LASER COOLING AND STOCHASTICS

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In the statistical analysis of cooling and trapping of atoms by a combination of laser and magnetic field technology, Aspect, Bardou, Bouchaud and Cohen-Tannoudji (1994) showed that Lévy flights is the key tool. A review of their analysis, from the point of view of renewal theory and occupation times for stochastic processes, is given here and some further analysis provided. Brief discussions of two related types of models are also given.

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1 Introduction

Cooling and trapping of atoms, by a combination of laser and magnetic field technology, is a subject area of great current interest in physics. By directing a number of laser beams towards a chosen point in space and setting up a suitable magnetic field around the point it is possible to hold a cloud of atoms largely concentrated in a very small region around the point, as indicated in Figure 1. The basis of the techniques is the fact that light acts mechanically on material objects, such as atoms, meaning that it can change their positions and velocities. Each single atom follows a random trajectory, but is staying most of the time near the centre of the trapping region; it moves very little and is therefore 'cold'.

Stochastic considerations have led to a substantially better understanding, and subsequently to a dramatic improvement in efficiency, of the cooling; see Bardou et al. (1994), Bardou (1995) and Reichel et al. (1995).

Of particular interest are the questions:

(i) how much of the total time of the experiment does the momentum (vector) of the atom belong to a small neighbourhood of the origin.

(ii) what is the distribution of the momentum given that it belongs to such a neighbourhood.

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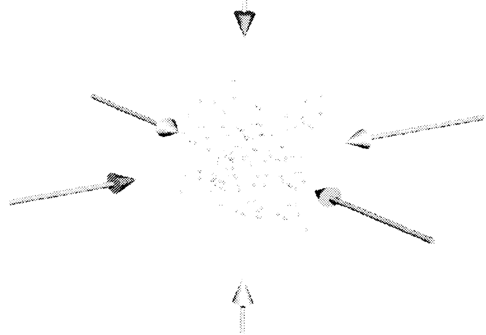

Figure 1. Experimental setup for laser cooling and trapping.

These and related questions are discussed in considerable detail in a forthcoming paper by Bardou, Bochaud, Aspect and Cohen-Tannoudji (1999), a preliminary version of which has kindly been provided to us by Francois Bardou. (See also Bardou and Castin (1998)). We shall refer to their treatment as the ABBC analysis.

The ABBC analysis led to the heart of the matter but through an approximate analysis, ab initio. In Sections 4 and 5 we review and extend that work in the light of the theory of renewals and occupation times for stochastic processes. In this we draw on well known results of that theory as expounded, for instance, in Bingham, Goldie and Teugels (1987). Section 2 outlines the physical setting in more detail and Section 3 specifies the resulting stochastic process model for a one-dimensional component of the momentum vector. Some analogous, but simpler, models that allow of a fairly detailed analysis are briefly treated in Section 6, and the final Section 7 contains concluding remarks.

2 Laser cooling

The four most prominent cooling techniques, listed in the order they arose chronologically, are Doppler cooling, Sisyphus cooling, VSCPT (Velocity-Selective Coherent Population Trapping) and Raman ccoling. Doppler cooling and Sisyphus cooling were capable of bringing the temperature down to 1 μK, approximately, but lower limits are not achievable by these methods due to a recoil effect. With VSCPT and Raman cooling temperatures of the order 1 nK are reached. These two methods rely heavily on the effects of what in the physics literature has become to be known as 'Lévy flights', and which play an important role in many other contexts in physics. In the language of stochastics the effects are those associated to the properties of the stable laws (cf. Sections 4 and 5).

For an atom subjected to VSCPT, the quantum mechanical description of its behaviour is as a wave function ψ and it is this function that undergoes a random trajectory in Hilbert space, the stochastic movements being caused by absorption and emission of photons. In this connection, see Cohen-Tannoudji, Bardou and Aspect (1992), Castin and Mølmer (1995) and Mølmer and Castin (1996).

The models to be described and discussed in the following refer mainly to the Raman method. Under that type of experimental setup we have in mind here the momentum of the atom is accurately determined, in the sense of having a narrow probability distribution (centered on zero) and hence, due to complementarity, the position is only vaguely determined. Correspondingly, the stochastic processes we shall be discussing in the following sections are to be conceived as models for the time behaviour of the momentum rather than the position of the atom. However, this still means (recall the atomic scales) that with high probability the position will be in a (roughly) spherical region with a diameter of the order of 1 mm or less (the central region in Figure 1).

Laser cooling and trapping makes it possible to measure important physical quantities with unprecedented precision and to study various types of fundamental questions in particle physics, for instance concerning atom optics, atom interferometry, atomic clocks, and high resolution spectroscopy. The 1997 Nobel Price in physics was given for research in this area, to Steven Chu, Claude Cohen-Tannoudji and William D. Phillips. The three Nobel Prize Lectures, by Chu (1998), Cohen-Tannoudji (1998) and Phillips (1998), are highly readable and informative. An earlier, less technical and very illuminating, discussion was given by Aspect and Dalibar (1994).

For the future, the techniques hold much promise for the study of 'pure' situations, such as systems of a small number of atoms in well-defined states exhibiting quantum features.

3 Stochastic momentum model

As indicated in Sections 1 and 2, the basic description of the behaviour of a single atom is in terms of the random 'path' of its wave function ψ. Under Raman cooling (and also under VSCPT) the description can for many purposes be reduced to the following type of model for the momentum of the atom, as a function of t.

Let Y_t be a Markov jump process with state space $\mathbb{R}^D$ and transition law $\mu(x, \mathrm{d}y)$ for jumps from x to y. The rate function for the waiting times will be denoted by λ; in other words, letting $\tau(y)$ be the generic notation for a waiting time in state y we have that the law of $\tau(y)$ is exponential with mean $\lambda(y)^{-1}$. We write $\Gamma_A(t)$ for the occupation time in a set A up till time t, i.e.

$$\Gamma_A(t) = \int_0^t 1_A(Y_s)\mathrm{d}s$$

and $B_x(\rho)$ will denote the ball in $\mathbb{R}^D$ with radius ρ and center x. We shall refer to $B_0(r)$, for some small r, as the 'trap', this corresponding to the cold states of the atom. Finally, let A_t be the random variable that is 0 if the atom is in the trap at time t and is 1 otherwise, and define q_t to be the conditional probability density of Y_t given that $A_t = 0$, i.e.

$$q_t(y) = p(y \ddagger Y_t | A_t = 0)$$

The dimensions $D = 1, 2$ or 3 are those of physical interest, and we shall mainly consider the one-dimensional case. The key experimental setting is such that, up to a scaling, which is unimportant in the present context (see further in Section 4, Footnote 3),

$$\lambda(y) = c|y|^\gamma \quad \text{for } y \in B_0(1)$$

for some $c > 0$ and some $\gamma > 0$. The parameter γ is determined by the experimental setup[2], the case $\gamma = 2$ being of some special interest[3]. For y outside the ball $B_0(1)$ various forms of $\lambda(y)$ are considered. We shall discuss three model types:

Model type I: For some $R > 1$ there is a reflecting barrier at the surface of the ball $B_0(R)$, and $\lambda(y) = c$ for $1 < |y| < R$.

Model type II: $\lambda(y) = c$ for all y with $1 < |y|$.

Model type III: $\lambda(y) = 1$ for $1 < |y| < R$ and $\lambda(y) = c(R/|y|)^\eta$ for $\eta > 0$ and all y with $R \leq |y|$.

Furthermore, under model types II and III $\mu(x, \mathrm{d}y)$ is of the form

$$\mu(x, \mathrm{d}y) = \phi(y - x)\mathrm{d}y$$

[2]The reason for the special power form of the intensity function in a neighborhood of zero comes from physical considerations of the atoms influenced by the laser. In VSCPT cooling $\gamma = 2$ because an atom which has absorbed a photon is in an unstable, excited state. Physical reasoning shows that the transmisson rate in this state will be proportional to the square of the momentum of the atom. This in turn leads to a transmission rate in the non-coupled, stable state with the same momentum dependency. For Raman cooling, on the other hand, γ can be tuned by the experimenter. In this set-up the reasoning goes via the Fourier transform of light pulses. For example, a Blackman pulse gives $\gamma = 4$, while for a rectangular pulse $\gamma = 2$. See Aspect et al. (1988) and (1989) for more detailed physical explanations.

[3]In some instances, however, a more realistic specification of λ inside the ball $B_0(1)$ is as

$$\lambda(y) = c|y|^\gamma + c_0$$

for $c_0 > 0$ but very small. We shall not consider this possibility further here, but take it up in connection with the discrete model formulation in Subsection 6.1

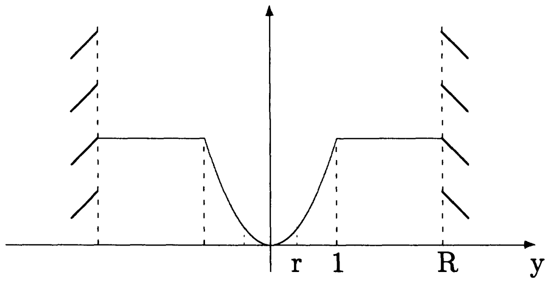

Figure 2. Shape of $\lambda(y)$ in model type I with $\gamma = 2$.

for some function ϕ, which in the one-dimensional case may be taken as

$$\phi(\cdot) = \delta^{-1}\mathbf{1}_{[-\delta/2,\delta/2]}(\cdot)$$

for some positive $\delta < 1$; that is, for $D = 1$ the jump sizes are uniformly distributed between $-\delta/2$ and $\delta/2$[4]. For model type I this form is modified near the reflecting barrier, in a natural fashion. (For model type I, only the behaviour within the ball $B_0(R)$ is studied.)

For investigations in physics, model type I is the most important and we shall mainly consider this. Furthermore, for simplicity, we largely restrict attention to the one-dimensional case, i.e. $D = 1$.

4 The ABBC analysis

As already indicated, in the ABBC approach one considers, in momentum space, a small ball $B_0(r)$ - the *trap* - centred at 0 and with radius r.

Let $\tau_1, \tau_2, ...$ and $\hat{\tau}_1, \hat{\tau}_2, ...$ denote the successive sojourn times in and out of the trap, respectively, the τ_i-s constituting an i.i.d. sequence and likewise for the $\hat{\tau}_i$-s. It is assumed that with sufficient accuracy one can think of these two sequences as being independent. The degree of accuracy of the implied approximation depends on the size of δ.

It is furthermore argued that provided $r << \delta/2$ one can, to good approximation, assume that when the atom jumps into the trap from outside, the attained momentum y will be uniformly distributed in $B_0(r)$. Letting $\lambda(y)$ denote the rate of the exponential waiting time distribution in momentum state y one therefore has that the τ_i-s follow the distribution with density

$$p(x \ddagger \tau) = |B_0(r)|^{-1} \int_{B_0(r)} \lambda(y) e^{-\lambda(y)x}\, dy$$

[4]In the units chosen here, δ is of the order of $\hbar|k|$ where $\hbar$ is Planck's constant and k is the optical wave-vector.

where $|B_0(r)|$ is the volume of $B_0(r)$ and τ is a generic random variable having the same distribution as the τ_i-s. Let $\alpha = D/\gamma$, then

$$p(x \ddagger \tau) \sim a|x|^{-(1+\alpha)}$$

for some constant a. Thus, provided $\alpha < 2$, the law of the τ_i-s belongs to the domain of attraction of a positive α-stable distribution with a scaling constant b depending on a. We denote the distribution function of this positive α-stable law[5] by $S_\alpha(x; b)$. In particular, if $\gamma = 2$ and $D = 1$, then $\alpha = 1/2$ and

$$p(x \ddagger \tau) = \frac{1}{2} r^{-1} \gamma(\frac{3}{2}, r^2 x) x^{-3/2}$$

where $\gamma(a, x)$ is the incomplete gamma function

$$\gamma(a, x) = \int_0^x s^{a-1} e^{-s}\, ds$$

Hence the τ_i-s are in the domain of attraction of the $\frac{1}{2}$-stable law with scaling constant $b = 2^{-3} r^{-2}$.

As regards what happens outside the trap, it is argued that under model type I the $\hat{\tau}_i$-s belong to the domain of attraction of the normal law, while under type II the domain of attraction is again that of a $\frac{1}{2}$-stable law with some scaling constant $\hat{b}$, as is indeed plausible in view of well-known probabilistic results. Under model type III, the distribution of the $\hat{\tau}_i$-s is argued to belong to the domain of attraction of a $\frac{1}{4}$-stable law when η is chosen equal to 2.

In the calculations below we will frequently refer to the distributions of τ_i and $\hat{\tau}_i$ as being α-stable and $\hat{\alpha}$-stable respectively, where $\hat{\alpha} \leq \alpha$ and $\hat{\alpha}, \alpha \in (0, 1)$. In model type II we have $\alpha = \hat{\alpha}$ while under model type III, $\hat{\alpha} < \alpha$. Mathematically, the most interesting cases are $\alpha = \hat{\alpha} = 1/2$ (model type II) and $\alpha = 1/2$, $\hat{\alpha} = 1/4$ (model type III).

4.1 Occupation times

We consider here the time spent by the atom in the trap between 0 and t.

Model type I: Let $\tau_{(t)}$ denote the longest of the periods spent in the trap before time t. For $t \to \infty$, $\tau_{(t)}$ is of the order of t (cf. a well-known property of the stable laws) and hence, in particular, $\Gamma_{B_0(r)}(t)/t \to 1$.

Model type II: In this case

$$\Gamma_{B_0(r)}(t)/t \to \frac{b}{b + \hat{b}}$$

where b and $\hat{b}$ are the scaling constants of τ and $\hat{\tau}$, respectively.

Model type III: In this model $\hat{\alpha} < \alpha$ and hence $\Gamma_{B_0(r)}(t)/t \to 0$.

[5]In a standard notation (see, e.g., Samorodnitsky and Taqqu (1994)) this law is denoted $S_\alpha(b, \beta, \mu)$ with $\beta = 1$ and $\mu = 0$.

4.2 The 'sprinkling distributions'

To obtain more precise information on the distribution of the momentum Y_t at time t the authors derive the 'sprinkling distributions' $\mathcal{S}_R$ and $\mathcal{S}_E$. In the traditional probabilistic terminology and assuming that the atom starts outside the trap, $\mathcal{S}_R$ and $\mathcal{S}_E$ are, in fact, the renewal measures corresponding to the sequences $\{\hat{\tau}_1 + ... + \tau_i + \hat{\tau}_i\}$ and $\{\hat{\tau}_1 + \tau_1 + ... + \hat{\tau}_i + \tau_i\}$, respectively. Denoting the corresponding renewal densities by u_R and u_E we have

$$u_E(t) = \sum_{i=1}^{\infty} p^{n*}(t \ddagger \hat{\tau} + \tau)$$

and

$$u_R(t) = p(t \ddagger \hat{\tau}) + \int_0^t u_E(t-x)p(x \ddagger \hat{\tau})\mathrm{d}x$$

The Laplace transforms of $u_E(t)$ and $u_R(t)$ are

$$\int_0^{\infty} e^{-\theta t} u_E(t)\, dt = \frac{\bar{L}\{\theta \ddagger \hat{\tau}\}\bar{L}\{\theta \ddagger \tau\}}{1 - \bar{L}\{\theta \ddagger \hat{\tau}\}\bar{L}\{\theta \ddagger \tau\}\}}$$

and

$$\int_0^{\infty} e^{-\theta t} u_R(t)\, dt = \frac{\bar{L}\{\theta \ddagger \hat{\tau}\}}{1 - \bar{L}\{\theta \ddagger \hat{\tau}\}\bar{L}\{\theta \ddagger \tau\}}$$

respectively, where $\bar{L}\{\theta \ddagger \chi\}$ is the Laplace transform of the random variable χ at θ.

Now consider model type I. Then $\hat{\tau}+\tau$ belongs to the domain of attraction of a positive α-stable law with scale parameter b and, as the authors show and as follows also from results of Dynkin and Lamperti (see further in Section 5), we then have

$$u_E(t), u_R(t) \sim \frac{1}{b\Gamma(\alpha)} t^{-(1-\alpha)}$$

for $t \to \infty$. In model type II $\hat{\tau}$ belongs to the domain of attraction of a positive α-stable law with scale parameter $\hat{b}$. Thus,

$$u_E(t), u_R(t) \sim \frac{1}{(\hat{b}+b)\Gamma(\alpha)} t^{-(1-\alpha)}$$

when $t \to \infty$. Under model type III, $\hat{\tau}$ belongs to the $\hat{\alpha}$-stable domain, where $\hat{\alpha} < \alpha$. Hence,

$$u_E(t), u_R(t) \sim \frac{1}{\hat{b}\Gamma(\hat{\alpha})} t^{-(1-\hat{\alpha})}$$

when $t \to \infty$.

4.3 Trapping probabilities

Next the authors discuss the probability of finding an atom in the trap. We give here a similar derivation of this probability: Let $Q(t) = \Pr\{A_t = 0\}$, i.e. be the probability of finding the atom in the trap $B_0(r)$ at time t. We have

$$Q(t) = G(t) + \int_0^t p(x \ddagger \hat{\tau}_1 + \tau_1) Q(t-x)\, dx \tag{1}$$

where

$$G(t) = \int_0^t p(x \ddagger \hat{\tau}) \Pr\{\tau > t - x\}\, dx$$

Relation (1) is a renewal equation, which has the solution

$$\begin{aligned} Q(t) &= \int_0^t G(t-x) u_E(x)\, dx \\ &= \int_0^t u_E(t-x) \int_0^x p(u \ddagger \hat{\tau}) \Pr\{\tau > x - u\}\, du\, dx \end{aligned} \tag{2}$$

The Laplace transform of $Q(t)$ takes the form

$$\int_0^\infty e^{-\theta t} Q(t)\, dt = \bar{L}\{\theta \ddagger \hat{\tau}\} \cdot \frac{1 - \bar{L}\{\theta \ddagger \tau\})}{\theta} \cdot \frac{\bar{L}\{\theta \ddagger \hat{\tau}\}\bar{L}\{\theta \ddagger \tau\}}{1 - \bar{L}\{\theta \ddagger \hat{\tau}\}\bar{L}\{\theta \ddagger \tau\}}$$

In order to study the asymptotics of $Q(t)$ we need to distinguish between the different model types. First, consider model type I. Since $\hat{\tau}$ has finite expectation and τ belongs to the domain of attraction of an α-stable law with scale b,

$$\int_0^\infty e^{-\theta t} Q(t)\, dt \sim \theta^{-1} - \mathrm{E}\{\hat{\tau}\}$$

when $\theta \to 0$. Hence, when $t \to \infty$, $Q(t) \sim 1$. In model type II both $\hat{\tau}$ and τ have distributions in the domain of attraction of an α-stable law with $\alpha \in (0,1)$ and scale parameter $\hat{b}$ and b, respectively. We have, for small θ,

$$\int_0^\infty e^{-\theta t} Q(t)\, dt \sim \frac{b}{\hat{b} + b} \theta^{-1}$$

which implies

$$Q(t) \sim \frac{b}{\hat{b} + b}$$

when $t \to \infty$. Finally, for model type III τ and $\hat{\tau}$ have distributions in the domain of an α-stable and an $\hat{\alpha}$-stable distribution, respectively, where $\hat{\alpha} < \alpha$ and $\hat{\alpha}, \alpha \in (0,1)$. In this case, the small θ behaviour will be

$$\int_0^\infty e^{-\theta t} Q(t)\, dt \sim \frac{b}{\hat{b}} \theta^{(\alpha - \hat{\alpha}) - 1}$$

which gives the large time asymptotics

$$Q(t) \sim \frac{b}{\hat{b}} \frac{1}{\Gamma(1-(\alpha-\hat{\alpha}))} \cdot t^{-(\alpha-\hat{\alpha})}$$

Note that formally for $\hat{\alpha} = \alpha$ this expression becomes $Q(t) \sim b/\hat{b}$, which differs from the correct results as given for model type II.

4.4 Momentum distribution

Finally the authors discuss the distribution of the momentum at time t inside the trap. We have,

$$\begin{aligned}
p(y, 0 \ddagger Y_t, A_t) &= (2r)^{-1} \sum_{i=1}^{\infty} \int_0^t p(x \ddagger \hat{\tau}_1 + \tau_2 + \ldots + \hat{\tau}_i) \Pr\{\tau(y) > t - x\}\, dx \\
&= (2r)^{-1} \int_0^t \sum_{i=1}^{\infty} p(x \ddagger \hat{\tau}_1 + \tau_2 + \ldots + \hat{\tau}_i) \Pr\{\tau(y) > t - x\}\, dx \\
&= (2r)^{-1} \int_0^t \Pr\{\tau(y) > t - x\} u_R(x)\, dx \\
&= (2r)^{-1} \int_0^t e^{-(t-x)\lambda(y)} u_R(x)\, dx \\
&= (2r)^{-1} \int_0^t e^{-(1-\frac{x}{t})t\lambda(y)} u_R(\frac{x}{t}t) \frac{dx}{t} \cdot t \\
&= (2r)^{-1} \int_0^1 e^{-(1-u)t\lambda(y)} u_R(tu)\, du \cdot t
\end{aligned}$$

The asymptotics for $p(y, 0 \ddagger Y_t, A_t)$ is easily studied in terms of the asymptotics of u_R:

$$u_R(x) \sim cx^{-(1-\tilde{\alpha})}$$

where $\tilde{\alpha} = \alpha$ in model types I and II and $\tilde{\alpha} = \hat{\alpha}$ in model type III. Furthermore, $c = (b\Gamma(\alpha))^{-1}$ and $c = ((b+\hat{b})\Gamma(\alpha))^{-1}$ in model types I and II, respectively, and $c = (\hat{b}\Gamma(\hat{\alpha}))^{-1}$ in model type III. Hence, for $t \to \infty$,

$$p(y, 0 \ddagger Y_t, A_t) \sim \frac{c}{2r} \mathcal{G}_{\tilde{\alpha}}(t\lambda(y))\, t^{\tilde{\alpha}} \tag{3}$$

where

$$\mathcal{G}_\eta(x) = \int_0^1 e^{-(1-u)x} u^{\eta-1}\, du$$

Our main interest lies in the large time behaviour of $q_t(y)$: First, notice that

$$q_t(y) = p(y \ddagger Y_t | A_t = 0) = Q^{-1}(t) p(y, 0 \ddagger Y_t, A_t)$$

Hence, we can give the asymptotic results for $q_t(y)$ under the three different model types appealing to the asymptotics for $Q(t)$ derived in the subsection above: Under model type I

$$q_t(y) \sim (2rb\Gamma(\alpha))^{-1} \cdot \mathcal{G}_\alpha\left(t\lambda(y)\right) t^\alpha, \quad t \to \infty \tag{4}$$

For model type II,

$$q_t(y) \sim \frac{b}{2rb\Gamma(\alpha)} \mathcal{G}_\alpha\left(t\lambda(y)\right) t^\alpha, \quad t \to \infty \tag{5}$$

and, finally, for model type III,

$$q_t(y) \sim \frac{\Gamma(1-(\alpha-\hat{\alpha}))}{2rb\Gamma(\alpha)} \mathcal{G}_{\hat{\alpha}}\left(t\lambda(y)\right) t^{2\alpha-\hat{\alpha}}, \quad t \to \infty \tag{6}$$

It follows that through rescaling by the transformation $u = \beta_t y$, where β_t is defined by $t\lambda(\beta_t^{-1}) = 1$, one obtains a limit law for u (conditional on $A_t = 0$) in all three cases.

5 Further analysis

We now return to the two first themes of Section 4 in order to discuss these further in the light of existing probabilistic results on occupation times and renewal theory.

5.1 Occupation times

Let us first consider the general momentum model introduced in Section 3. From Ethier and Kurtz (1986), p.162, we know that Y_t is a time-homogeneous Markov process with generator given by

$$\mathcal{A}f(x) = \lambda(x) \int_{\mathbb{R}^D} (f(x) - f(y))\, \mu(x, dy) \tag{7}$$

The domain of $\mathcal{A}$ is the space of real-valued measurable functions on $\mathbb{R}^D$ which are integrable with respect to the measure $\mu(x, dy)$.

Assume $\lambda(\cdot) \geq 0$ is bounded, and denote $\lambda := \sup_y \lambda(y)$. Introduce a modification of the transition probabilities μ in the following manner:

$$\tilde{\mu}(x, A) = \left(1 - \frac{\lambda(x)}{\lambda}\right) \delta_A(x) + \frac{\lambda(x)}{\lambda} \mu(x, A) \tag{8}$$

Let $\{x_k\}$ be the Markov chain with transition law $\tilde{\mu}$. According to Ethier and Kurtz (1986), Y_t has the same finite dimensional probability distributions as the process $X_t := x_{P_t}$, where P_t is a Poisson process with intensity λ

independent of $\{x_k\}$. The transition probabilites for X_t are easily derived to be

$$\Pr\{X_{t+s} \in A \,|\, X_t = x\} = e^{\lambda s} \sum_{n=0}^{\infty} \frac{(\lambda s)^n}{n!} \Pr\{x_n \in A \,|\, x_0 = x\} \tag{9}$$

Our main object of interest is relative occupation times for Y_t. Denote the occupation time in a Borel set A given that $Y_0 = x$ by

$$\Gamma_A^x(t) = \int_0^t 1_A(Y_s^x)\, ds \tag{10}$$

The relative occupation time in question for laser cooling is

$$\Gamma_{\mathrm{rel}}^x(t) := \Gamma_{B_0(r)}^x / \Gamma_{B_0^c(r)}^x \tag{11}$$

We consider the occupation time distribution of Y_t by exploiting the equivalence between the processes X_t and Y_t: Let $A \in \mathcal{B}(\mathbb{R}^D)$. For $n \in \mathbb{N}_0$ define

$$N_A(n) = \#\{x_i \in A \,:\, 0 \le i \le n\} \tag{12}$$

i.e. $N_A(n)$ is the number of visits to the set A of the Markov chain $\{x_i\}$ up till time n. Denote the number of jumps of X_t between 0 and t by N_t and define

$$\tilde{N}_A(t) := N_A(N_t) = \#\{x_i \in A \,:\, 0 \le i \le N_t\} \tag{13}$$

With these objects at hand, we can start to calculate an expression for the (defective) probability density of the occupation time of Y_t in a set A. For $s \le t$, let $p^{x_0}(s \ddagger \Gamma_A(t))$ be the (defective) probability density of $\Gamma_A(t)$ at s when $Y_0 = x_0 \notin A$. If $\tilde{N}_A(t) = k$ we know that X_s has spent k exponentially distributed time periods in the set A on the time interval $[0, t]$. These exponential waiting times are independent with intensity λ, and the sum of k periods will thus be gamma distributed with parameters k and λ. Hence,

$$p^{x_0}(s \ddagger \Gamma_A(t)) = p^{x_0}(0 \ddagger \tilde{N}_A(t))\delta_0(s) + \sum_{k=1}^{\infty} p^{x_0}(k \ddagger \tilde{N}_A(t)) g(s; k, \lambda) \tag{14}$$

where $g(s; k, \lambda) = \frac{\lambda^k}{\Gamma(k)} s^{k-1} e^{-\lambda s}$, is the density of the gamma distribution. A straightforward calculation with conditional probabilities shows that

$$\begin{aligned} p^{x_0}(k \ddagger \tilde{N}_A(t)) &= \sum_{n=k}^{\infty} p^{x_0}(k \ddagger N_A(n)) p(n \ddagger N_t) \\ &= e^{-\lambda t} \sum_{n=k}^{\infty} \frac{(\lambda t)^n}{n!} p^{x_0}(k \ddagger N_A(n)) \end{aligned}$$

since N_t is Possion distributed with intensity λ. Thus

$$(15)\qquad p^{x_0}(s \ddagger \Gamma_A(t)) = e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}p^{x_0}(0 \ddagger N_A(n))\delta_0(s)$$
$$+ e^{-\lambda(t+s)}\sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\frac{\lambda^{n+k}t^n s^{k-1}}{n!(k-1)!}p^{x_0}(k \ddagger N_A(n))$$
$$= e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}p^{x_0}(0 \ddagger N_A(n))\delta_0(s)$$
$$+ e^{-\lambda(s+t)}\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{\lambda^{n+k}t^n s^{k-1}}{n!(k-1)!}p^{x_0}(k \ddagger N_A(n))$$
$$= e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}p^{x_0}(0 \ddagger N_A(n))\delta_0(s)$$
$$+ e^{-\lambda(s+t)}\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}\sum_{k=1}^{n}\frac{\lambda^k s^{k-1}}{(k-1)!}p^{x_0}(k \ddagger N_A(n))$$

Hence, we see that the problem of calculating the occupation time of Y_t is reduced to finding the occupation time in A for the chain $\{x_k\}$.

We now consider the asymptotics for the occupation time, in the framework provided by Takacs (1959): Assume we have a stochastic process which enters states A and B alternately. The states A and B are disjoint subsets of the state space of the process, and their union constitutes the whole state space. The sequences of the successive sojourn times spent in the two states are assumed to be independent positive random variables. Under some asymptotic assumptions for the sums of the sojourn times in the two states, Takacs (1959) provides explicit asymptotic results for the total sojourn time in state B (or A) during the time interval $(0, t)$. His results are directly applicable to the laser cooling framework. We consider this in further detail:

Let state $B = B_0(r)$ where $r << 1$ and, as in the Section above, $\{\tau_i\}$ denotes the sequence of sojourn times in B, while the sojourn times in state $A = B^c$ are denoted $\{\hat{\tau}_i\}$. As we saw in Section 4, the τ_i-s will belong to the domain of attraction of a positive stable distribution of index α and scale parameter b. I.e.,

$$(16)\qquad \lim_{n\to\infty}\Pr\left\{\frac{\sum_{i=1}^{n}\tau_i}{n^{1/\alpha}} \le x\right\} = S_\alpha(x;b)$$

Concerning the shape of the distribution of the $\hat{\tau}_i$-s, this will depend on the choice of model. In model type I, the $\hat{\tau}_i$-s will belong to the domain of attraction of the normal distribution:

$$(17)\qquad \lim_{n\to\infty}\Pr\left\{\frac{\sum_{i=1}^{n}\hat{\tau}_i - \mathrm{E}\{\hat{\tau}\}n}{\sigma n^{1/2}} \le x\right\} = \Phi(x)$$

where σ^2 is the variance of the generic variable $\hat{\tau}$ with the same distribution as the $\hat{\tau}_i$-s. $\Phi(x)$ is the standard normal distribution. In model type II, the $\hat{\tau}_i$-s belong to the domain of attraction of a stable law of index α with scale parameter $\hat{b}$:

$$\lim_{n\to\infty} \Pr\left\{\frac{\sum_{i=1}^{n}\hat{\tau}_i}{n^{1/\alpha}} \le x\right\} = S_\alpha(x;\hat{b}) \tag{18}$$

Finally, for model type III, the sojourn times in the 'hot' state are in the domain of attraction of a stable law $S_{\hat{\alpha}}(x;\hat{b})$

$$\lim_{n\to\infty} \Pr\left\{\frac{\sum_{i=1}^{n}\hat{\tau}_i}{n^{1/\hat{\alpha}}} \le x\right\} = S_{\hat{\alpha}}(x;\hat{b}) \tag{19}$$

The conditions (16) and (17-19) are exactly what is needed in order to state the following result by Takacs (1959):

Theorem 5.1 *The asymptotics of $\Gamma_{B_0(r)}(t)$ is given by*

$$\lim_{t\to\infty} \Pr\left\{\frac{\Gamma_{B_0(r)}(t) - M_j t}{\tilde{M}_j t^{m_j}} \le s\right\} = Q_j(s), \quad j = 1,2,3 \tag{20}$$

where,

Model type I. $m_1 = \alpha$, $M_1 = 1$ and $\tilde{M}_1 = E\{\hat{\tau}\}$. $Q_1(s)$ is the distribution of $-\chi^{-1/2}$ where χ is distributed as $S_\alpha(s;b)$.

Model type II. $m_2 = 1$, $M_2 = 0$ and $\tilde{M}_2 = 1$. $Q_2(s)$ is the distribution of $\chi/(\zeta+\chi)$ where ζ is distributed as $S_\alpha(s;\hat{b})$ and χ as $S_\alpha(s;b)$.

Model type III. $m_3 = \hat{\alpha}/\alpha$, $M_3 = 0$ and $\tilde{M}_3 = 1$. $Q_3(s)$ is the distribution of $\chi\zeta^{-1/2}$ where ζ is distributed as $S_{\hat{\alpha}}(s;\hat{b})$ and χ as $S_\alpha(s;b)$.

5.2 The 'sprinkling' distribution

Again, let $\{\hat{\tau}_i\}_1^\infty$ and $\{\tau_i\}_1^\infty$ be independent random variables denoting the sojourn times in the 'hot' and 'cold' states respectively. Let $F(t)$ and $\hat{F}(t)$ be the distribution functions of τ_i and $\hat{\tau}_i$ respectively, where we assume the tail behaviour

$$1 - F(t) \sim \frac{b\ell(t)}{\Gamma(1-\alpha)} \cdot t^{-\alpha}, \quad t \to \infty \tag{21}$$

and

$$1 - \hat{F}(t) \sim \frac{\hat{b}\ell(t)}{\Gamma(1-\alpha)} \cdot t^{-\alpha}, \quad t \to \infty \tag{22}$$

for positive constants b, $\hat{b}$ and α. Note that in the case of model type II we have $\alpha = 1/2$, $\ell(t) = 1$, the constants b and $\hat{b}$ being the scale parameters of the stable distribution. For convenience, we denote the distribution function of $\sigma := \hat{\tau}+\tau$ by F_σ. Further, $\hat{\tau}$ and τ are generic random variables distributed as $\hat{\tau}_i$ and τ_i. Introduce the two renewal processes

$$\hat{M}_t = \max\{k \mid \hat{\tau}_1 + \tau_1 + \hat{\tau}_2 + \ldots + \hat{\tau}_k \le t\} \tag{23}$$

$$M_t = \max\{k \mid \hat{\tau}_1 + \tau_1 + \hat{\tau}_2 + \ldots + \hat{\tau}_k + \tau_k \le t\} \tag{24}$$

By convention, we let $\hat{M}_t = 0$ and $M_t = 0$ if the sets of k's to maximize are empty. Define the process

$$S_t = \sum_{i=1}^{\hat{M}_t} \hat{\tau}_i + \sum_{i=1}^{M_t} \tau_i \tag{25}$$

The processes $\hat{M}_t$ and M_t decide the state of the cooling process. To see this, introduce the times $R_1 = \hat{\tau}_1$, $E_1 = \hat{\tau}_1 + \tau_1$, $R_2 = \hat{\tau}_1 + \tau_1 + \hat{\tau}_2$, $E_2 = \hat{\tau}_1 + \tau_1 + \hat{\tau}_2 + \tau_2, \ldots$. The E_i-s denote the *exit* times, i.e. the times when the process exits the cooling state. On the other hand, the R_i-s are the times the atom returns to the cooling state. It is easy to see that if $\hat{M}_t = n$ and $M_t = n-1$, then $t \in [R_n, E_n)$, while if $\hat{M}_t = n$ and $M_t = n$, $t \in [E_n, R_{n+1})$. Thus, M_t is either equal to or one less than $\hat{M}_t$. In the former case the process S_t is in the hot state (i.e. the waiting time to next change is distributed as $\hat{\tau}_n$), while in the latter S_t is in the cooled state (i.e. waiting time to next change is distributed according to τ_{n+1}). In the previously introduced notation, $A_t = 0$ if and only if $\hat{M}_t = M_t + 1$ while $A_t = 1$ if and only if $\hat{M}_t = M_t$.

We consider the asymptotic behaviour of the residual time $R_t := t - S_t$ when we are in the 'cool' state, i.e. when $A_t = 0$. Motivated by the Dynkin-Lamperti theorem (see Bingham et al. (1987), p.361), it is natural to consider R_t/t and show that this has a limiting distribution. We adopt the argument in Bingham et al. (1987), p.361, to our case of *two* independent sequences of waiting times: Let $u \le v$ and $u, v \in [0,1]$. We have that $ut \le R_t \le vt$ and $A_t = 0$ if and only if for some $n \in \mathbb{N}_0$ and $y \in [1-v, 1-u]$,

$$\sum_{i=1}^{n} \sigma_i + \hat{\tau}_{n+1} = ty$$

and $\tau_{n+1} \ge t(1-y)$. Summing over n and integrating over y we get

$$\Pr\{u \le R_t/t < v,\ A_t = 0\} = \int_{1-v}^{1-u} (1 - F(t(1-y)))\,(\hat{F} * U_\sigma)(tdy) \tag{26}$$

where $U_\sigma(t)$ is the renewal measure associated to σ. Observe that $\hat{F} * U_\sigma = U_R$, the 'sprinkling' distribution with density $u_R(t)$. The right hand side of

(26) can be written

$$\int_{1-v}^{1-u} (1 - F(t(1-y)))\, U_R(tdy) = \int_{1-v}^{1-u} \frac{1 - F(t(1-y))}{1 - F(t)} (1 - F(t)) \times U_\sigma(t) \frac{U_R(tdy)}{U_\sigma(t)}$$

From Feller (1971), p. 271, we have

$$1 - F * \hat{F}(t) \sim \frac{(b + \hat{b})\ell(t)}{\Gamma(1-\alpha)} \cdot t^{-\alpha}$$

when $t \to \infty$, and by Tauberian theory this yields

$$U_R(t) \sim \frac{\Gamma(1+\alpha)}{(b+\hat{b})\ell(t)} \cdot t^{\alpha}$$

when $t \to \infty$. Hence, for $t \to \infty$,

$$(1 - F(t))U_\sigma(t) \sim \frac{b}{b + \hat{b}} \frac{\sin \pi\alpha}{\pi\alpha}$$

and

$$\frac{U_R(ty)}{U_\sigma(t)} \sim \frac{\ell(t)}{\ell(ty)} y^{\alpha} \sim y^{\alpha}$$

implying

$$\frac{U_R(tdy)}{U_\sigma(t)} \sim \alpha y^{\alpha-1}$$

Finally,

$$\frac{1 - F(t(1-y))}{1 - F(t)} \sim \frac{1}{(1-y)^{\alpha}}$$

In conclusion, we get

$$\begin{aligned}\int_{1-v}^{1-u} (1 - F(t(1-y)))U_R(tdy) &\to \int_{1-v}^{1-u} \frac{1}{(1-y)^{\alpha}} \frac{b}{b+\hat{b}} \frac{\sin \pi\alpha}{\pi\alpha} \cdot \alpha y^{\alpha-1}\, dy \\ &= \frac{b}{b+\hat{b}} \frac{\sin \pi\alpha}{\pi} \int_u^v y^{-\alpha}(1-y)^{-(1-\alpha)}\, dy\end{aligned}$$

Similar calculations can be worked through for the case when $A_t = 1$. Hence, we have the following version of the Dynkin-Lamperti Theorem in the case of *two* independent sequences $\{\tau_i\}$ and $\{\hat{\tau}_i\}$:

Theorem 5.2 *Assume F and $\hat{F}$ have tail behaviour as in (21) and (22). Then the normalized residual time $R_t/t := 1 - S_t/t$ when we are in the 'cool' state has a limiting distribution*

$$\lim_{t\to\infty} \Pr\{R_t/t \in (u,v), A_t = 0\} = \frac{b}{b+\hat{b}} \int_u^v g_\alpha(y)\, dy$$

and, when we are in the 'hot' state,

$$\lim_{t\to\infty} \Pr\{R_t/t \in (u,v), A_t = 1\} = \frac{\hat{b}}{b+\hat{b}} \int_u^v g_\alpha(y)\,dy$$

where

$$g_\alpha(x) = \frac{\sin \pi\alpha}{\pi} y^{-\alpha}(1-y)^{-(1-\alpha)}$$

6 Some analogous models

The approach considered in Sections 4 and 5 is based on certain approximations in relation to the basic model for the momentum, specified in Section 3. While these approximations appear highly plausible, any precise assessment of their accuracy is not available at present.

In this section we briefly discuss two model types that can be considered as alternative approximations to the momentum model and that allow fairly detailed analysis. The first is purely discrete and the other is of the diffusion type.

6.1 Discrete circular models

Let $x(t)$ be a semi-Markov process with a finite state space $\mathcal{S}$ consisting of $m+1$ points that we may view as positioned equidistantly around a circle. We talk of $x(t)$ as the position of the atom at time t, and one of the points in $\mathcal{S}$, denoted 0, will be considered as the 'trap'. We index the other points in $\mathcal{S}$ as $i = \pm 1, ..., \pm k$ if $m = 2k$ and as $i = \pm 1, ..., \pm k, k+1$ if $m = 2k+1$. Let q_i denote the density of the waiting time distribution at site i and suppose that $q_{-i} = q_i$, $i = 1, ..., k$. Furthermore, we let σ_0 be the recurrence time to $i = 0$, i.e. the time it takes for the atom, having just left 0, to return to the trap, and ξ_0 will denote the mean value of σ_0.

For brevity we shall consider here only the case $m = 2$ and we write q for q_1. More general settings and more detailed analyses will be discussed in a forthcoming paper. Also, we assume that the transitions between states follow the symmetric random walk pattern, i.e. transition can take place only to one of the two neighbouring sites on the circle, with equal probaility $\frac{1}{2}$. Finally, suppose the process starts at site $i = 1$ and we let $\Gamma_0(t) = \int_0^t \mathbf{1}_{\{0\}}(x(s))\,ds$. The distribution of $\Gamma_0(t)$ then has an atom of size $\Pr\{\sigma_0 > t\}$ at 0 while at $u > 0$ the probability density of $\Gamma_0(t)$ is

$$\begin{aligned} p(u \ddagger \Gamma_0(t)) &= \sum_{\nu=1}^{\infty} \left(\Pr\{\sigma_0 > \cdot\} * p^{*\nu}(\cdot \ddagger \sigma_0)\right)(t-u) q_0^{*\nu}(u) \\ &\quad + \bar{Q}_0(u) \sum_{\nu=1}^{\infty} p^{*\nu}(t-u \ddagger \sigma_0) q_0^{*(\nu-1)}(u) \end{aligned} \tag{27}$$

where

$$\bar{Q}_0(u) = \int_u^\infty q_0(v)\mathrm{d}v$$

We can mimic the ABBC treatment of model type I with $c_0 = 0$ by letting q_0 and q be the densities of a stable law of index $\frac{1}{2}$ and a negative exponential law with parameter λ, while the case where $c_0 > 0$ may be mimicked by instead letting q_0 be the density of the negative exponential distribution with a parameter $\lambda_0 < \lambda$. In the latter case,

$$q_0^{*\nu}(u) = \Gamma(\nu)^{-1}\lambda_0^\nu u^{\nu-1}e^{-\lambda_0 u}$$

and

$$p^{*\nu}(t \ddagger \sigma_0) = \lambda e^{-\lambda t} \cdot \frac{1}{2}\sum_{x=0}^{\infty}\binom{x+\nu-1}{x}\frac{1}{x!}(\lambda t/2)^x \tag{28}$$

It follows that (27) may be rewritten as

$$p(u \ddagger \Gamma_0(t)) = R(t-u,u) + R_0(t-u,u) \tag{29}$$

where $R_0(t-u,u)$ tends to 0 at an exponential rate as $t \to \infty$ and

$$\begin{aligned} R(t-u,u) &= \xi_0 \sum_{\nu=1}^{\infty} p^{*\nu}(t-u \ddagger \sigma_0) q_0^{*\nu}(u) \\ &\quad + \bar{Q}_0(u)\sum_{\nu=1}^{\infty} p^{*\nu}(t-u \ddagger \sigma_0) q_0^{*(\nu-1)}(u) \\ &= (\xi_0 + \mu) S(t-u,u) \end{aligned}$$

with $\mu = \lambda^{-1}$ and

$$\begin{aligned} S(t-u,u) &= \sum_{\nu=1}^{\infty} p^{*\nu}(t-u \ddagger \sigma_0) q_0^{*(\nu-1)}(u) \\ &= \frac{1}{2}\lambda_0\lambda e^{-\{\lambda_0 u + \lambda(t-u)\}} \\ &\quad \cdot \sum_{x=0}^{\infty}\frac{(\lambda(t-u)/2)^x}{x!^2}\sum_{\nu=0}^{\infty}\frac{(x+\nu)!}{\nu!}\frac{(\lambda_0 u)^\nu}{\nu!} \\ &= \frac{1}{2}\lambda_0\lambda e^{-\{\lambda_0 u + \lambda(t-u)\}} \\ &\quad \cdot \sum_{x=0}^{\infty}\frac{(\lambda(t-u)/2)^x}{x!} M(x+1,1,\lambda_0 u) \end{aligned}$$

The last factor is a special case of Kummerer's function[6] $M(a, c, z)$ defined for $c \neq 0, -1, -2, \ldots$ by

$$M(a, c, z) = \Gamma(c)\mathbf{M}(a, c, z) \tag{30}$$

and

$$\mathbf{M}(a, c, z) = \sum_{s=0}^{\infty} \frac{a(a+1)\ldots(a+s-1)}{c(c+1)\ldots(c+s-1)} \frac{z^s}{s!} \tag{31}$$

By formulae (9.03), (9.04) and Subsection 10.4 in Olver (1974; Chapter 7) we have, for z real and tending to ∞,

$$M(a, c, z) \sim z^{a-c} e^z / \Gamma(a) \tag{32}$$

provided neither a nor $1 + a - c$ is a negative integer or 0. Consequently, for $t \to \infty$ we have

$$\begin{aligned} S(t-u, u) &\sim \frac{1}{2}\lambda_0 \lambda e^{-\lambda(t-u)} \sum_{x=0}^{\infty} \frac{(\lambda_0 \lambda u(t-u)/2)^x}{x!^2} \\ &= \frac{1}{2}\lambda_0 \lambda e^{-\lambda(t-u)} I_0(t\sqrt{\{2\lambda_0 \lambda u(t-u)\}}) \end{aligned}$$

where I_0 is the Bessel function

$$I_0(z) = \sum_{s=0}^{\infty} \frac{(z^2/4)^s}{s!^2}$$

Hence, since for z real and tending to ∞

$$I_0(z) \sim (2\pi)^{-1/2} z^{-1/2} e^z$$

we find

$$S(t-u, u) \sim (2\pi)^{-1/2} \lambda_0 \lambda e^{-\lambda(t-u)} \{4\lambda_0 \lambda u(t-u)\}^{-1/4} t^{-1/2} e^{\{2\lambda_0 \lambda u(t-u)\}^{-1/2}}$$

All in all we therefore have

(33)
$$p(u \ddagger \Gamma_0(t)) \sim \frac{(\xi_0 + \mu)\lambda_0 \lambda}{\sqrt{2\pi}(4\lambda_0 \lambda)^{1/4}} (u(t-u))^{-1/4} \cdot t^{-1/2} \cdot e^{(2\lambda_0 \lambda u(t-u))^{-1/2} - \lambda(t-u)}$$

for $t \to \infty$.

[6]This function is also referred to as a degenerate hypergeometric function (cf. Gradstheyn and Ryzhik (1965; p. 1058) who use the notation $\Phi(a, c; z)$ instead of $M(a, c, z)$).

6.2 A diffusion model

We introduce a diffusion model for the atomic momentum which is a geometric Brownian motion in a neighbourhood of zero and a reflected Brownian motion elsewhere. The process will be reflected at R and $-R$ (for $R > 1$). Consider the diffusion

$$dX_t = \sqrt{\lambda(X_t)}\, dB_t, \quad X_0 = x \geq 0 \tag{34}$$

where

$$\lambda(x) = x^\alpha \mathbf{1}_{(-1,1)}(x) + \mathbf{1}_{(-R,-1)\cup(1,R)}(x) + \infty \cdot \mathbf{1}_{(-\infty,-R)\cup(R,\infty)}(x) \tag{35}$$

The diffusion process is symmetric around 0 and converges to zero a.s. when $t \to \infty$. By construction, the invariant measure of X_t is $\lambda^{-1}(x)$. This model for the laser cooling and trapping process does not take jumps into account. Observe that when $x \in (0,1]$ the first passage time for X_t to 1 will have heavy tails. Indeed, since $X_t^x = x\exp(B_t - t/2)$, the first passage time to 1 will be the same in distribution as the first passage time to zero of a Brownian motion with drift $-1/2$ starting at $\ln x$. Hence, as is well known, the tail of the distribution will go like $bt^{-3/2}$.

We consider the occupation time for X_t in $[-1,1]$. It is sufficient to consider only positive values of x since the process is symmetric around zero and its paths will never cross the axis. Let,

$$\Gamma^x(t) := \int_0^t \mathbf{1}_{[0,1]}(X_s^x)\, ds$$

The Laplace transform $z(x) = \int_0^\infty \exp(-\alpha t)\mathrm{E}[\exp(-\beta\Gamma^x(t))]\, dt$ is a piecewise C^2-solution to

$$\begin{cases} x^2 z''(x) = 2(\alpha+\beta)z(x) - 1, & x \in (0,1) \\ z''(x) = 2\alpha z(x) - 1, & x \in (1,R) \end{cases}$$

with boundary condition

$$z'(R) = 0$$

(cf. Karatzas and Shreve (1991)). The solution is: For $x \in (0,1]$,

$$z(x) = \frac{\beta}{\alpha(\alpha+\beta)} \cdot \frac{2\sqrt{2\alpha}\sinh((R-1)\sqrt{2\alpha})}{(1+\sqrt{1+8(\alpha+\beta)})\cosh((R-1)\sqrt{2\alpha})} \times$$
$$\frac{1}{1 + \frac{2\sqrt{2\alpha}\sinh((R-1)\sqrt{2\alpha})}{(1+\sqrt{1+8(\alpha+\beta)})\cosh((R-1)\sqrt{2\alpha})}} \cdot \exp(\frac{1}{2}\ln x \cdot (1+\sqrt{1+8(\alpha+\beta)})) +$$
$$(\alpha+\beta)^{-1}$$

and for $x \in (1, R]$.

$$z(x) = \frac{-\beta}{\alpha(\alpha+\beta)} \cdot \frac{1}{1 + \frac{2\sqrt{2\alpha}\sinh((R-1)\sqrt{2\alpha})}{(1+\sqrt{1+8(\alpha+\beta)})\cosh((R-1)\sqrt{2\alpha})}} \cdot \frac{\cosh((R-x)\sqrt{2\alpha})}{\cosh((R-1)\sqrt{2\alpha})} + \frac{1}{\alpha}$$

We can invert these transforms with respect to α:

Theorem 6.1 *For $x \in (1, R]$ we have,*

$$\begin{aligned} E^x\left[e^{-\beta\Gamma(t)}\right] &= 1 + \int_0^t (e^{-\beta(t-s)} - 1) \\ &\times\Big\{\sum_{n=0}^{\infty}(-1)^n q_{x,R,+}(\cdot) * \{h(\cdot;\beta) * q'_{1,R,-}(\cdot)\}^{*n}(\cdot)\Big\}(s)\, ds \end{aligned} \tag{36}$$

and for $x \in (0, 1]$,

$$\begin{aligned} E^x\left[e^{-\beta\Gamma(t)}\right] &= e^{-\beta t} + \sqrt{x}\int_0^t (1 - e^{-\beta(t-s)})\Big\{\sum_{n=0}^{\infty}(-1)^n q_{x,R,+}(\cdot) * \\ &\{h(\cdot;\beta) * q'_{1,R,-}(\cdot)\}^{*(n+1)} * p_x(\cdot;\beta)\Big\}(s)\, ds \end{aligned} \tag{37}$$

where

$$h(t;\beta) = \int_0^t e^{-\beta s}\Big\{e^{-s/8}\frac{2}{\sqrt{2\pi s}} + \frac{1}{4}\int_0^s e^{-u/8}\frac{du}{\sqrt{2\pi u}} - \frac{1}{2}\Big\}\frac{ds}{\sqrt{2\pi(t-s)}} \tag{38}$$

$$q_{x,R,\pm}(t) = \sum_{n=0}^{\infty}(-1)^n\Big\{p_{x<R}(\cdot) \pm p_{x<R}(\cdot) * p_{1<R}(\cdot)\Big\}(\cdot) * p_{1<R}^{*2n}(\cdot)(t) \tag{39}$$

$$p_x(t;\beta) = \frac{1}{4}e^{-\beta t}e^{-t/8}p_{\frac{1}{2}\ln x<0}(t/4) \tag{40}$$

and $p_{x<R}(t)$ is the density for the first passage time in R for a Brownian motion starting at $x < R$.

We note that it is possible to invert these transforms with respect to β. In a forthcoming paper we will do this and investigate the asymptotic properties of the distributions.

7 Concluding remarks

We hope in the future to address some of the following points.

The ultimate aim would be to give a detailed probabilistic treatment of the path properties of the jump process of the wave function ψ, which is a stochastic process in an infinite dimensional Hilbert space whose precise properties are determined by quantum mechanics (cf. Section 2).

A less ambitious aim is to analyze the basic momentum model (outlined in Section 3) more directly, that is without the type of initial approximation that lie in treating the successive sojourns in and out of the trap as if they were independent. In particular, it seems of some considerable interest (in dimensions $D = 1, 2, 3$) to obtain more accurate information about: (i) the momentum distribution in the trap (ii) the effect of having the size of the jumps (which is of the order of δ in the notation we have adopted) comparable to the size of the trap (iii) the relation between time behaviour and ensemble behaviour.

Several of these points seem quite challenging, but we realize that the interest in them may be largely mathematical rather than motivated by essential physical questions.

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