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MODELLING BY LÉVY PROCESSES

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1 Introduction

A considerable body of recent work uses Lévy processes to model and analyse financial time series. Section 2 provides a brief review of this work. The review is to a large extent based on two papers Barndorff-Nielsen and Shephard (2001a,b) where more detailed information may be found. See also Barndorff-Nielsen and Shephard (2001c,d,e).

The models in question aim to incorporate one or more of the main stylised features of financial series, be they stock prices, foreign exchange rates or interest rates. A summary of these stylised features, and a comparison with related empirical findings in the study of turbulence, is given in Section 3. (In fact, the intriguing similarities between finance and turbulence have given rise to a new field of study coined 'econophysics'.)

2 Lévy Processes in Finance

A Lévy process is a stochastic process (in continuous time) with independent and homogeneous increments. The study of such processes, as part of probability theory generally, is currently attracting a great deal of attention, see Bertoin (1996,1999), Sato (1999), Barndorff-Nielsen, Mikosch and Resnick (2001), and references given there.

It is by now well recognised that Brownian motion generally provides a poor description of log price processes of stocks and other financial assets. Improved descriptions are obtained by substituting Brownian motion by suitably chosen alternative Lévy processes, for instance hyperbolic Lévy motion, normal inverse Gaussian Lévy motion and, more generally, one of the

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generalised hyperbolic Lévy motions. See Eberlein (2001), Prause (1999), Mantegna and Stanley (1999), Barndorff-Nielsen and Prause (2001).

Merely changing from Brownian motion to another, more suitable, Lévy process does not, however, provide a modelling of the important quasi long range dependencies (cf. Section 3) that pervades the financial markets. But such dependencies may be captured by further use of Lévy processes, as innovation processes driving volatility processes in the framework of SV (Stochastic Volatility) models. Discrete time models of this kind were considered in Barndorff-Nielsen (1998b). That approach has since been developed, in joint work with Neil Shephard, into the continuous time setting, and the rest of the present note consists mainly in a summary of that work (Barndorff-Nielsen and Shephard (2001a,b; cf. also 2001c,d,e)).

The stochastic volatility models considered are of the form

$$\mathrm{dx}^*(\mathrm{t}) = \left\{\mu + \beta\sigma^2(\mathrm{t})\right\}\mathrm{dt} + \sigma(\mathrm{t})\mathrm{dw}(\mathrm{t}) \tag{2.1}$$

where, for concretenes, we may think of $x^*(t)$ as the log price process of a given stock. In (2.1), $w(t)$ is Brownian motion and $\sigma^2(t)$, which represents the fluctuating and time dependent volatility, is a stationary stochastic process, for simplicity assumed independent of $w(t)$. Of particular interest are cases where $\sigma^2(t)$ is of OU type (Ornstein-Uhlenbeck type) or is a superposition of such processes. In the former instance, $\sigma^2(t)$ satisfies a stochastic differential equation of the form

$$\mathrm{d}\sigma^2(\mathrm{t}) = -\lambda\sigma^2(\mathrm{t})\mathrm{dt} + \mathrm{dz}(\lambda\mathrm{t}) \tag{2.2}$$

where $z(t)$ is a Lévy process with positive increments; thus $z(t)$ is a *subordinator.* Because of its role in (2.2), $z(t)$ is referred to as the *Background Driving Lévy Process* (BDLP, for short). The correlation function $r(u)$ of the (stationary) solution $\sigma^2(t)$ of (2.2) has the exponential form $r(u) = \exp(-\lambda u)$.

In particular, choosing the volatility process so that $\sigma^2(t)$ follows the inverse Gaussian law $IG(\delta,\gamma)$ with probability density

$$\frac{\delta}{\sqrt{2\pi}}e^{\delta\gamma}x^{-3/2}\exp\{-\frac{1}{2}(\delta^2x^{-1}+\gamma^2x)\}$$

one obtains that the increments of the log returns over a lag Δ, i.e. $x^*(t+\Delta) - x^*(t)$, are approximately distributed according to a normal inverse Gaussian law, and these laws are known to describe the distributions of log returns well. More generally, one may consider the generalized inverse Gaussian distribution $GIG(\lambda,\delta,\gamma)$

$$\frac{(\gamma/\delta)^\lambda}{2K_\lambda(\delta\gamma)}x^{\lambda-1}\exp\{-\frac{1}{2}(\delta^2x^{-1}+\gamma^2x)\},$$

where K_λ is a Bessel function, as the law for the volatility $\sigma^2(t)$. The corresponding approximate laws of the increments $x^*(t+\Delta) - x^*(t)$ are then of the generalized hyperbolic type which, besides the hyperbolic and normal inverse Gaussian distributions, inter alia includes the variance gamma laws, the Student distributions, and the Laplace distributions. It is important here to note that it is essential for the construction of these OU processes that the $GIG(\lambda, \delta, \gamma)$ distributions are *selfdecomposable* (cf. Barndorff-Nielsen (1998b)).

Furthermore, whatever the choice of the process $\sigma^2(t)$, if the parameter β is (approximately) 0 then the autocorrelations of the sequence of log returns will be (approximately) 0, reflecting another important stylized fact.

The dependency structure in the log price process (as it manifests itself for instance in the autocorrelations of the absolute or squared returns) may be modelled by endowing the $\sigma^2(t)$ process with a suitable correlation structure. This can be done by taking $\sigma^2(t)$ to be a superposition of independent OU processes, while keeping the chosen marginal law of $\sigma^2(t)$. Already the superposition of just two OU processes (with different regression parameters λ_1 and λ_2) may go a long way in describing the observed dependency structure of $x^*(t)$ (see, for instance, Barndorff-Nielsen (1998b; Figure 1)). However, even processes with real long range dependence can be constructed in this way (Barndorff-Nielsen (2001)).

Finally, the so-called leverage effect (see Section 3) can be modelled by adding an extra term in equation (2.1), defined using again the BDLP $z(t)$.

3 Stylized Features of Finance and Turbulence

A number of characteristic features of observational series from finance and from turbulence are summarised in table 1. The features are widely recognized as being esssential for understanding and modelling within these two, quite different, subject areas. In finance the observational series concerned consist of values of assets such as stocks or (logarithmic) stock returns or exchange rates, while in wind turbulence the series typically give the velocities or velocity derivatives (or differences), in the mean wind direction of a large Reynolds number wind field. For some typical examples of empirical probability densities of logarithmic asset returns, on the one hand, and velocity differences in large Reynolds number wind fields, on the other, see, for instance, Eberlein and Keller (1995) and Shephard (1996), respectively Barndorff-Nielsen (1998a).

A very characteristic trait of time series from turbulence as well as finance is that there seems to be a kind of switching regime between periods of relatively small random fluctuations and periods of high 'activity'. In turbulence this phenomenon is known as intermittency whereas in finance one speaks of stochastic volatility or conditional heteroscedasticity. For cumulative processes $x^*(t)$ in finance a basic expression of the volatility is given by the *quadratic variation* process $[x^*](t)$, defined as

$$[x^*](t) = p\text{-}\lim \sum_{i=1}^{n} (x^*(t_i) - x^*(t_{i-1}))^2$$

where $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$ and the limiting procedure is for the grid size $\max(t_i - t_{i-1})$ tending to 0. Similarly, in turbulence intermittency is expressed as the energy dissipation rate per unit mass at position ξ:

$$e_r(\xi) = r^{-1} \int_{\xi - r/2}^{\xi + r/2} \left(\frac{\partial u}{\partial x}\right)^2 \mathrm{dx}$$

Here $u = u(x)$ is the velocity at position x in the mean direction of the wind field. For detailed and informative discussions of the concepts of intermittency and energy dissipation, see Frisch (1995).

	Finance	Turbulence
varying activity	volatility	intermittency
semiheavy tails	+	+
asymmetry	+	+
aggregational Gaussianity	+	+
0 autocorrelation	+	−
quasi long range dependence	+	[+]
scaling/selfsimilarity	[+]	+

TABLE 1. Stylised features.

The term 'semiheavy tails', in table 1, is intended to indicate that the data suggest modelling by probability distributions whose densities behave, for $x \to \pm\infty$, as

$$const.\, |x|^{\rho_\pm} \exp(-\sigma_\pm |x|)$$

for some $\rho_+, \rho_- \in \mathbf{R}$ and $\sigma_+, \sigma_- \geq 0$. The generalised hyperbolic laws exhibit this type of behaviour.

Velocity differences in turbulence show an inherent asymmetry consistent with Kolmogorov's modified theory of homogeneous high Reynolds number turbulence (cf. Barndorff-Nielsen, 1986). Distributions of financial asset returns are generally rather close to being symmetric around 0, but for stocks there is a tendency towards asymmetry stemming from the fact that the equity market is prone to react differently to positive as opposed to negative returns, cf. for instance Shephard (1996; Subsection 1.3.4). This reaction pattern, or at least part of it, is referred to as a 'leverage effect' whereby increased volatility tends to be associated with negative returns.

By aggregational Gaussianity is meant the fact that long term aggregation of financial asset returns, in the sense of summing the returns over longer periods, will lead to approximately normally distributed variates, and similarly in the turbulence context[2]. For illustrations of this, see for instance Eberlein and Keller (1995) and Barndorff-Nielsen (1998a).

The estimated autocorrelation functions based on log price differences on stocks or currencies are generally (closely) consistent with an assumption of zero autocorrelation.

Nevertheless, this type of financial data exhibit 'quasi long range dependence' which manifests itself *inter alia* in the empirical autocorrelation functions of the absolute values or the squares of the returns, which stay positive for many lags.

For discussions of scaling phenomena in turbulence we refer to Frisch (1995). As regards finance, see Barndorff-Nielsen and Prause (2001) and references given there.

In addition, it is relevant to mention the one-dimensional Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

This nonlinear partial differential equation may be viewed as a 'toy model' version of the Navier-Stokes equations of fluid dynamics and, as such, have been the subject of extensive analytical and numerical studies, see for instance Frisch (1995; p.142-143) and Bertoin (2001), and references given there. In finance, Burgers' equation has turned up in work by Hodges and Carverhill (1993) and Hodges and Selby (1997). However, the interpretation of the equation in finance does not appear to have any relation to the role of the equation in turbulence.

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[2]However, in turbulence a small skewness generally persists, in agreement with Kolmogorov's theory of isotropic turbulence.

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