Minimal Expected Ranks for the Secretary Problems With Uncertain Selection

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Abstract

A secretary problem which allows the applicant to refuse an offer of acceptance with probability $1-p$ $(0 < p < 1)$ is considered with the objective of minimizing the expected rank of the applicant selected. The optimal rule is derived and an explicit solution to the problem, as the number of applicants becomes infinite, is obtained. The memory-length-one rules of the problem are also discussed.

keywords. optimal stopping, relative ranks, memory-length-one rule.

1. Introduction and summary

Before discussing our problems, we state the basic framework of the secretary problem and review briefly the two standard problems. A set of n rankable applicants (1 being the best and n the worst) appear before us one at a time in random order with all $n!$ permutations equally likely. When the i-th applicant appears, we must decide either to accept (select) or reject it based on the observed rank of the applicant relative to those preceding it,$1 \leq i < n$. If no selection has been made prior to the n-th applicant, then the last one must be selected. An offer of selection is accepted with certainty by the applicant.

According to the criterion of optimality, the secretary problem is often distinguished into two standard problems ; the *rank minimization problem*, in which the objective is to minimize the expected (absolute) rank of the applicant selected and the *probability maximization problem*, in which the objective is to maximize the probability of selecting the best applicant. Chow et al.(1964) showed that, as $n \to \infty$, the minimal expected rank for the rank minimization problem tends to the value $\prod_{j=1}^{\infty}(1+\frac{2}{j})^{1/1+j} \approx 3.8695$. Bruss and Ferguson(1993) also considered this problem in the full information setting where the decision is based on the actual values associated with the applicants, assumed to be independent and identically distributed from a known distribution (see also Assaf and Samuel-Cahn(1996)). As for the probability maximization problem, Lindley(1961) showed that the limiting maximal probability is e^{-1}. See also Gilbert and Mosteller(1966) and Dynkin and Yushkevich(1969).

In this note, we first generalize the rank minimization problem to include the possibility of an applicant refusing an offer. We assume for simplicity that each applicant only accepts an offer of selection with a known probability p $(0 < p < 1)$, independent of the rank of the applicant and all else. In other words, an offer is refused with probability $q = 1-p$. The availability of the applicant can be only ascertained by making an offer and stopping when it accepts. For this problem to make sense it is assumed that, if the process terminates with no applicant selected, the risk is n, which corresponds to choosing the worst. In Section 2, we show that, the limiting minimal expected rank for this problem tends to the value

$$\prod_{j=1}^{\infty}\left\{1+\frac{2}{j}\left(\frac{1+pj}{1+q+pj}\right)\right\}^{\frac{1}{1+pj}}. \tag{1.1}$$

Smith(1975) solved the probability maximization problem with uncertain selection, showing that the limiting maximal probability of selecting the best becomes $p^{1/q}$. Thus the result (1.1) would complete a two by two factorial design of the standard secretary problems.

Rubin and Samuels(1977) considered the standard secretary problems (with certain selection) with severe memory constraint, called the *memory-length-one rule*, where we are only allowed to remember just one of the previously observed applicants. That is, the only thing we can observe about a current applicant is whether it is better or worse than the currently remembered one. The remarkable result is that the limiting expected rank can be kept finite even with such a severe constraint. The nature of truly optimal memory-length-one rule remains unknown, but Rubin and Samuels(1977) obtained, through the argument of the *infinite secretary problem* (see, e.g., Gianini and Samuels(1976)), the value 7.414 as an upper bound for the limiting minimal expected rank. In Section 3, we attempt to obtain such an upper bound for the problem with uncertain selection. Two kinds of memory-length-one rules (MODEL 1 and MODEL 2) will be examined and a simple upper bound

$$U(p)=\frac{(1+p)^{2(1+p)/p}-1}{p(1+p)} \tag{1.2}$$

could be obtained as a function of p from MODEL 1.

For further information of the secretary problem, the reader is referred to Ferguson(1989) and Samuels(1991).

2. Minimal rank for the problem with uncertain selection

Let $X_r, 1 \le r \le n$, denote the absolute rank of the r-th applicant and let also $Y_r, 1 \le r \le n$, denote the rank of the r-th applicant relative to those preceding it, that is, $Y_r = 1+$ number of $X_1, \cdots, X_{r-1}$ which are less than X_r. It is easy to see that Y_r are independent random variables having its probability distribution $P\{Y_r = j\} = 1/r$ for $1 \le j \le r, 1 \le r \le n$ and that, for $j \le i \le n - r + j$,

$$P\{X_r = i \mid Y_r = j\} = \binom{i-1}{j-1}\binom{n-i}{r-j} \Big/ \binom{n}{r}.$$

Thus $E[X_r \mid Y_r = j]$, which is the conditional expected rank of the r-th applicant given that its relative rank is j and simply denoted by $Q_{r,j}$, is calculated as

$$Q_{r,j} = \sum_{i=j}^{n-r+j} iP\{X_r = i \mid Y_r = j\} = \left(\frac{n+1}{r+1}\right) j. \tag{2.1}$$

Our problem is to find a rule among all rules which minimizes the expected rank of the applicant selected.

Let $v_r, 0 \le r < n$, denote the minimal expected rank attainable when we haven't stopped by time r. Suppose that the r-th applicant has appeared and $Y_r = j$ is just observed. Since the applicant only accepts an offer with probability p, the expected risk incurred is $pQ_{r,j} + qv_r$ or v_r, depending on whether we make an offer or not. Hence, the backward induction equations for generating an optimal rule are given by, for $r \le n$,

$$v_{r-1} = \frac{1}{r}\sum_{j=1}^{r} \min\{pQ_{r,j} + qv_r,\ v_r\},$$

or equivalently

$$v_{r-1} = \frac{p}{r}\sum_{j=1}^{r} \min\{Q_{r,j} - v_r,\ 0\} + v_r \tag{2.2}$$

starting with the end condition $v_n = n$.

Let

$$s_r = \left[\frac{r+1}{n+1} v_r\right], \qquad 1 \le r \le n,$$

where $[x]$ denotes the greatest integer $\le x$. s_r is non-decreasing in r because v_r is, from (2.2), non-decreasing in r, while $Q_{r,j}$ is decreasing in r and increasing in j. From Eq.(2.2), an optimal rule makes an offer to the r-th applicant if $Y_r \le s_r$ until an offer is in effect accepted or the process terminates with no offer accepted. v_0 is the minimal expected rank under an optimal rule. Though Eq.(2.2) enables us to calculate v_0 for any n, a closed form expression for v_0 for sufficiently large n will shed light on the solution.

Theorem 2.1

Let $v^* = \lim_{n\to\infty} v_0$. *Then*

$$v^* = \prod_{j=1}^{\infty} \left\{1 + \frac{2}{j}\left(\frac{1+pj}{1+q+pj}\right)\right\}^{\frac{1}{1+pj}}.$$

Proof. We appeal to the heuristic method of Robbins (1991). Eq.(2.2) can be written as

$$v_r - v_{r-1} = \frac{p}{r}\sum_{j=1}^{s_r}(v_r - Q_{r,j}) = \frac{p}{r}\left\{s_r v_r - \frac{n+1}{r+1}\frac{s_r(s_r+1)}{2}\right\}. \tag{2.3}$$

Dividing both the left and the right sides of (2.3) by $1/n$ and letting $n \to \infty$ and $r \to \infty$ in such a way that $r/n \to t$, we obtain, if we put $v'(t) = \lim_{n,r\to\infty}(v_r - v_{r-1})/n^{-1}$,

$$v'(t) - \frac{ps}{t}v(t) + \frac{ps(s+1)}{2t^2} = 0, \tag{2.4}$$

which is valid for all t such that $s \le tv(t) \le s+1$ which follows from the definition of s_r. Define the sequence of $t's$, $0 < t_1 < t_2 < \cdots < 1$ by the equations :

$$t_s v(t_s) = s, \qquad s = 1, 2, \cdots, \tag{2.5}$$

so that the differential equation (2.4) is satisfied in the interval $t_s \le t \le t_{s+1}$. Eq.(2.4) can be integrated to yield

$$v(t) = \frac{ps(s+1)}{2(1+ps)t} + A_s t^{ps}$$

or

$$tv(t) = \frac{ps(s+1)}{2(1+ps)} + A_s t^{1+ps}, \tag{2.6}$$

where A_s is an integration constant.
We have, from (2.5) and (2.6)

$$\begin{aligned} s &= t_s v(t_s) = \frac{ps(s+1)}{2(1+ps)} + A_s t_s^{1+ps} \\ s+1 &= t_{s+1} v(t_{s+1}) = \frac{ps(s+1)}{2(1+ps)} + A_s t_{s+1}^{1+ps}. \end{aligned}$$

Thus

$$\frac{t_{s+1}}{t_s} = \left\{1 + \frac{2}{s}\left(\frac{1+ps}{1+q+ps}\right)\right\}^{\frac{1}{1+ps}},$$

and the repeated use of this leads to

$$t_{s+1} = t_1 \prod_{j=1}^{s} \left\{1 + \frac{2}{j}\left(\frac{1+pj}{1+q+pj}\right)\right\}^{\frac{1}{1+pj}},$$

and hence

$$t_1^{-1} = \prod_{j=1}^{\infty} \left\{1 + \frac{2}{j}\left(\frac{1+pj}{1+q+pj}\right)\right\}^{\frac{1}{1+pj}},$$

because $t_s \to 1$ as $s \to \infty$.
Thus the theorem is immediate from the fact that $v^* = v(t_1) = t_1^{-1}$.

Table 1 gives some numerical values of v^*.

Table 1. The limiting minimal expected rank v^*.

p	v^*
0.1	23.2635
0.5	6.2101
0.9	4.1389

3. Memory-length-one rules

We start with describing the finite problem to give a feeling for a memory-length-one rule, though our concerns consist not in the finite problem but in the limiting problem. On arrival of an applicant, we have three choices ; to *offer*, to *ignore* or to *remember and discard the previously remembered one.* A rule of the problem can be described by a sequence of choices $\{W_r/B_r; r = 1, 2, \cdots, n\}$, where W_r and B_r denote any of offer, ignore or remember, W_r being the choice if the r-th applicant is worse than the previously remembered one and B_r the choice if it is better. Since our problem includes the uncertainty of selection, the choice "offer" leads to the following two cases depending on the possible subsequent actions to be taken if the current applicant refuses an offer:

(1) The current applicant is newly remembered, and so the previously remembered
one is discarded.

(2) The current applicant is discarded, and the previously remembered one remains

unchanged.

Smith(1975) showed that, for the probability maximization problem with uncertain selection, the memory-length-one rule does not prevent us from using the optimal rule in case(1), namely the optimal rule is, for some r^*

$$W_r/B_r = \begin{cases} \text{ignore/remember} & (r \leq r^*) \\ \text{ignore/offer} & (r > r^*). \end{cases}$$

For the rank minimization problem, we distinguish the problem into two models, MODEL 1 and MODEL 2, according to which case ((1) or (2)) is assumed. It is not difficult to show that, as in the problem with certain selection, the optimal memory-length-one rule among the class that only includes three out of the nine possible choices is of the following form for both models

$$W_r/B_r = \begin{cases} \text{ignore/remember} & (r < a_n) \\ \text{ignore/offer} & (a_n \leq r < r_n) \\ \text{remember/remember} & (r = r_n) \\ W'_{r-r_n}/B'_{r-r_n} & (r > r_n), \end{cases}$$

where $\{W'_i/B'_i\ ; i = 1, \cdots, n - r_n\}$ is the optimal rule for the same problem with $n - r_n + 1$ applicants.

We now turn to the corresponding infinite problem. As in Gianini and Samuels(1976), the arrival times of the best, the second best, etc., applicants are an infinite sequence of independent random variables, each uniformly distributed on the interval (0, 1). It is noted that the minimal expected rank of the infinite problem gives an upper bound for the minimal expected rank of any finite problem (see Rubin and Samuels(1977) for reasoning). As a continuous analogue of the 3-action rule for the finite problem, we consider only the class of rules that choose numbers

$$0 = R_0 < A_1 < R_1 < \cdots < A_k < R_k < \cdots < 1$$

and alternately remembers the best applicant in each (R_{k-1}, A_k) and then makes an offer successively to the applicant in (A_k, R_k) better than the remembered one. Let the $A's$ and $R's$ be chosen so that, for all $i = 0, 1, \cdots$

$$(R_{i+1} - R_i) = R_1(1 - R_1)^i \tag{3.1}$$

and

$$\frac{A_{i+1} - R_i}{R_{i+1} - R_i} \equiv d. \tag{3.2}$$

Let T be the stopping time and X be the rank of the applicant selected. Then it is easy to see that the following useful relation, as derived by Rubin and Samuels(1977) for their problem,

$$E(X) = E(XI_{\{T<R_1\}}) + P\{T > R_1\}\frac{E(X)}{1-R_1} \tag{3.3}$$

still holds for our problem.
From (3.3), $E(X)$ is finite if and only if $P\{T > R_1\} < 1 - R_1$ and written as

$$E(X) = \frac{(1-R_1)E(XI_{\{T<R_1\}})}{1-R_1-P\{T > R_1\}}. \tag{3.4}$$

Our task is to find the two parameters d and R_1 which minimize the $E(X)$ subject to (3.1) and (3.2). Hereafter we discuss two models separately.

3.1. MODEL 1

Let $f_T(t), A_1 < t < R_1$, be the density function of T. To derive $f_T(t)$, we rely on the following lemma, which is interesting on its own.

Lemma 3.1
Let $Z_1, Z_2, \cdots$ be a sequence of random variables with Z_k uniformly distributed on the interval $(0, Z_{k-1}), Z_0 \equiv t \leq 1$.
(i) The density function and the distribution function of $Z_i, i \geq 1$, are respectively given by

$$f_i(z) = \frac{1}{(i-1)!t}\{\log(t/z)\}^{i-1}$$

and

$$F_i(z) = \left(\frac{z}{t}\right)\sum_{j=0}^{i-1}\frac{\{\log(t/z)\}^j}{j!}.$$

(ii) Let $N(s,t)$ denote the number of $Z_1, Z_2, \cdots$ whose value exceed s for $0 < s < t$, namely

$$N(s,t) = \max\{k : Z_k > s\},$$

where $\max \phi = 0$. *Then $N(s,t)$ is distributed as a Poisson random variable with parameter* $\log(t/s)$.

Proof. (i) By induction on i.
(ii) Since the event $N(s,t) = m$ occurs if and only if $Z_{m+1} \leq s < Z_m$ occurs, we have from (i) that

$$P\{N(s,t) = m\} = P\{Z_{m+1} \leq s < Z_m\}$$

$$\begin{aligned} &= F_{m+1}(s) - F_m(s) \\ &= \left(\frac{s}{t}\right)\frac{\{\log(t/s)\}^m}{m!}, \end{aligned}$$

which completes the proof.

From the assumption of the infinite problem, Z_k defined in Lemma 3.1 can be interpreted as the arrival time of the last *candidate* that appears prior to Z_{k-1} (for simplicity, we refer to a relatively best applicant as a candidate in this model). So the number of candidates that appear in time interval (s,t) has the same distribution as $N(s,t)$ defined in Lemma 3.1 (ii). Thus, by conditioning on $N(A_1,t)$, we have, for $A_1 < t < R_1$,

$$\begin{aligned} f_T(t) &= \sum_{m=0}^{\infty} f_T(t \mid N(A_1,t)=m)P\{N(A_1,t)=m\} \\ &= \sum_{m=0}^{\infty} q^m\left(\frac{p}{t}\right)P\{N(A_1,t)=m\} \\ &\quad (\textit{from the assumption of the infinite problem}) \\ &= \frac{p(dR_1)^p}{t^{1+p}}. \qquad (\textit{from Lemma } 3.1(ii)) \end{aligned} \tag{3.5}$$

We have from (3.5)

$$P\{T > R_1\} = 1 - \int_{A_1}^{R_1} f_T(t)dt = d^p$$

and

$$\begin{aligned} E(XI_{\{T<R_1\}}) &= \int_{A_1}^{R_1} E(X \mid T=t)f_T(t)dt \\ &= \int_{dR_1}^{R_1} \frac{1}{t} f_T(t)dt \\ &= \left(\frac{p}{1+p}\right)\left(\frac{d^{-1}-d^p}{R_1}\right), \end{aligned}$$

where the second equality follows since MODEL 1 makes an offer only to a candidate and since the k-th best applicant in $(0,t]$ has expected rank k/t (now $k=1$).

Thus we have from (3.4),

$$E(X) = \frac{p(d^{-1}-d^p)(1-R_1)}{(1+p)(1-R_1-d^p)R_1}. \tag{3.6}$$

For fixed d, the right side of (3.6) is minimized at $1-R_1 = d^{p/2}$. Substituting and minimizing with respect to d, we find that $1 - R_1$ is the unique root $x \in (0,1)$ of the equation

$$px^{2(1+1/p)} = (1+p)x - 1. \tag{3.7}$$

The upper bound for $E(X)$, as given in (1.2), can be obtained by approximating $1-R_1 = d^{p/2}$ by $(1+p)^{-1}$ in the right side of (3.6) ($1-R_1 > (1+p)^{-1}$ from (3.7), but the difference is rather small). Table 2 gives the optimal values of R_1, d and the corresponding $E(X)$ and $U(p)$ for given p. $U(p)$ gives a good approximation to $E(X)$.

Table 2. The optimal values of the two parameters R_1 and d, and the corresponding value of $E(X)$ in MODEL 1. $U(p)$ is an approximation to $E(X)$

p	R_1	d	$E(X)$	$U(p)$
0.1	0.074	0.214	63.02	64.91
0.5	0.291	0.253	13.60	13.85
0.9	0.430	0.288	8.10	8.20

3.2. MODEL 2

In this model, during the time interval (A_1, R_1) the remembered applicant, best in $(0, A_1)$, remains unchanged. Let M be the rank of this (remembered) applicant among those that appear in $(0, R_1)$. Consider now the $m+1$ best applicants in $(0, R_1)$. Then from the model assumption, $M = m+1$ occurs if and only if the $(m+1)$-st best applicant appears in $(0, A_1)$ and m best ones in (A_1, R_1). Thus

$$P\{M = m+1\} = \left(\frac{A_1}{R_1}\right)\left(1 - \frac{A_1}{R_1}\right)^m, \qquad m = 0, 1, \cdots \tag{3.8}$$

because the arrival times of the above $m+1$ best applicants are independent random variables, each uniformly distributed on the interval $(0, R_1)$.

Suppose that $M = m+1$. Then, for $A_1 < t < R_1$, in order for $T > t$ to occur, each of m best applicants in (A_1, R_1) must refuse an offer if it were to appear prior to t, and vice versa. Thus

$$P\{T > t \mid M = m+1\} = \left\{1 - p\left(\frac{t - A_1}{R_1 - A_1}\right)\right\}^m. \tag{3.9}$$

Therefore, from (3.9), the density of T conditional on $M = m+1$ is given by

$$f_T(t \mid M = m+1) \quad = \quad \frac{d}{dt}P\{T \leq t \mid M = m+1\}$$

$$= \frac{mp}{(1-d)R_1}\left\{1-\frac{p(t-dR_1)}{(1-d)R_1}\right\}^{m-1}. \qquad (3.10)$$

By conditioning on M, we have from (3.10)

$$\begin{aligned} E(XI_{\{T\le R_1\}}) &= E(E(XI_{\{T\le R_1\}} \mid M)) \\ &= \sum_{m=1}^{\infty} P\{M=m+1\}\sum_{k=1}^{m}\int_{A_1}^{R_1}\left(\frac{k}{R_1}\right)\left(\frac{1}{m}\right) f_T(t \mid M=m+1)dt \\ &= \frac{d}{2R_1}\{d^{-2}-(p+qd)^{-2}\} \end{aligned}$$

and

$$\begin{aligned} P\{T>R_1\} &= \sum_{m=0}^{\infty} P\{T>R_1 \mid M=m+1\}P\{M=m+1\} \\ &= \frac{d}{p+qd}. \end{aligned}$$

Thus we have from (3.4)

$$E(X)=\frac{p(d^{-1}-1)(1-R_1)\{d(p+qd)^{-1}+1\}}{2R_1\{(1-R_1)(p+qd)-d\}}. \qquad (3.11)$$

For fixed d, the right side of (3.11) is minimized at $1-R_1=\sqrt{d/(p+qd)}$. Substituting and minimizing with respect to d, we find that $1-R_1$ is the root $x \in (0,1)$ of the equation

$$qx^4(x^2-x-1)+px^3+x^2+x-1=0.$$

Table 3 gives the optimal values of R_1, d and the corresponding $E(X)$ for given p.

Table 3. The optimal values of the two parameters R_1 and d, and the corresponding value of $E(X)$ in MODEL 2

p	R_1	d	$E(X)$
0.1	0.266	0.105	47.95
0.5	0.395	0.225	12.39
0.9	0.447	0.284	7.99

3.3. Comparison between two models

Tables 2 and 3 show that, for each p, MODEL 2 has larger value of R_1, but smaller value of d, compared with MODEL 1 and that MODEL 2 gives better performance than MODEL 1. In additions to the comparison of R_1 and d between two models, it seemes interesting to compare the number of potential applicants in (A_1, R_1) to which an offer could be made if the process hasn't stopped before its arrival. Let N_i be the corresponding quantity in MODEL $i, i = 1, 2$. The distribution of N_2 is immediate from (3.8), that is,

$$\begin{aligned} P\{N_2 = k\} &= P\{M = k+1\} \\ &= d(1-d)^k, \qquad k = 0, 1, \cdots. \end{aligned}$$

Thus

$$E(N_2) = \frac{1-d}{d}.$$

By the way, a bit of consideration yields

$$E(N_1 \mid N_2) = \sum_{j=1}^{N_2} 1/j.$$

Hence we have by conditioning on N_2

$$\begin{aligned} E(N_1) &= E(E(N_1 \mid N_2)) \\ &= \sum_{k=1}^{\infty} \left(\sum_{j=1}^{k} 1/j \right) d(1-d)^k \\ &= -\log d. \end{aligned}$$

Table 4 gives the numerical values of $E(N_1)$ and $E(N_2)$. $E(N_2)$ is much larger than $E(N_1)$ especially for small p and very sensitive to the change of p.

We conclude this note by pointing out from Tables 1 and 3 that $E(X)$ for MODEL 2 is approximately two times as large as v^*.

Table 4. Comparison between $E(N_1)$ and $E(N_2)$

p	$E(N_1)$	$E(N_2)$
0.1	1.54	8.57
0.5	1.38	3.45
0.9	1.25	2.52

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