# A NONPARAMETRIC ASYMPTOTIC VERSION OF THE CRAMÉR-RAO BOUND

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The paper presents a genuinely asymptotic version of the Cramér-Rao bound, replacing the assumption of unbiasedness by locally uniform asymptotic unbiasedness, and the bound for the variance by a bound for the asymptotic variance. Bounds of this type are useful to obtain asymptotic results for estimator sequences which do not necessarily converge to a limit distribution. Under a condition slightly stronger than LAN, the minimal asymptotic variance obtained from the Convolution Theorem for regular estimator sequences turns out to be also a bound for the asymptotic variance of estimator sequences which are asymptotically unbiased, uniformly on shrinking $\chi^2$-neighbourhoods. For nonparametric models with a convergence rate slower than $n^{1/2}$, the asymptotic variance of such estimator sequences is necessarily infinite.

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## 1 Introduction

The Cramér-Rao bound is one of the standard topics in textbooks on mathematical statistics, ranging from elementary to advanced. In view of its limited applicability this is hard to explain. The question whether a given unbiased estimator has minimal variance can be answered by the Cramér-Rao bound only in one particular case: If the family is exponential, say $p(x,\vartheta) = c(\vartheta)\exp[\vartheta T(x)]$, and if the functional to be estimated is $\vartheta \to \int T(x)P_\vartheta(dx)$ (see Müller-Funk et al. (1989) for minimal regularity conditions). In all other cases the Cramér-Rao bound is not attainable, hence not a suitable standard for judging the optimality of an unbiased estimator.

Some authors make a point of the fact that the Cramér-Rao bound can be attained *asymptotically.* However, conditions under which the Cramér-Rao bound is a bound for the asymptotic variance are of a totally different nature, and so are the proofs.

There is no straight way from the Cramér-Rao bound to a bound for the asymptotic variance. This can be seen from examples showing the following properties. (i) For every sample size there exists an unbiased estimator with minimal convex risk, (ii) the sequence of these estimators is asymptotically normal with a variance larger than the Cramér-Rao bound, (iii) there exists an asymptotically normal estimator sequence with variance equal to the Cramér-Rao bound. Since the estimators specified under (i) are unique a.e., this shows the existence of models where the Cramér-Rao bound cannot be attained even asymptotically by sequences of unbiased estimators. For a natural example of this kind, based on a curved exponential family, see Pfanzagl (1994, p. 96, Example 2.7.3).

The purpose of the present paper is to establish a bound of Cramér-Rao or Chapman-Robbins type which is genuinely asymptotic. It presents a bound for the asymptotic variance of estimator sequences which are asymptotically unbiased, uniformly on shrinking sequences of neighbourhoods of a given $P_0$, or asymptotically unbiased along certain sequences converging to $P_0$.

For LAN-sequences this turns out to be the usual bound obtained from the Convolution Theorem for regular estimator sequences. Hence we obtain a slightly weaker assertion (a bound for the asymptotic variance instead of the Convolution Theorem) from a slightly weaker assumption (locally uniform asymptotic unbiasedness instead of locally uniform convergence to a limit distribution).

The main application is, however, to nonparametric models with an optimal convergence rate slower than $n^{1/2}$. In many such cases, it can be shown that locally uniform convergence to a limit distribution is impossible (see Pfanzagl, 2000). The results of the present paper are used to establish a variant of this result: If an estimator sequence has, at the optimal rate, a finite asymptotic variance at $P_0$ (but not necessarily a limit distribution!), then it cannot be asymptotically unbiased, locally uniformly on a shrinking sequence of neighbourhoods of $P_0$. This improves a result of Liu and Brown (1993).

The basic theorem will be presented in Section 2. Section 3 contains the application to LAN-families, Section 4 contains some nonparametric examples. Auxiliary results are collected in Section 5.

## 2 The general result

Let $\mathfrak{P}$ be a family of mutually absolutely continuous probability measures $P$ on some measurable space $(X, \mathcal{A})$. The problem is to estimate a functional $\kappa : \mathfrak{P} \to \mathbf{R}$, based on a sample of size $n$. The estimator $\kappa^{(n)}$ is a measurable map from $X^n$ to $\mathbf{R}$. The deviation of the estimate $\kappa^{(n)}(x_1, \ldots, x_n)$ from $\kappa(P)$ will be standardized by $c_n > 0$.

In the following $X(P_0; P)$ denotes the $\chi^2$-distance of $P$ from $P_0$, defined by

$$X(P_0; P) = \left[ \int \left( \frac{dP}{dP_0} \right)^2 dP_0 - 1 \right]^{1/2}.$$

Furthermore, $H(P_0, P)$ is the Hellinger metric defined by

$$H(P_0, P) = \left[1 - \int \sqrt{\frac{dP}{dP_0}} dP_0\right]^{1/2}.$$

Let

$$\pi_0 := \lim_{u\to\infty} \liminf_{n\to\infty} P_0^n\{c_n|\kappa^{(n)} - \kappa(P_0)| \le u\}. \tag{2.1}$$

We remark that $\pi_0$ is the same for every rate sequence equivalent to $c_n$, $n \in \mathbf{N}$, i.e. if $c_n$, $n \in \mathbf{N}$, is replaced by a sequence $c_n'$, $n \in \mathbf{N}$, fulfilling

$$0 < \liminf_{n\to\infty} c_n'/c_n \le \limsup_{n\to\infty} c_n'/c_n < \infty,$$

then

$$\lim_{u\to\infty} \liminf_{n\to\infty} P_0^n\{c_n'|\kappa^{(n)} - \kappa(P_0)| < u\} = \pi_0.$$

The estimator sequence $\kappa^{(n)}$, $n \in \mathbf{N}$, attains at $P_0$ the rate $c_n$, $n \in \mathbf{N}$, iff $\pi_0 = 1$. Observe that $\pi_0 = 1$ follows from $\limsup_{n\to\infty} c_n \int |\kappa^{(n)} - \kappa(P_0)| dP_0^n < \infty$. For applications in Section 4 a weaker condition suffices, namely $\pi_0 > 0$.

For $y \in \mathbf{R}$ and $u > 0$ let

$$L_u(y) := y1_{[-u,u]}(y). \tag{2.2}$$

**Definition 2.1** *The estimator sequence $\kappa^{(n)}$, $n \in \mathbf{N}$, is asymptotically unbiased with rate $c_n$, $n \in \mathbf{N}$, along the sequence $P_n$, $n \in \mathbf{N}$, if*

$$\limsup_{u\to\infty} \limsup_{n\to\infty} \left| \int L_u[c_n(\kappa^{(n)} - \kappa(P_n))] dP_n^n \right| = 0. \tag{2.3}$$

*If $P_n = P_0$ for $n \in \mathbf{N}$ we speak of "asymptotic unbiasedness at $P_0$".*

**Theorem 2.2** *Let $P_n \in \mathfrak{P}$, $n \in \mathbf{N}$, be a sequence such that*

$$0 < r := \liminf_{n\to\infty} c_n|\kappa(P_n) - \kappa(P_0)| \le \limsup_{n\to\infty} c_n|\kappa(P_n) - \kappa(P_0)| < \infty. \tag{2.4}$$

*Let $\kappa^{(n)}$, $n \in \mathbf{N}$, be an estimator sequence which is asymptotically unbiased at $P_0$ and along $P_n$, $n \in \mathbf{N}$, and fulfills $\pi_0 > 0$. Then, if $a := \limsup_{n\to\infty} X(P_0^n; P_n^n) < \infty$,*

$$\lim_{u\to\infty} \liminf_{n\to\infty} \int L_u[c_n(\kappa^{(n)} - \kappa(P_0))]^2 dP_0^n \ge \frac{\pi_0^2 r^2}{a^2}\left(1 - \frac{a^2}{\pi_0}\right). \tag{2.5'}$$

*Furthermore, if* $b := \limsup_{n\to\infty} 2H(P_0^n, P_n^n) < \infty$, *then*

$$\lim_{u\to\infty} \liminf_{n\to\infty} \int L_u[c_n(\kappa^{(n)} - \kappa(P_0))]^2 dP_0^n$$

$$(2.5'') \qquad + \lim_{u\to\infty} \limsup_{n\to\infty} \int L_u[c_n(\kappa^{(n)} - \kappa(P_n))]^2 dP_n^n \geq \frac{\pi_0^2 r^2}{b^2}\left(1 - \frac{b^2}{\pi_0}\right).$$

*The assertions remain true if* lim inf *is replaced by* lim sup *in (2.4) and (2.5).*

To obtain a bound which is as sharp as possible, one has to choose the sequence $P_n$, $n \in \mathbf{N}$, such that $X(P_0^n; P_n^n)$ is small, and $c_n|\kappa(P_n) - \kappa(P_0)|$ large. This is the way how the theorem can be applied to particular problems. Moreover, a version with sequences is necessary for the application to differentiable paths. From the aesthetic point of view, the following version based on sequences of neighbourhoods may be more satisfying.

**Corollary 2.3** *Let* $\mathfrak{P}_n \subset \mathfrak{P}$, $n \in \mathbf{N}$, *be a nonincreasing sequence of sets containing* $P_0$. *Assume that the estimator sequence* $\kappa^{(n)}$, $n \in \mathbf{N}$, *is with rate* $c_n$, $n \in \mathbf{N}$, *asymptotically unbiased, uniformly on* $\mathfrak{P}_n$, $n \in \mathbf{N}$, *i.e.*

$$(2.6) \qquad \lim_{u\to\infty} \limsup_{n\to\infty} \sup_{P\in\mathfrak{P}_n} \left| \int L_u[c_n(\kappa^{(n)} - \kappa(P))] dP^n \right| = 0.$$

*Let*

$$(2.7) \qquad \hat{r} := \liminf_{n\to\infty} \sup_{P\in\mathfrak{P}_n} c_n|\kappa(P) - \kappa(P_0)| > 0.$$

*Then, if* $\hat{a} := \limsup_{n\to\infty} \sup_{P\in\mathfrak{P}_n} X(P_0^n, P^n) < \infty$,

$$(2.8') \qquad \lim_{u\to\infty} \liminf_{n\to\infty} \int L_u[c_n(\kappa^{(n)} - \kappa(P_0))]^2 dP_0^n \geq \frac{\pi_0 \hat{r}^2}{\hat{a}^2}\left(1 - \frac{\hat{a}^2}{\pi_0}\right).$$

*Furthermore, if* $\hat{b} := \limsup_{n\to\infty} \sup_{P\in\mathfrak{P}_n} 2H(P_0^n, P^n) < \infty$, *then*

$$(2.8'') \qquad \lim_{u\to\infty} \liminf_{n\to\infty} \int L_u[c_n(\kappa^{(n)} - \kappa(P_0)]^2 dP_0^n$$

$$+ \lim_{u\to\infty} \limsup_{n\to\infty} \sup_{P\in\mathfrak{P}_n} \int L_u[c_n(\kappa^{(n)} - \kappa(P))]^2 dP^n \geq \frac{\pi_0 \hat{r}^2}{\hat{b}^2}\left(1 - \frac{\hat{b}^2}{\pi_0}\right).$$

**Proof** Apply Theorem 2.2 for a sequence $P_n \in \mathfrak{P}_n$, $n \in \mathbf{N}$, such that

$$\liminf_{n\to\infty} c_n|(P_n) - \kappa(P_0)| = \liminf_{n\to\infty} \sup_{P\in\mathfrak{P}_n} c_n|\kappa(P) - \kappa(P_0)|.$$

This concludes the proof. ∎

**Proof of Theorem 2.2** (i). To simplify our notations, let

(2.9′) $$K_n := c_n(\kappa^{(n)} - \kappa(P_0))$$

(2.9″) $$\hat{K}_n := c_n(\kappa^{(n)} - \kappa(P_n))$$

(2.10′) $$G_{n,u} := |\int L_u[K_n] dP_0^n|$$

(2.10″) $$\hat{G}_{n,u} := |\int L_u[\hat{K}_n] dP_n^n|$$

and

$$r_n := c_n(\kappa(P_n) - \kappa(P_0)).$$

We have

$$r = \liminf_{n\to\infty} |r_n|.$$

Let

$$\sigma^2 := \lim_{u\to\infty} \liminf_{n\to\infty} \int L_u[K_n]^2 dP_0^n.$$

Since assertion (2.5′) is trivial for $\sigma^2 = \infty$ and for $a^2 \geq \pi_0$, we assume $\sigma^2 < \infty$ and $a^2 < \pi_0$ in the following.

(ii). For every $\varepsilon > 0$ there exists $u_\varepsilon > 0$ and $n_\varepsilon \in \mathbf{N}$ such that the following relations hold true for $u \geq u_\varepsilon$

(2.11′) $$\limsup_{n\to\infty} G_{n,u} < \varepsilon$$

(2.11″) $$\limsup_{n\to\infty} \hat{G}_{n,u} < \varepsilon$$

(2.12) $$P_0^n\{|K_n| \leq u\} > (1-\varepsilon)\pi_0, \qquad \text{for } n \geq n_\varepsilon.$$

Relations (2.11) follow from unbiasedness as defined in (2.3). Relation (2.12) follows from (2.1) and $\pi_0 > 0$. For reasons which will become clear later on, we assume that $\varepsilon \in (0, 3/4)$.

(iii). The following relation for an arbitrary probability measure $Q \mid \mathbf{B}$ will be used repeatedly

(2.13) $$\int |x|^\alpha 1_{[y,z]}(|x|) Q(dx) \leq y^{\alpha-2} \int x^2 1_{[0,z]}(|x|) Q(dx) \quad \text{for } 0 < y < z \text{ and } \alpha \in [0,2].$$

(iv). Let now $\varepsilon > 0$, $v > \sup_{n\in\mathbf{N}} |r_n|$ and $u > \max\{u_\varepsilon, v\}$ be fixed. If $\sigma^2 < \infty$, there exists an infinite subset $\mathbf{N_0} \subset \mathbf{N}$ (depending on $u, v$, and $\varepsilon$) such that

(2.14) $$\int L_{u+v}[K_n]^2 dP_0^n < (1+\varepsilon)\sigma^2, \qquad \text{for } n \in \mathbf{N_0}.$$

Since

$$|1_{[-u,u]}(\hat{K}_n) - 1_{[-u,u]}(K_n)| \le 1_{[u-v,u+v]}(|K_n|), \qquad \text{for } n \in \mathbf{N},$$

we obtain

$$(2.15) \qquad \begin{aligned} |\hat{K}_n 1_{[-u,u]}(\hat{K}_n) - \hat{K}_n 1_{[-u,u]}(K_n)| &\le |\hat{K}_n| 1_{[u-v,u+v]}(|K_n|) \\ &\le (|K_n| + v) 1_{[u-v,u+v]}(|K_n|). \end{aligned}$$

From (2.13), applied with $Q = P_0^n \circ K_n$, $z = u + v$, $y = u - v$ and $\alpha = 0$, and $\alpha = 1$, we obtain (use (2.14)) for $n \in \mathbf{N_0}$

$$(2.16) \qquad \left| \int \left( \hat{K}_n 1_{[-u,u]}(\hat{K}_n) - \hat{K}_n 1_{[-u,u]}(K_n) \right) dP_0^n \right| \le u(u-v)^{-2}(1+\varepsilon)\sigma^2.$$

Moreover,

$$(2.17) \qquad \left| \int (L_u[K_n] - L_u[\hat{K}_n]) dP_0^n \right| \le G_{n,u} + \hat{G}_{n,u} + B_{n,u},$$

with

$$B_{n,u} := \left| \int L_u[\hat{K}_n] dP_n^n - \int L_u[\hat{K}_n] dP_0^n \right|.$$

Since $K_n - \hat{K}_n = r_n$, we have

$$(2.18) \qquad \begin{aligned} r_n 1_{[-u,u]}(K_n) &= (K_n - \hat{K}_n) 1_{[-u,u]}(K_n) \\ &= K_n 1_{[-u,u]}(K_n) - \hat{K}_n 1_{[-u,u]}(\hat{K}_n) + \hat{K}_n 1_{[-u,u]}(\hat{K}_n) - \hat{K}_n 1_{[-u,u]}(K_n). \end{aligned}$$

From (2.18), (2.16) and (2.17) we obtain for $n \in \mathbf{N_0}$, $n \ge n_\varepsilon$

$$(2.19) \qquad |r_n| P_0^n\{|K_n| \le u\} \le u(u-v)^{-2}(1+\varepsilon)\sigma^2 + G_{n,u} + \hat{G}_{n,u} + B_{n,u}.$$

(v). By Lemma 5.3, applied with $P_0^n$ and $P_n^n$ in place of $P_0$ and $P$, respectively, and with $f = L_u[\hat{K}_n]$ we obtain

$$(2.20') \qquad B_{n,u} \le \left( \int L_u[\hat{K}_n]^2 dP_0^n \right)^{1/2} X(P_0^n; P_n^n)$$

and

$$(2.20'') \qquad B_{n,u} \le 2 \left( \int L_u[\hat{K}_n]^2 dP_n^n + \int L_u[\hat{K}_n]^2 dP_0^n \right)^{1/2} H(P_0^n, P_n^n).$$

From

$$1_{[-u,u]}(\hat{K}_n) \le 1_{[-(u+v),u+v]}(K_n),$$

we obtain

$$\begin{aligned}(2.21)\quad &\int L_u[\hat{K}_n]^2 dP_0^n \\ &\leq \int (K_n - r_n)^2 1_{[-(u+v),u+v]}(K_n) dP_0^n \\ &= \int L_{u+v}[K_n]^2 dP_0^n - 2r_n \int L_{u+v}[K_n] dP_0^n + r_n^2 P_0\{|K_n| \leq u+v\} \\ &\leq \int L_{u+v}[K_n]^2 dP_0^n + 2v\left|\int L_{u+v}[K_n] dP_0^n\right| \\ &\qquad + r_n^2 P_0^n\{|K_n| \leq u\} + v^2 P_0^n\{u < |K_n| \leq u+v\}.\end{aligned}$$

From (2.13), applied with $Q = P_0^n \circ K_n$, $z = u+v$, $y = u$ and $\alpha = 0$ we obtain (use (2.14))

$$P_0^n\{u < |K_n| \leq u+v\} \leq u^{-2}(1+\varepsilon)\sigma^2.$$

Together with (2.20′) and (2.21) this implies for $n \in \mathbf{N_0}$, $n \geq n_\varepsilon$ (recall $a := \limsup_{n\to\infty} X(P_0^n; P_n^n)$)

$$(2.22)\quad B_{n,u} \leq a((1+v^2u^{-2})(1+\varepsilon)\sigma^2 + 2vG_{n,u+v} + r_n^2 P_0^n\{|K_n| \leq u\})^{1/2}.$$

Together with (2.19) this implies

$$\begin{aligned}&|r_n| P_0^2\{|K_n| \leq u\} \leq u(u-v)^{-2}(1+\varepsilon)\sigma^2 + G_{n,u} + \hat{G}_{n,u} \\ (2.23)\quad &\qquad + a((1+v^2u^{-2})(1+\varepsilon)\sigma^2 + 2vG_{n,u+v} + r_n^2 P_0^n\{|K_n| \leq u\})^{1/2}.\end{aligned}$$

Since $t \mapsto r_n t - a(A + r_n^2 t)^{1/2}$ is increasing on $(a^2/4, \infty)$ if $A \geq 0$, relation (2.23) remains valid if we replace $P_0^n\{|K_n| \leq u\}$ by $(1-\varepsilon)\pi_0$. (Recall that $(1-\varepsilon)\pi_0 > \pi_0/4 > a^2/4$). Taking now the limit over a subsequence $\mathbf{N_1} \subset \mathbf{N_0}$ for which $\hat{r} := \lim_{n\in\mathbf{N_1}} |r_n|$ exists, we obtain (hint: use (2.11))

$$(2.24)\quad \begin{aligned}&\hat{r}(1-\varepsilon)\pi_0 \leq u(u-v)^{-2}(1+\varepsilon)\sigma^2 + 2\varepsilon \\ &\qquad + a((1+v^2u^{-2})(1+\varepsilon)\sigma^2 + 2v\varepsilon + \hat{r}^2(1-\varepsilon)\pi_0)^{1/2}.\end{aligned}$$

This inequality does not depend on $\mathbf{N_0}$ any more. Since it holds for all $u > \max\{u_\varepsilon, v\}$ and all $\varepsilon \in (0, 3/4)$ we obtain

$$\hat{r}\pi_0 \leq a(\sigma^2 + \hat{r}^2\pi_0)^{1/2},$$

or

$$(2.25)\quad \sigma^2 \geq \pi_0^2 \frac{\hat{r}^2}{a^2}\left(1 - \frac{a^2}{\pi_0}\right).$$

Since $\hat{r} \geq \liminf_{n\to\infty} |r_n| =: r$, relation (2.5′) follows.

(vi). The proof of (2.5″) is the same, using instead of the bound (2.22) for $B_{n,u}$, resulting from (2.20′), the following bound resulting from (2.20″)

$$B_{n,u} \leq b((1+\varepsilon) \lim_{u\to\infty} \limsup_{n\to\infty} \int L_u[\hat{K}_n]^2 dP_n^n + (1+v^2u^{-2})(1+\varepsilon)\sigma^2 + 2vG_{n,u+v} + r_n^2 P_0^n\{|K_n| \leq u\})^{1/2}.$$

This concludes the proof. ∎

## 3 Applications to LAN-families

Assume that for $P_0 \in \mathfrak{P}$ there is a linear space $T_0$ of functions $g : X \to \mathbf{R}$ such that $\int g dP_0 = 0$ and $\int g^2 dP_0 < \infty$, and an element $\kappa^* \in T_0$ such that the following holds true.

For every $g \in T_0$ there exists a path $P_{n,g} \in \mathfrak{P}$, $n \in \mathbf{N}$, such that

$$P_0^n \circ \log \frac{dP_{n,g}^n}{dP_0^n} \Longrightarrow N_{(-\frac{1}{2}\sigma^2(g), \sigma^2(g))}, \tag{3.1}$$

with

$$\sigma^2(g) := \int g(x)^2 P_0(dx)$$

and

$$\lim_{n\to\infty} n^{1/2}(\kappa(P_{n,g}) - \kappa(P_0)) = \int g\kappa^* dP_0. \tag{3.2}$$

Condition (3.1) is, in particular, fulfilled if there exists a density of the type

$$\frac{dP_{n,g}}{dP_0} = 1 + n^{-1/2} g + n^{-1/2} r_n, \tag{3.3}$$

with $r_n$ fulfilling the conditions

$$P_0\{|r_n| > \varepsilon n^{1/2}\} = o(n^{-1}), \qquad \text{for } \varepsilon > 0, \tag{3.4′}$$

$$\int r_n 1_{\{|r_n| \leq n^{1/2}\}} dP_0 = o(n^{-1/2}), \tag{3.4″}$$

$$\int r_n^2 1_{\{|r_n| \leq n^{1/2}\}} dP_0 = o(n^0). \tag{3.4‴}$$

Observe that (3.3) implies $\int r_n dP_0 = 0$.

As pointed out by LeCam (see Pfanzagl, 1985, p. 25–27), condition (3.4) is equivalent to Hellinger differentiability of the path $P_{n,g}$ with derivative $g/2$.

In most examples, the path $P_{n,g}$, $n \in \mathbf{N}$, can be chosen such that the sequence $r_n$, $n \in \mathbf{N}$, of remainder terms fulfills a condition slightly stronger than (3.4), namely

$$\int r_n^2 dP_0 = o(n^0). \tag{3.5}$$

**Theorem 3.1** *Let $\kappa^{(n)}$, $n \in \mathbf{N}$, be an estimator sequence such that*

$$\lim_{u\to\infty} \limsup_{n\to\infty} P_0^n\{c_n|\kappa^{(n)} - \kappa(P_0)| \leq u\} = 1. \tag{3.6}$$

*Assume, moreover, that $\kappa^{(n)}$, $n \in \mathbf{N}$, is asymptotically unbiased with the rate $n^{1/2}$, uniformly on all paths fulfilling conditions (3.2) and (3.3) with $r_n$, $n \in \mathbf{N}$, fulfilling (3.5). Then*

$$\lim_{u\to\infty} \liminf_{n\to\infty} \int L_u[n^{1/2}(\kappa^{(n)} - \kappa(P_0))]^2 dP_0^n \geq \sigma^2(\kappa^*). \tag{3.7}$$

In this theorem, the uniformity along paths refers to asymptotic unbiasedness only. If the uniformity along paths holds for convergence to a limit distribution ("regular estimator sequences") and if this limit distribution has expectation zero, then this implies asymptotic unbiasedness and (3.6), hence also (3.7). This is, however, of no interest, since for regular estimator sequences the Convolution Theorem yields a much stronger assertion. The attractive feature of Theorem 3.1 lies in the fact that it requires asymptotic unbiasedness only (without reference to any limit distribution). Actually, the asymptotic unbiasedness along the paths $P_{n,a\kappa^*}$, $n \in \mathbf{N}$ with $a > 0$ is all we need in the proof of Theorem 3.1. Though this condition is weaker, it is not more plausible than asymptotic unbiasedness along all paths $P_{n,g}$, $n \in \mathbf{N}$, with $g \in T_0$. For readers who find it difficult to understand the meaning of "uniformity along paths" — as is the case with the author — we add the remark that this condition may, of course, be replaced by the stronger condition of "uniformity on $\{P \in \mathfrak{P} : X(P_0^n, P^n) \leq a\}$ for some $a > 0$".

**Remark 3.1** Now we specialize Theorem 3.1 to the case of a 1-parametric family $\{P_\vartheta : \vartheta \in \Theta\}$, $\Theta \subset \mathbf{R}$ such that (see relation (3.3))

$$\frac{dP_{\vartheta_0+n^{-1/2}t}}{dP_{\vartheta_0}} = 1 + n^{-1/2}t\ell'(x, \vartheta_0) + n^{-1/2}r_n(x, \vartheta_0, t),$$

with $r_n$ fulfilling relation (3.5). In this case,

$$\kappa^*(x, \vartheta_0) = \frac{\ell'(x, \vartheta_0)}{\int \ell'(\cdot, \vartheta_0)^2 dP_{\vartheta_0}}.$$

Let $\vartheta^{(n)}$, $n \in \mathbf{N}$, be an estimator sequence which is asymptotically unbiased on the paths $P_{\vartheta_0+n^{-1/2}t}$, $n \in \mathbf{N}$. Since $y^2 \geq L_u(y)^2$ we obtain from (3.7)

$$(3.8) \qquad \limsup_{n\to\infty} n \int (\vartheta^{(n)} - \vartheta_0)^2 dP^n_{\vartheta_0} \geq 1 / \int \ell'(x, \vartheta_0)^2 dP_{\vartheta_0}.$$

This is the classical version of the Cramér-Rao bound, established here under a condition weaker than "unbiasedness for every $n \in \mathbf{N}$".

Condition (3.5), required in Theorem 3.1, is slightly stronger than needed for LAN. Therefore, we supply another Theorem (with a slightly more complex result).

**Theorem 3.2** *Let $\kappa^{(n)}$, $n \in \mathbf{N}$, be an estimator sequence fulfilling (3.6). Assume, moreover, that $\kappa^{(n)}$, $n \in \mathbf{N}$, is asymptotically unbiased with the rate $n^{1/2}$, uniformly on all paths fulfilling conditions (3.1) and (3.2). Then*

$$\lim_{u\to\infty} \liminf_{n\to\infty} \int L_u[n^{1/2}(\kappa^{(n)} - \kappa(P_0))]^2 dP^n_0 < \sigma^2(\kappa^*)$$

*implies for t sufficiently small*

$$\lim_{u\to\infty} \limsup_{n\to\infty} \int L_u[n^{1/2}(\kappa^{(n)} - \kappa(P_{n,t\kappa^*}))]^2 dP^n_{n,t\kappa^*} > \sigma^2(\kappa^*).$$

Since

$$\lim_{u\to\infty} \liminf_{n\to\infty} \int L_u[n^{1/2}(\kappa^{(n)} - \kappa(P_0))]^2 dP^n_0 \geq \sigma^2(\kappa^*)$$

under the conditions of Theorem 3.1, the use of Theorem 3.2 is, roughly speaking, restricted to the case that there exists a path from direction $t\kappa^*$ fulfilling the conditions (3.4), but not the stronger condition (3.5).

**Remark 3.2** Comparable bounds exist for the concentration of estimator sequences which are asymptotically median unbiased, uniformly on paths fulfilling (3.1) and (3.2). This is proved in Pfanzagl (1994, p. 271, Corollary 8.2.5) for parametric families, but the proof extends immediately to the more general case of a family with paths fulfilling (3.1) and (3.2).

**Proof of Theorem 3.1** For paths $P_{n,g}$ fulfilling (3.3) and (3.5) we obtain from Lemma 5.2 that

$$\lim_{n\to\infty} X^2(P^n_0; P^n_{n,g}) = \exp[\sigma^2(g)] - 1.$$

Moreover, from (3.2),

$$\lim_{n\to\infty} n^{1/2}(\kappa(P_{n,g}) - \kappa(P_0)) = \int g(x)\kappa^*(x) P_0(dx) \quad .$$

Applied with $g = t\kappa^*$ we obtain from (2.5′)

$$\lim_{u\to\infty} \liminf_{n\to\infty} \int L_u[n^{1/2}(\kappa^{(n)} - \kappa(P_0))]^2 dP_0^n \geq \sigma^2(\kappa^*)\Psi(t^2\sigma^2(\kappa^*)),$$

with

$$\Psi(z) = z(2 - \exp[z])/(\exp[z] - 1).$$

Since $\Psi$ attains its maximal value 1 for $z \to 0$, the assertion follows. ∎

**Proof of Theorem 3.2** For paths $P_{n,g}$ fulfilling (3.1) we obtain from Lemma 5.1, applied with $M = N_{(-\sigma^2(g)/2,\ \sigma^2(g))}$, that

$$\lim_{n\to\infty} H^2(P_0^n, P_{n,g}^n) = 1 - \exp[-\sigma^2(g)/8].$$

Moreover, from (3.2)

$$\lim_{n\to\infty} n^{1/2}(\kappa(P_{n,g}) - \kappa(P_0)) = \int g(x)\kappa^*(x)P_0(dx).$$

Applied with $g = t\kappa^*$ we obtain from relation (2.5″)

$$\begin{aligned}&\lim_{u\to\infty} \liminf_{n\to\infty} \int L_u[n^{1/2}(\kappa^{(n)} - \kappa(P_0))]^2 dP_0^n \\ &\qquad + \lim_{u\to\infty} \limsup_{n\to\infty} \int L_u[n^{1/2}(\kappa^{(n)} - \kappa(P_{n,t\kappa^*}))]^2 dP_{n,t\kappa^*}^n \\ &\geq 2\sigma^2(\kappa^*)\Psi(t^2\sigma^2(\kappa^*)/8),\end{aligned}$$

with

$$\Psi(z) = z(4\exp[-z] - 3)/(1 - \exp[-z]).$$

Since $\Psi$ attains its maximal value 1 for $z \to 0$, the assertion follows. ∎

## 4 Applications to nonparametric problems

In most nonparametric problems the optimal convergence rate is slower than $n^{1/2}$. Usually it is of the type $n^\alpha L(n)$ with $\alpha \in (0, 1/2)$ and $L$ a slowly varying function, such as $(\log n)^a$. In these cases, estimator sequences converging locally uniformly to a limit distribution usually do not exist. (See Pfanzagl, 2000, for details.)

In the present paper we shall apply Theorem 3.1 to show that estimator sequences which have at $P_0$ a finite asymptotic variance with this rate cannot be asymptotically unbiased.

**Theorem 4.1** *Let*

$$\mathfrak{P}_{n,a} := \{P \in \mathfrak{P} : X(P_0; P) \leq n^{-1/2}a\} \tag{4.1}$$

*and*

$$r_n(a) := \sup\{c_n|\kappa(P) - \kappa(P_0)| : P \in \mathfrak{P}_{n,a}\}. \tag{4.2}$$

*Assume that*

$$\limsup_{a\to 0} a^{-1} \liminf_{n\to\infty} r_n(a) = \infty. \tag{4.3}$$

*Let $\kappa^{(n)}$, $n \in \mathbf{N}$, be an estimator sequence with the following properties.*

$$\lim_{u\to\infty} \liminf_{n\to\infty} P_0^n\{c_n|\kappa^{(n)} - \kappa(P_0)| \leq u\} > 0 \tag{4.4}$$

*and*

$$\sigma_0^2 := \lim_{u\to\infty} \liminf_{n\to\infty} \int L_u[c_n(\kappa^{(n)} - \kappa(P_0))]^2 dP_0^n < \infty. \tag{4.5}$$

*Then the estimator sequence $\kappa^{(n)}$, $n \in \mathbf{N}$, cannot be asymptotically unbiased with the rate $c_n$, $n \in \mathbf{N}$, uniformly on $\{P \in \mathfrak{P} : X(P_0^n; P^n) \leq a\}$ for some $a > 0$.*

Observe that conditions (4.4) and (4.5) refer to the performance of the estimator sequence at $P_0$ only.

If condition (4.4) is fulfilled for some sequence $c_n$, $n \in \mathbf{N}$, it is also fulfilled for every sequence $c_n'$, $n \in \mathbf{N}$, such that $\limsup_{n\to\infty} c_n'/c_n < \infty$. It is condition (4.3) which guarantees that the sequence $c_n$, $n \in \mathbf{N}$, increases quickly enough.

**Proof** If $\kappa^{(n)}$, $n \in \mathbf{N}$, is asymptotically unbiased, uniformly on $\{P \in \mathfrak{P} : X(P_0^n; P^n) \leq a\}$, relation (2.8′) implies that

$$\sigma_0 \geq \pi_0 \left(1 - \frac{a^2}{\pi_0}\right)^{1/2} a^{-1} \liminf_{n\to\infty} \tilde{r}_n(a), \tag{4.6}$$

with

$$\tilde{r}_n(a) := \sup\{c_n|\kappa(P) - \kappa(P_0)| : X(P_0^n; P^n) \leq a\}. \tag{4.7}$$

Since

$$X^2(P_0^n; P^n) = (1 + X^2(P_0; P))^n - 1,$$

$X(P_0; P) \leq n^{-1/2}a/2$ implies

$$X^2(P_0^n; P^n) \leq (1 + n^{-1}a^2/4)^n - 1 < a^2, \qquad \text{for } a \in (0, 1).$$

Therefore,

$$r_n(a/2) \le \tilde{r}_n(a), \qquad \text{for } a \in (0,1).$$

Hence (4.6) implies

$$\sigma_0 \ge \pi_0 \left(1 - \frac{a^2}{\pi_0}\right)^{1/2} a^{-1} \liminf_{n\to\infty} r_n(a/2).$$

Since this relation holds for every $a \in (0,1)$, relation (4.3) is in contradiction to the assumption that $\sigma_0 < \infty$. ∎

The impossibility result expressed in Theorem 4.1 is in close relationship to another impossibility result, Theorem 4.1 in Pfanzagl (2000), which reads as follows: Under condition (4.3), there exists no estimator sequence such that $P^n \circ c_n(\kappa^{(n)} - \kappa(P))$, $n \in \mathbf{N}$, converges to a fixed limit distribution, uniformly on $\mathfrak{P}_{n,a}$, $n \in \mathbf{N}$, for some $a > 0$. This does not exclude the possibility that $P_0^n \circ c_n(\kappa^{(n)} - \kappa(P_0))$, $n \in \mathbf{N}$, converges to some limit distribution, say $Q_0$. If $Q_0$ has expectation 0, then $\kappa^{(n)}$, $n \in \mathbf{N}$, is asymptotically unbiased at $P_0$. If the convergence to $Q_0$ is uniform on $\mathfrak{P}_{n,a}$, $n \in \mathbf{N}$, say, then the asymptotic unbiasedness, too, is uniform on $\mathfrak{P}_{n,a}$, $n \in \mathbf{N}$. The impossibility of uniform convergence to a limit distribution on $\mathfrak{P}_{n,a}$, $n \in \mathbf{N}$, does, however, not exclude the existence of estimator sequences which are asymptotically unbiased, uniformly on $\mathfrak{P}_{n,a}$, $n \in \mathbf{N}$. According to Theorem 4.1 this is impossible if the variance of $Q_0$ is finite. $P_0^n \circ c_n(\kappa^{(n)} - \kappa(P_0)) \Rightarrow Q_0$ with $\int u Q_0(du) = 0$ and $\int u^2 Q_0(du) < \infty$ excludes asymptotic unbiasedness uniformly on $\{P \in \mathfrak{P} : X(P_0^n; P^n) \le a\}$ for some $a > 0$, hence also uniform convergence to $Q_0$. As far as uniform convergence to a limit distribution is concerned, the result in Pfanzagl (2000, Theorem 4.1) is stronger in that it excludes uniform convergence to any limit distribution (and not only to limit distributions $Q_0$ fulfilling $\int u Q_0(du) = 0$ and $\int u^2 Q_0(du) < \infty$).

The impossibility assertion of Theorem 4.1 is based upon condition (4.3). To grasp the meaning of this condition, consider a situation in which for some sequence $c_n$, $n \in \mathbf{N}$, the limit of $r_n(a)$, $n \in \mathbf{N}$, exists in $(0, \infty)$ for every $a > 0$. Then, according to Theorem 3.1 in Pfanzagl (2000), no estimator sequence for $\kappa$ can converge, uniformly on $\mathfrak{P}_{n,a}$, at a rate better than $c_n$, $n \in \mathbf{N}$. Because of the special structure of $\mathfrak{P}_{n,a}$ (as defined in (4.1)), the existence of $\lim_{n\to\infty} r_n(a)$ for $a > 0$ implies that $c_n = n^\alpha L(n)$ and $\lim_{n\to\infty} r_n(a) = a^{2\alpha}$ for some $\alpha \ge 0$, with $L$ slowly varying as $n$ tends to infinity. (Hint: apply the results of section 6 in Pfanzagl (2000) to $r_n(a) = s_n(1)^{-1} s_n(a)$.) This is the situation we met with in various non- and semiparametric models. More specifically, we have in these models $c_n = n^\alpha$ and

$$\inf_{a>0} a^{-2\alpha} \liminf_{n\to\infty} r_n(a) > 0 \quad . \tag{4.8}$$

If $\alpha \in [0, 1/2)$, this implies (4.3) and excludes, therefore, the existence of estimator sequences which are with the (optimal) rate $n^\alpha$ asymptotically unbiased and have, at $P_0$, a finite truncated asymptotic variance (in the sense of (4.5)).

This will be illustrated by some examples taken from Pfanzagl (2000).

**Example 4.1** Let $\mathfrak{P}$ be the family of all Lebesgue densities, admitting $k$ derivatives, $k \geq 0$, which fulfill a Lipschitz condition of order 1 with a given Lipschitz constant. The functional to be estimated is $p(x_0)$, with $x_0$ fixed. According to relation (7.8) in Pfanzagl (2000) relation (4.8) holds with $\alpha = (k+1)/(2k+3)$.

For $k = 0$, the rate bound is $n^{1/3}$. The same rate bound holds for a smaller family, namely the family of all probability measures with monotone densities with $p'(x_0) < 0$. According to Prakasa Rao (1969, p. 35, Theorem 6.3) the sequence of maximum likelihood estimators $p^{(n)}$, $n \in \mathbf{N}$, attains this rate for every $P_0$ with $p_0'(x_0) < 0$. More precisely,

$$P_0^n \circ n^{1/3}(4p_0(x_0)|p_0'(x_0)|)^{-1/3}(p^{(n)} - p(x_0)), \qquad n \in \mathbf{N},$$

converges to a limit distribution, independent of $P_0$, which is symmetric about 0 and has finite moments of all orders (see Groeneboom, 1989, Corollary 3.4, p. 94 for properties of this limit distribution). According to Theorem 4.1, this estimator sequence cannot be asymptotically unbiased, uniformly on $\{P \in \mathfrak{P} : X(P_0^n, P^n) \leq a\}$ for some $a > 0$.

**Example 4.2** Let $\mathfrak{P}$ be the family of all Lebesgue densities which have a continuous 2$^{\text{nd}}$ derivative and a unique mode. The functional to be estimated is the mode.

According to relation (7.12) in Pfanzagl (2000) relation (4.8) holds with $\alpha = 1/5$.

**Example 4.3** Let $\mathfrak{P}$ be the family of all distributions $P_{\beta,r}$ over $(0,1)$ with a Lebesgue density of the following type

$$K(\beta, r)x^{\beta-1}(1 + r(x)), \qquad \text{with } \beta \in (0, \infty).$$

In this representation, $r$ is a continuous function fulfilling

$$\sup_{x \in (0,1)} |r(x)|x^{-\varrho\beta} < \infty,$$

where $\varrho \in (0, \infty)$ is known. The functional to be estimated is $\kappa(P_{\beta,r}) = \beta$, the extreme value index.

Let $P_0$ be a probability measure with density $\beta x^{\beta-1}$. According to relation (7.39) in Pfanzagl (2000) relation (4.8) holds with $\alpha = \varrho/(2\varrho + 1)$.

All the examples mentioned above make use of the $\chi^2$-distance. Recall that this has a definite advantage. The requirement of uniformity on $\{P \in \mathfrak{P} : X(P_0^n, P^n) \leq a\}$ for some $a > 0$ is weaker than the corresponding condition based on $H$. Moreover, assertions like (2.5′) and (2.8′), based on the $\chi^2$-distance, refer to the asymptotic variance at $P_0$. Using the Hellinger metric, the lower bounds given in (2.5″) and (2.8″) refer to

$$(4.9'') \qquad \lim_{u\to\infty} \limsup_{n\to\infty} \sup_{\{P: H(P_0^n, P^n)\leq a\}} \int L_u[c_n(\kappa^{(n)} - \kappa(P))]^2 dP^n,$$

rather than

$$(4.9') \qquad \lim_{u\to\infty} \liminf_{n\to\infty} \int L_u[c_n(\kappa^{(n)} - \kappa(P_0))]^2 dP_0^n.$$

A lower bound for (4.9″) is not only weaker in the mathematical sense. It also refers to a quantity which is not so easy to interpret as (4.9′).

Since in nonparametric models there is much freedom in the choice of the sequence $P_n$, $n \in \mathbf{N}$, underlying relations (2.5′) and (2.5″), it seems questionable whether there are models where the version with $H$ is applicable, and the version with $X$ is not. It is, therefore, just for the reason of completeness that we add the following theorem.

**Theorem 4.2** *Let*

$$(4.10) \qquad \mathfrak{Q}_{n,a} := \{P \in \mathfrak{P} : \ H(P_0, P) \leq n^{-1/2} a\}$$

*and*

$$(4.11) \qquad s_n(a) := \sup\{c_n |\kappa(P) - \kappa(P_0)| : \ P \in \mathfrak{Q}_{n,a}\}.$$

*Assume that*

$$(4.12) \qquad \limsup_{a\to 0} a^{-1} \liminf_{n\to\infty} s_n(a) = \infty.$$

*Let $\kappa^{(n)}$, $n \in \mathbf{N}$, be an estimator sequence fulfilling (4.4) and, for some $a > 0$,*

$$(4.13) \qquad \sigma_*^2 := \lim_{u\to\infty} \limsup_{n\to\infty} \sup_{P\in\mathfrak{Q}_{n,a}} \int L_u[c_n(\kappa^{(n)} - \kappa(P))]^2 dP^n < \infty.$$

*Then the estimator sequence $\kappa^{(n)}$, $n \in \mathbf{N}$, cannot be asymptotically unbiased with the rate $c_n$, $n \in \mathbf{N}$, uniformly on $\{P \in \mathfrak{P} : \ H(P_0^n, P^n) \leq a\}$ for some $a > 0$.*

**Proof** If $\kappa^{(n)}$, $n \in \mathbf{N}$, is asymptotically unbiased, uniformly on $\{P \in \mathfrak{P} : H(P_0^n, P^n) \leq a\}$, relation (2.8″) implies that

$$(4.14) \qquad \sigma_* \geq 8^{-1/2} \pi_0 \left(1 - 4\frac{a^2}{\pi_0}\right)^{1/2} a^{-1} \liminf_{n\to\infty} \tilde{s}_n(a),$$

with

$$\tilde{s}_n(a) = \sup\{c_n|\kappa^{(n)} - \kappa(P)| : \ H(P_0^n, P^n) \le a\}.$$

Since $H^2(P_0^n, P^n) \le n^{1/2} H^2(P_0, P)$, we have $\tilde{s}_n(a) \ge s_n(a)$. Hence (4.14) holds with $\tilde{s}_n(a)$ replaced by $s_n(a)$. Since this relationship holds for every $a > 0$, relation (4.12) is in contradiction to (4.13). ∎

Theorem 4.2 is closely related to Theorem 2 in Liu and Brown (1993, p. 4). In the expressive terminology of these authors their result reads as follows: At a singular point of irregular infinitesimality an estimator sequence cannot be both locally asymptotically unbiased and locally asymptotically informative. Theorem 4.2 differs from this result of Liu and Brown only in two aspects of minor importance, namely the use of $L_u$ instead of $\ell_u[y] = L_u[y] + u(1_{(u,\infty)}(y) - 1_{(-\infty,-u)}(y))$, and the use of condition (4.12) in place of "irregular infinitesimality" (see Definition 2.6, p. 4, in the paper by Liu and Brown). We remark that Theorem 4.2 is stronger than the corresponding theorem with $\ell_u$ in place of $L_u$ (in relation (4.13) and in the definition of asymptotic unbiasedness). (4.13) with $\ell_u$ in place of $L_u$ is stronger, since $L_u[y]^2 \le \ell_u[y]^2$. Moreover,

$$uP\{c_n|\kappa^{(n)} - \kappa(P)| > u\} \le u^{-1} \int \ell_u[c_n(\kappa^{(n)} - \kappa(P))]^2 dP^n. \tag{4.15}$$

Hence the definitions of asymptotic unbiasedness, uniformly on $\mathfrak{Q}_{n,\alpha}$, based on $L_u$ or $\ell_u$, are equivalent if (4.13) holds with $\ell_u$ in place of $L_u$.

## 5 Auxiliary results

**Lemma 5.1** *Let $P_n, Q_n$ be sequences of probability measures on $(X, \mathcal{A})$ with $q_n \in dQ_n/dP_n$. If*

$$P_n \circ \log q_n \Rightarrow M, \tag{5.1}$$

*with $M|\mathbf{B}$ a probability measure fulfilling $\int \exp[v] M(dv) = 1$, then*

$$\lim_{n\to\infty} H(P_n, Q_n)^2 = 1 - \int \exp[v/2] M(dv). \tag{5.2}$$

**Proof** If $q \in dQ/dP$, we have

$$H(P, Q)^2 := \frac{1}{2} \int (\sqrt{q} - 1)^2 \, dP = -\int (\sqrt{q} - 1) \, dP. \tag{5.3}$$

With

$$\Pi_n := P_n \circ (\sqrt{q_n} - 1), \tag{5.4}$$

relations (5.3) may be rewritten as

$$H(P_n, Q_n)^2 = \frac{1}{2}\int y^2 \Pi_n(dy) = -\int y \Pi_n(dy). \tag{5.5}$$

Since $\sqrt{u} - 1 = \exp\left[\frac{1}{2}\log u\right] - 1$, we have

$$\Pi_n = (P_n \circ \log q_n) \circ (v \mapsto \exp[v/2] - 1).$$

Since $v \mapsto \exp[v/2] - 1$ is continuous, relation (5.1) implies

$$\Pi_n \Rightarrow \Pi_0 := M \circ (v \mapsto \exp[v/2] - 1). \tag{5.6}$$

We have

$$\liminf_{n\to\infty} \int y^2 \Pi_n(dy) \geq \int y^2 \Pi_0(dy)$$

and, since $y \geq -1$,

$$\liminf_{n\to\infty} \int y \Pi_n(dy) \geq \int y \Pi_0(dy).$$

This implies

$$\liminf_{n\to\infty} H(P_n, Q_n)^2 \geq \frac{1}{2}\int y^2 \Pi_0(dy)$$

and

$$\limsup_{n\to\infty} H(P_n, Q_n)^2 \leq -\int y \Pi_0(dy).$$

Since

$$\frac{1}{2}\int y^2 \Pi_0(dy) = -\int y \Pi_0(dy) = 1 - \int \exp[v/2] M(dv),$$

the assertion follows. ∎

**Lemma 5.2** *Let $P_0, P_n$, $n \in \mathbf{N}$, be probability measures with $\mu$-densities $p_0$ and $p_n$, respectively. Assume there exists a function $g \in \mathcal{L}_2(P_0)$ with $\int g(x) P_0(dx) = 0$ such that*

$$\frac{p_n}{p_0} = 1 + n^{-1/2} g + n^{-1/2} r_n \tag{5.7}$$

*with*

$$\int r_n^2(x) P_0(dx) \to 0. \tag{5.8}$$

*Then*

$$\lim_{n\to\infty} X^2(P_0^n; P_n^n) = \exp[\int g^2 dP_0] - 1 \ . \tag{5.9}$$

**Proof** Since

$$X^2(P_0;P_n) = \int \left(\frac{p_n}{p_0} - 1\right)^2 dP_0 = n^{-1} \int (g + r_n)^2 dP_0,$$

we have

$$\lim_{n\to\infty} nX^2(P_0;P_n) = \int g^2 dP_0.$$

Since

$$X^2(P_0^n;P_n^n) = (1 + X^2(P_0;P_n))^n - 1,$$

the assertion follows. ∎

The following relations are used implicitly by various authors.

**Lemma 5.3** *For every function* $f : X \to \mathbf{R}$ *which is integrable with respect to* $P$ *and* $P_0$,

(5.10′) $$|\int f dP - \int f dP_0| \le (\int f^2 dP_0)^{1/2} X(P_0;P)$$

*and*

(5.10″) $$|\int f dP - \int f dP_0| \le 2(\int f^2 dP + \int f^2 dP_0)^{1/2} H(P_0,P).$$

**Proof** Let $p$ and $p_0$ be densities with respect to some dominating measure $\mu$. By Schwarz's inequality

(5.11) $$|\int f(p - p_0)d\mu|^2 \le \int f^2 h d\mu \int \frac{(p - p_0)^2}{h} d\mu$$

for any measurable function $h : X \to (0,\infty)$. Relation (5.10′) follows by application of (5.11) with $h = p_0$. Relation (5.10″) follows by application with $h = p + p_0$, since $(p - p_0)^2/(p + p_0) \le 2(\sqrt{p} - \sqrt{p_0})^2$. ∎

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