^2(u)du \text{ as } \varepsilon \to 0;$$

Under the above condition , it follows that the estimators $\theta_{i\varepsilon}(t), 1 \leq i \leq N$ are independent estimators of $\theta(t)$ such that

$$\sup_{1\leq i\leq N} |E[\hat{\theta}_{i\varepsilon}(t) - \theta(t)]| \leq C_8 \varepsilon^{\frac{2\beta}{2\beta+1}} \tag{4.37}$$

and

$$\sup_{1\leq i\leq N} E[\hat{\theta}_{i\varepsilon}(t) - \theta(t)]^2 \leq C_9 \varepsilon^{\frac{4\beta}{2\beta+1}} \tag{4.38}$$

where C_8 and C_9 are constants depending on the kernel $G(.)$, the Lipschitz constant L_0 and N. Note that the estimators $\theta_{i\varepsilon}(t), 1 \leq i \leq N$ are the best estimators of $\theta(t)$ as far as the rate of mean square error are concerned by Theorem 4.6 in Kutoyants (1994). We now combine these estimators in an optimum fashion to get an estimator using all the information available.

It is easy to check that

$$\gamma_\varepsilon[\hat{\theta}_{i\varepsilon}(t) - \theta(t)] = \chi(A_i)\gamma_\varepsilon\phi_\varepsilon^{-1} \int_0^T G(\frac{t-s}{\phi_\varepsilon})\frac{\varepsilon}{\sqrt{\lambda_i+1}} u_{i\varepsilon}^{-1}(s)\chi(A_s^{(i)})dW_i(s) + \tilde{J}_{2i\varepsilon} \tag{4.39}$$

$$= \chi(A_i)\tilde{J}_{1i\varepsilon} + \tilde{J}_{2i\varepsilon} \text{ (say)} \tag{4.40}$$

Note that $E(\tilde{J}_{1i\varepsilon}^2) = O(\gamma_\varepsilon^2 J_{1i\varepsilon}^2)$ where

$$J_{1i\varepsilon}^2 = \varepsilon^2\phi_\varepsilon^{-2} \int_0^T G^2(\frac{t-s}{\phi_\varepsilon})\frac{1}{\lambda_i+1} E(u_{i\varepsilon}^{-2}(s)\chi(A_s^{(i)}))ds \tag{4.41}$$

as defined earlier. In addition to (C_1), assume that

$$(C_2)\tilde{J}_{2i\varepsilon} = o_p(1) \text{ as } \varepsilon \to 0$$

for $1 \leq i \leq N$.

Since $P(A_i) \to 1$ as $\varepsilon \to 0$, it follows by the Central limit theorem for stochastic integrals (cf. Kutoyants (1994), Prakasa Rao (1999a)) that

$$\gamma_\varepsilon[\hat{\theta}_{i\varepsilon}(t) - \theta(t)] \to^{\mathcal{L}} N(0, \frac{1}{u_i^2(t)(\lambda_i+1)} \int_{-\infty}^{\infty} G^2(u)du) \tag{4.42}$$

as $\varepsilon \to 0$ for $1 \le i \le N$. Define

$$\tilde{\theta}_{N\varepsilon}(t) = \frac{\sum_{i=1}^{N} \hat{\theta}_{i\varepsilon}(t)(\lambda_i + 1)u_i^2(t)}{\sum_{i=1}^{N}(\lambda_i + 1)u_i^2(t)}. \tag{4.43}$$

Note that the random variable $\tilde{\theta}_{N\varepsilon}(t)$ is not an estimator of $\theta(t)$ as the functions $u_i(t)$ depend on the function $\theta(t)$. However the random variable $\tilde{\theta}_{N\varepsilon}(t)$ is a linear function of independent random variables $\hat{\theta}_{i\varepsilon}(t), 1 \le i \le N$. ¿From the earlier calculations , it can be checked that

$$\begin{aligned} E(\tilde{\theta}_{N\varepsilon}(t) - \theta(t))^2 &= Var(\tilde{\theta}_{N\varepsilon}(t)) + (E(\tilde{\theta}_{N\varepsilon}(t) - \theta(t))^2 \\ &\le C_8 \varepsilon^{\frac{4\beta}{2\beta+1}} + C_9 \varepsilon^{\frac{4\beta}{2\beta+1}} \\ &\le C_{10} \varepsilon^{\frac{4\beta}{2\beta+1}}. \end{aligned} \tag{4.44}$$

As a consequence, we have the following result.

Theorem 4.1: Under the conditions stated earlier, for $0 < t < T$,

(i) $\tilde{\theta}_{N\varepsilon}(t) \xrightarrow{p} \theta(t)$ as $\varepsilon \to 0$;

(ii) $E(\tilde{\theta}_{N\varepsilon}(t)) \to \theta(t)$ as $\varepsilon \to 0$;

(iii) $\lim_{\varepsilon \to 0} E(\tilde{\theta}_{N\varepsilon}(t) - \theta(t))^2 \to$ as $\varepsilon \to 0$;

(iv) $\limsup_{\varepsilon \to 0} E(\tilde{\theta}_{N\varepsilon}(t) - \theta(t))^2 \varepsilon^{\frac{-4\beta}{2\beta+1}} < \infty$.

(v) $\varepsilon^{\frac{-2\beta}{2\beta+1}}(\tilde{\theta}_{N\varepsilon}(t) - \theta(t)) \xrightarrow{\mathcal{L}} N(0, \sigma^2(t))$ as $\varepsilon \to 0$

where $N(0, \sigma^2(t))$ denotes the normal distribution with mean zero and variance $\sigma^2(t))$ given by

$$\sigma^2(t) = \frac{1}{\sum_{i=1}^{N} u_i^2(t)(\lambda_i + 1)} \int_{-\infty}^{\infty} G^2(u)du. \tag{4.45}$$

Let

$$\theta^*_{N\varepsilon}(t) = \frac{\sum_{i=1}^{N} \hat{\theta}_{i\varepsilon}(t)(\lambda_i + 1)\hat{u}_{i\varepsilon}^2(t)}{\sum_{i=1}^{N}(\lambda_i + 1)\hat{u}_{i\varepsilon}^2(t)} \tag{4.46}$$

where

$$\hat{u}_{i\varepsilon}(t) = v_i e^{\int_0^t (\hat{\theta}_{i\varepsilon}(s) - \lambda_i)ds}. \tag{4.47}$$

Note that for any $1 \leq i \leq N$,

$$\begin{aligned} E[\int_0^t \hat{\theta}_{i\varepsilon}(s)ds - \int_0^t \theta(s)ds]^2 &= E[\int_0^t (\hat{\theta}_{i\varepsilon}(s) - \theta(s))ds]^2 \\ &\leq E[t\int_0^t (\hat{\theta}_{i\varepsilon}(s) - \theta(s))^2 ds] \\ &= t\int_0^t E[(\hat{\theta}_{i\varepsilon}(s) - \theta(s))^2]ds \\ &\leq C_9 t \varepsilon^{\frac{4\beta}{2\beta+1}} \end{aligned}$$

and hence

$$\int_0^t \hat{\theta}_{i\varepsilon}(s)ds - \int_0^t \theta(s)ds \xrightarrow{p} 0 \text{ as } \varepsilon \to 0 \tag{4.48}$$

for $1 \leq i \leq N$. This in turn implies that, for $0 < t < T$,

$$\hat{u}_{i\varepsilon}(t) \xrightarrow{p} u_i(t) \text{ as } \varepsilon \to 0 \tag{4.49}$$

for $1 \leq i \leq N$. In view of (4.30), it follows that the estimator $\theta^*_{N\varepsilon}(t)$ is a consistent estimator of $\theta(t)$.

Theorem 4.2: Under the conditions stated above, for $0 < t < T$,

$$\theta^*_{N\varepsilon}(t) \xrightarrow{p} \theta(t) \text{ as } \varepsilon \to 0. \tag{4.50}$$

Note that

$$\begin{aligned} \gamma_\varepsilon[\theta^*_{N\varepsilon}(t) - \theta(t)] &= \gamma_\varepsilon[\frac{\sum_{i=1}^N \hat{\theta}_{i\varepsilon}(t)(\lambda_i + 1)\hat{u}^2_{i\varepsilon}(t)}{\sum_{i=1}^N (\lambda_i + 1)\hat{u}^2_{i\varepsilon}(t)} - \theta(t)] \\ &= \frac{\sum_{i=1}^N \gamma_\varepsilon(\hat{\theta}_{i\varepsilon}(t) - \theta(t))(\lambda_i + 1)\hat{u}^2_{i\varepsilon}(t)}{\sum_{i=1}^N (\lambda_i + 1)\hat{u}^2_{i\varepsilon}(t)}. \end{aligned}$$

Since

(i) $\gamma_\varepsilon(\hat{\theta}_{i\varepsilon}(t) - \theta(t)) \xrightarrow{\mathcal{L}} N(0, \sigma^2(t))$ as $\varepsilon \to 0$ for $1 \leq i \leq N$,

(ii) $\hat{u}_{i\varepsilon}(t) \xrightarrow{p} u_i(t)$ as $\varepsilon \to 0$ for $1 \leq i \leq N$, and since the estimators $\hat{\theta}_{i\varepsilon}(t), 1 \leq i \leq N$ are independent random variables, it follows that the estimator $\theta^*_{N\varepsilon}(t)$ is asymptotically normal and we have the following theorem.

Theorem 4.3: Under the conditions stated earlier, for $0 < t < T$,

$$\gamma_\varepsilon(\theta^*_{N\varepsilon}(t) - \theta(t)) \xrightarrow{\mathcal{L}} N(0, \sigma^2(t)) \text{ as } \varepsilon \to 0 \tag{4.51}$$

where

$$\gamma_\varepsilon = \varepsilon^{-\frac{2\beta}{2\beta+1}} \tag{4.52}$$

and

$$\sigma^2(t) = \frac{1}{\sum_{i=1}^N u_i^2(t)(\lambda_i + 1)} \int_{-\infty}^{\infty} G^2(u)du. \tag{4.53}$$

Remarks 1: If $k = 0$ and $\beta = 1$, that is, the function $\theta(.) \in \Theta$ where Θ is the class of uniformly bounded functions which are Lipschitzian of order one, then it follows that

$$\varepsilon^{-\frac{2}{3}}(\theta^*_{N\varepsilon}(t) - \theta(t)) \xrightarrow{\mathcal{L}} N(0, \sigma^2(t)) \text{ as } \varepsilon \to 0. \tag{4.54}$$

Remarks 2: It is known that the probability measures generated by stochastic processes satisfying the SPDE given by (4.1) are absolutely continuous with respect to each other when $\theta(.)$ is a constant (cf. Huebner et al.(1993)). There are classes of SPDE which generate probability measures which are singular with respect to each other when $\theta(.)$ is a constant. One can study the problem of nonparametric inference for a linear multiplier for such a class of SPDE by the above methods(cf. Prakasa Rao (2000b)).

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