# A Class of Best–Choice Problems on Sequences of Continuous Bivariate Random Variables*

SAKAGUCHI, Minoru† and HAMADA, Toshio‡
*Kobe University of Commerce*‡

**Abstract**

An employer interviews a finite number $n$ of applicants for a position. They are interviewed one by one sequentially in random order. As each applicant $i$ is interviewed, two attributes are evaluated by the amounts $X_i$ and $Y_i$, where $X_i$'s and $Y_i$'s are not necessarily mutually independent, but $\{(X_i, Y_i)\}_{i=1}^{n}$ is iid sequence of continuous bivariate random variables and the common distribution of $(X_i, Y_i)$'s is known. Suppose that the employer is under the condition of full-information secretary problem without recall. We consider two kinds of the employer's objective and for each of the objectives the problems are formulated by dynamic programming and the optimal policy is explicitly derived.

## 1 Introduction

The present study is a continuation of the previous work by Sakaguchi and Szajowski[6]. An employer interviews a finite number of applicants for a position. They are interviewed one by one sequentially in random order. Each applicant has two attributes $X_i$ and $Y_i$, which are not necessarily independent. The employer observes $(X_i, Y_i)$ sequentially one by one, as each applicant appears, and he must choose (=stop at) one applicant without recall (*i.e.* if applicant is once not chosen, she is rejected and cannot be recalled later).

Let $\tau$ be the stopping time. Then the objective of the employer is to find the stopping rule which derive $\tau^*$ such that

$$(1^\circ) \qquad E\left[X_\tau I(Y_\tau \geq a)\right] \longrightarrow \max_{\tau}$$

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where $I(e)$ is the indicator of the event $e$, and $a > 0$ is a given fixed constant, and

$$(2^\circ) \qquad \Pr.\left\{X_\tau = \max_{1\le t\le n} X_t \quad \& \quad Y_\tau \ge a\right\} \longrightarrow \max_\tau .$$

These two problems are studied for the cases, where $\{(X_i, Y_i)\}_{i=1}^n$ is an *iid* sequence of r.v.s with common bivariate uniform and normal distributions. Each of the problems is formulated by dynamic programming and the optimal stopping rule is explicitly derived. The problem ($2^\circ$) belongs to the so-called monotone case (see, for example, [3; 137~139] in the optimal stopping theory, but the problem ($1^\circ$) doesn't so.

Four examples of the solutions are given, and one of them shows that if $(X_i, Y_i)$ is distributed as $\mathbf{N}\left(0,0;\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)$, then, for $n = 10$, we get $\max_\tau E\left[X_\tau I(Y_\tau \ge 0.84)\right] = 0.517$ and $\max_\tau E\left[X_\tau I(Y_\tau \ge -\infty)\right] = 1.276$.

An important and classical literature in secretary problems is Gilbert and Mosteller [2]. Recent look for the secretary problem and its various extensions can be found in Samuels [7]

# 2 Selecting better $X$ under the required condition that $Y \ge a$.

Let $\{(X_i, Y_i)\}_{i=1}^n$ be a sequence of independent bivariate r.v.s as given in the previous section. Observing the sequence $(X_i, Y_i), i = 1, 2, \cdots, n$, one by one sequentially, we want to maximize $EX_\tau$, where $\tau$ is the stopping time, under the required condition that $Y_\tau \ge a$, for a given $a > 0$. If we fail to stop (=choose) until the $(n-1)$st observation, then we must stop at $(X_n, Y_n)$, with reward $X_n I(Y_n \ge a)$.

Denoting, by $V_n$, the expected reward obtained by employing the optimal stopping rule where $n$ is the number of remaining objects, the Optimality Equation is

$$(1) \quad V_n = E\left[XI(Y \ge a) \vee V_{n-1}\right] \qquad (n \ge 2;\ V_1 = E\left[XI(Y \ge a)\right])$$

or equivalently,

$$(2) \qquad V_n = T_a(V_{n-1}) + V_{n-1}, \qquad (n = 2, 3, 4, \cdots)$$

where

$$(3) \qquad T_a(s) \equiv E\left[\{XI(Y \ge a) - s\}^+\right]$$

is a convex non-increasing function of $s$, for any fixed $a > 0$.

We consider the cases where $(X, Y)$ has bivariate uniform distribution and bivariate normal distribution.

## 2.1 Bivariate uniform distribution

Let the r.v. $(X, Y)$ be uniformly distributed on $[0,1]^2$ with pdf

$$p(x,y) = 1 + \gamma(1-2x)(1-2y), \quad |\gamma| \leq 1. \tag{4}$$

The correlation coefficient is equal to $(1/3)\gamma$. Since

$$q(x) \equiv \int_a^1 p(x,y)dy = \bar{a} \left\{1 - \gamma a(1-2x)\right\},$$

$$\bar{q}\,(x) \equiv \int_0^a p(x,y)dy = a\left\{1 + \gamma\,\bar{a}\,(1-2x)\right\},$$

and $q(x)+\bar{q}\,(x) = 1$, we have

$$T_a(s) = \int_0^1 dx \int_0^1 \{xI(y \geq a) - s\}^+ p(x,y)dy$$

$$= \textstyle\int_0^1 (x-s)^+ q(x)dx, \quad if \quad s \geq 0.$$

Direct computation gives

$$\int_0^1 (x-s)^+ q(x)dx = \bar{a} \int_s^1 (x-s)\left\{1 - \gamma a(1-2x)\right\}dx \tag{5}$$

$$= (1-s)^2 \left\{\frac{1}{2}\,\bar{a} + \frac{1}{6}\gamma a\,\bar{a}\,(1+2s)\right\}.$$

From (2) and (5), $\{V_n\}$ is an increasing sequence starting from

$$V_1 = E\left[XI(Y \geq a)\right] = \bar{a}\left(\frac{1}{2} + \frac{1}{6}\gamma a\right),$$

and is given by

$$V_n = \frac{1}{2}\,\bar{a}\,(1 - V_{n-1})^2 \left\{1 + \frac{1}{3}\gamma a\,(1 + 2V_{n-1})\right\} + V_{n-1} \qquad (n \geq 1;\ V_0 = 0) \tag{6}$$

Summarizing the above findings, we finally get

**Theorem 1** *The optimal stopping rule for the problem* (1°) *for bivariate uniform distribution* (4) *is to:*

$$\begin{cases} Stop, & if X \geq V_{n-1} \quad and \quad Y \geq a; \\ Continue, & if \quad otherwise, \end{cases}$$

*where* $\{V_n\}$ *is determined by the recursion* (6). *The optimal expected reward for the n-object problem is* $V_n$.

If $a \downarrow 0$, the problem approaches the univariate version, and (6) becomes $V_n = \frac{1}{2}\left(1 + V_{n-1}^2\right)$ which is the well-known Moser's sequence. If $0 < a < 1$, and $\gamma = 0$, the problem is the independent bivariate version, and (6) becomes

$$V_n = \frac{1}{2}\,\bar{a}\left(1 + V_{n-1}^2\right) + aV_{n-1}.$$

As a function of $0 < a < 1$, and for fixed $n$, $V_n$ is decreasing and $\left\{\begin{array}{c}\text{concave}\\ \text{linear}\\ \text{convex}\end{array}\right\}$ if $\left\{\begin{array}{c}0 < \gamma \le 1\\ \gamma = 0\\ -1 \le \gamma < 0\end{array}\right\}$, with values $\frac{1}{2}\left(1 + V_{n-1}^2\right)$ at $a = 0$, and $V_{n-1}$ at $a = 1$.

The values of $V_n$, for $n = 1(1)10, 15, 20$, $\gamma = -1.0(0.5)1.0$ and $a = 0.6, 0.7$ are given by Table 1.

As is seen by (6) and actually shown by Table 1, positive (negative) $\gamma$ makes the values of $V_n$ larger (smaller), for all $n$ and $a$, as $|\gamma|$ goes to unity.

## 2.2 Bivariate normal distribution

Let the r.v. $(x, y)$ be normally distributed on $(-\infty, \infty)^2$, with pdf

$$\begin{aligned}(7)\quad p(x,y) &= \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left(x^2 - 2\rho xy + y^2\right)\right\}\\ &= \phi(x)\frac{1}{\sqrt{1-\rho^2}}\phi\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right) = \phi(y)\frac{1}{\sqrt{1-\rho^2}}\phi\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right)\end{aligned}$$

where $\phi(x) \equiv (2\pi)^{-1/2}e^{-(1/2)x^2}$. From (3), we have

$$\begin{aligned}T_a(s) &= \int_{-\infty}^{\infty} dx\,\{xI(y \ge a) - s\}^+ p(x,y)dy\\ &= \int_a^{\infty} dy \int_s^{\infty} (x-s)p(x,y)dx + (-s)^+ \int_{-\infty}^{a} dy \int_{-\infty}^{\infty} p(x,y)dx\\ (8)\qquad &= \int_a^{\infty} \phi(y)dy \int_s^{\infty} (x-s)\frac{1}{\sqrt{1-\rho^2}}\phi\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right)dx + (-s)^+\Phi(a)\end{aligned}$$

where

$$\Phi(a) = \int_{-\infty}^{a} \phi(y)dy.$$

Since

$$(9)\qquad \int_s^{\infty}(x-s)\phi(x)dx = \phi(s) - s\,\bar{\Phi}\,(s) \equiv \Psi(s),$$

| $a$ | 0.6 | | | | |
|---|---|---|---|---|---|
| n \ $\gamma$ | −1.0 | −0.5 | 0.0 | 0.5 | 1.0 |
| 1 | 0.1600 | 0.1800 | 0.2000 | 0.2200 | 0.2400 |
| 2 | 0.2639 | 0.2962 | 0.3280 | 0.3592 | 0.3897 |
| 3 | 0.3391 | 0.3795 | 0.4183 | 0.4554 | 0.4907 |
| 4 | 0.3972 | 0.4429 | 0.4860 | 0.5261 | 0.5631 |
| 5 | 0.4438 | 0.4933 | 0.5388 | 0.5802 | 0.6175 |
| 6 | 0.4823 | 0.5345 | 0.5814 | 0.6231 | 0.6599 |
| 7 | 0.5148 | 0.5688 | 0.6164 | 0.6579 | 0.6937 |
| 8 | 0.5428 | 0.5981 | 0.6458 | 0.6867 | 0.7215 |
| 9 | 0.5672 | 0.6233 | 0.6709 | 0.7110 | 0.7446 |
| 10 | 0.5886 | 0.6453 | 0.6926 | 0.7317 | 0.7641 |
| 15 | 0.6673 | 0.7239 | 0.7680 | 0.8023 | 0.8293 |
| 20 | 0.7181 | 0.7729 | 0.8133 | 0.8433 | 0.8662 |
| $a$ | 0.7 | | | | |
| $n$ \ $\gamma$ | −1.0 | −0.5 | 0.0 | 0.5 | 1.0 |
| 1 | 0.1150 | 0.1325 | 0.1500 | 0.1675 | 0.1850 |
| 2 | 0.1988 | 0.2287 | 0.2584 | 0.2876 | 0.3165 |
| 3 | 0.2637 | 0.3028 | 0.3409 | 0.3778 | 0.4133 |
| 4 | 0.3160 | 0.3620 | 0.4060 | 0.4477 | 0.4869 |
| 5 | 0.3595 | 0.4108 | 0.4590 | 0.5036 | 0.5446 |
| 6 | 0.3963 | 0.4518 | 0.5029 | 0.5492 | 0.5909 |
| 7 | 0.4281 | 0.4869 | 0.5399 | 0.5872 | 0.6288 |
| 8 | 0.4559 | 0.5173 | 0.5717 | 0.6192 | 0.6603 |
| 9 | 0.4805 | 0.5439 | 0.5992 | 0.6466 | 0.6870 |
| 10 | 0.5025 | 0.5675 | 0.6233 | 0.6704 | 0.7098 |
| 15 | 0.5852 | 0.6545 | 0.7097 | 0.7532 | 0.7876 |
| 20 | 0.6407 | 0.7108 | 0.7634 | 0.8028 | 0.8326 |

Table 1: Bivariate uniform version of Problem (1°). Values of $V_n$ in (6) for $a = 0.6$ and $a = 0.7$

say, the first term of (8) becomes

$$\sqrt{1-\rho^2}\int_a^\infty \phi(y)\Psi\left(\frac{s-\rho y}{\sqrt{1-\rho^2}}\right)dy. \tag{10}$$

The function $\Psi(s)$ is convex, decreasing, and approaches when $s \to \pm\infty$ to the polygon $(-s)\vee 0$ from upside. Also note that $\Psi(0) = (2\pi)^{-1/2}$ and $\Psi(-s) = s + \Psi(s)$ for all $s$. By (9), it is clear that the integral (10) converges. For more details about the function $\Psi(s)$, see DeGroot [1; Section 13.4~13.6], and Sakaguchi [4].

From (2) and (3), $\{V_n\}$ is an increasing sequence starting from

$$V_1 = E\{XI(Y \geq a)\} = \int_{-\infty}^{\infty} x\phi(x)\,\bar{\Phi}\left(\frac{a-\rho x}{\sqrt{1-\rho^2}}\right)dx = \rho\phi(a).$$

By (2) and (8)~(10) we get

**Theorem 2** *The optimal stopping rule for the problem (1°) for bivariate normal distribution (7) is to:*

$$\begin{cases} \textit{Stop}, & \textit{if } X \geq V_{n-1} \textit{ and } Y \geq a \\ \textit{Continue}, & \textit{if otherwise}, \end{cases}$$

*where $\{V_n\}$ is determined by the recursion*

$$V_n = \sqrt{1-\rho^2}\int_a^\infty \phi(y)\Psi\left(\frac{V_{n-1}-\rho y}{\sqrt{1-\rho^2}}\right)dy + (-V_{n-1})^+\,\Phi(a) + V_{n-1} \tag{11}$$

*The optimal expected reward for the n-object problem is $V_n$.*

If $\rho = 0$, the problem is independent bivariate version, and (11) becomes

$$V_n = \bar{\Phi}(a)\{\Psi(V_{n-1}) + V_{n-1}\} + \Phi(a)V_{n-1}^+ \tag{12}$$

For $\rho \neq 0$, the infinite integral in (10) seems to have no simpler expression. Table 2 shows the values of $V_n$, given by (12) for $n = 1(1)10, 15, 20$, when $\rho = 0$ and $a = 0.52, 0.60, 0.70, 0.84, 1.00, 1.28, 1.65$.

This table shows, for example, that for bivariate normal distribution with mean vector (0,0) and covariance matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, you have $\max_\tau E[X_\tau I(Y_\tau \geq 0.84)] = 0.517$, when $n = 10$. However, if you disregard the requirement that $Y_\tau \geq 0.84$, and consider the corresponding univariate version $EX_\tau \to \max_\tau$, then you can get $V_{10} = 1.276$, since you have, in this case, the recursion $V_n = \Psi(V_{n-1}) + V_{n-1}$ $(n = 0, 1, 2, \cdots; V_1 \equiv 0)$.

| $a$ | 0.52 | 0.60 | 0.70 | 0.84 | 1.00 | 1.28 | 1.65 |
|---|---|---|---|---|---|---|---|
| $n \setminus \bar{\Phi}(a)$ | 0.6985 | 0.7257 | 0.7580 | 0.7995 | 0.8413 | 0.8997 | 0.9505 |
| 1 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 2 | 0.120 | 0.109 | 0.097 | 0.080 | 0.063 | 0.040 | 0.020 |
| 3 | 0.223 | 0.205 | 0.182 | 0.152 | 0.122 | 0.078 | 0.039 |
| 4 | 0.313 | 0.288 | 0.258 | 0.218 | 0.176 | 0.114 | 0.058 |
| 5 | 0.392 | 0.363 | 0.327 | 0.278 | 0.226 | 0.149 | 0.076 |
| 6 | 0.462 | 0.430 | 0.389 | 0.333 | 0.273 | 0.182 | 0.094 |
| 7 | 0.526 | 0.490 | 0.446 | 0.384 | 0.317 | 0.213 | 0.112 |
| 8 | 0.583 | 0.545 | 0.498 | 0.431 | 0.359 | 0.244 | 0.129 |
| 9 | 0.635 | 0.596 | 0.546 | 0.476 | 0.397 | 0.273 | 0.145 |
| 10 | 0.683 | 0.642 | 0.590 | 0.517 | 0.434 | 0.300 | 0.162 |
| 15 | 0.877 | 0.831 | 0.773 | 0.689 | 0.591 | 0.425 | 0.238 |
| 20 | 1.019 | 0.972 | 0.911 | 0.822 | 0.716 | 0.529 | 0.308 |

Table 2: Bivariate normal version of Problem (1°). Values of $V_n$ in (12) for various $a$

# 3 Selecting best $X$ with the condition that $Y \geq a$

In this section we want to select the bivariate continuous r.v. $(X_\tau, Y_\tau)$ that satisfies

$$\Pr\left\{\max_{1\leq i\leq n} X_i = X_\tau \text{ and } Y_\tau \geq a\right\} \to \max_\tau,$$

where $\tau$ is the stopping time, for a given fixed $0 < a < 1$. If $(X_i, Y_i)$ satisfies $X_i = \max_{1\leq t\leq i} X_t = x$ and $Y_i \geq a$, then on-and-after the $(i+1)$-st r.v., any $(X_j, Y_j)$ with $X_j < x$, are rejected even if $Y_j \geq a$. If we fail to stop (=select) until the $(n-1)$-st r.v., then we must stop at $(X_n, Y_n)$, with reward $I\left(\max_{1\leq i\leq n} X_i = X_n \;\;\&\;\; Y_n \geq a\right)$.

## 3.1 Bivariate uniform distribution with pdf (4)

Define state $(x|n, i)$ to mean that no stop has yet been made, and we face the $i$-th r.v. with $X_i = \max(X_1, X_2, \cdots, X_i) = x$ and $Y_i \geq a$. Denote by $v_{n,i}(x)$ the expected reward obtained by employing the optimal rule for the $n$-object problem at state $(x|n, i)$. Since the common marginal distribution of $X_i$'s is uniform on $[0, 1]$, we easily have the Optimality Equation

$$(13) \qquad v_{n,i}(x) = \max\left[x^{n-i},\ \sum_{j=i+1}^{n} x^{j-i-1}\int_x^1 q(z)v_{n,j}(z)dz\right],$$

$$(i = 1, 2, \cdots, n; 0 \leq x \leq 1; v_{n,n}(x) \equiv 1)$$

where $q(z) \equiv \int_a^1 p(z,y)dy = \bar{a}\ (1-\gamma a\,(1-2z))$ is the same one used in the previous Subsection 2.1.

The one-step stopping region ([3]; pp.137-139) corresponding to this optimality equation is

$$B \equiv \left\{ (x|n,i) \left| x^{n-i} \geq \sum_{j=i+1}^{n} x^{j-i-1} \int_x^1 q(z) z^{n-j} dz \right. \right\},$$

which becomes, after simplification,

$$B = \left\{ (x|n,i) \left| 1/\ \bar{a} \geq (1-\gamma a) \sum_{m=1}^{n-i} \frac{x^{-m}-1}{m} + 2\gamma a x \sum_{m=2}^{n-i+1} \frac{x^{-m}-1}{m} \right. \right\},$$

or equivalently,

$$(14)\quad B = \left\{ (x|j) \left| 1/\ \bar{a} \geq (1-\gamma a) \sum_{m=1}^{j} \frac{x^{-m}-1}{m} + 2\gamma a x \sum_{m=2}^{j+1} \frac{x^{-m}-1}{m} \right. \right\},$$

by using $j = n - i$. Note that $B$ does not involve $n$.

**Lemma 3.1** *For $|\gamma| \leq 1$, the region $B$ given by (14) is "closed" i.e. if once a state enters $B$, the state never leaves $B$ as the process goes on.*

Proof. Let

$$K_j(x) \equiv (1-\gamma a) \sum_{m=1}^{j} m^{-1} \left(x^{-m}-1\right) + 2\gamma a x \sum_{m=2}^{j+1} m^{-1} \left(x^{-m}-1\right).$$

We have to prove $K_j(x) \geq K_{j-1}(z)$, for any $0 \leq x \leq z \leq 1$, and $|\gamma| \leq 1$. For $0 \leq \gamma \leq 1$, we have $K_j(x) \geq K_j(z)$, since $\sum_{m=1}^{j} m^{-1}(x^{-m}-1)$ and $x\sum_{m=2}^{j+1} m^{-1}(x^{-m}-1)$ are both decreasing in $0 \leq x \leq 1$. For $-1 \leq \gamma \leq 0$, we have by direct differentiation,

$$K_j^{'}(x) = -(1+\gamma a) \sum_{m=2}^{j+1} x^{-m} + 2\gamma a \sum_{m=2}^{j+1} m^{-1} \left(x^{-m}-1\right) < 0,$$

*i.e.* $K_j(x)$ is decreasing in $0 \leq x \leq 1$. Hence we have

$$(*)\qquad K_j(x) \leq K_j(z) \text{ for } 0 \leq x \leq z \leq 1, \text{ and } |\gamma| \leq 1.$$

We also have

$$K_j(z) - K_{j-1}(z) = (1-\gamma a) \sum_{m=1}^{j} m^{-1} \left(z^{-m}-1\right) + 2\gamma a z \sum_{m=2}^{j+1} m^{-1} \left(z^{-m}-1\right)$$

$$-(1-\gamma a)\sum_{m=1}^{j-1} m^{-1}\left(z^{-m}-1\right)-2\gamma az\sum_{m=2}^{j} m^{-1}\left(z^{-m}-1\right)$$

$$=(1-\gamma a)\left(z^{-j}-1\right)/j+2\gamma a\left(z^{-j}-z\right)/(j+1)$$

$$=\frac{1}{j(j+1)}\left[\{j+1+(j-1)\gamma a\}z^{-j}-2j\gamma az-(j+1)(1-\gamma a)\right]$$

The function of $z$ in $[\cdots]$ above is convex and decreasing in $0 \le z \le 1$, for all $|\gamma| \le 1$, with the values $+\infty$ at $z=0$, and 0 at $z=1$. Hence we have $K_j(z) \ge K_{j-1}(z)$. Combining this result with (*), completes the proof of the lemma.□

Let $d_j$ $(j=1,2,3,\cdots)$ be a unique root in $[0,1]$ of the equation

$$(15)\ \bar{a}\left\{(1-\gamma a)\sum_{m=1}^{j} m^{-1}\left(x^{-m}-1\right)+2\gamma ax\sum_{m=2}^{j+1} m^{-1}\left(x^{-m}-1\right)\right\}=1.$$

From Lemma 3.1, the l.h.s. of the equation (15) is smaller than unity, if and only if $x > d_j$. Note that the values of $d_j$ involve $0 < a < 1$ and $-1 \le \gamma \le 1$. If $a \downarrow 0$, (15) reduces to the equation $\sum_{m=1}^{j} m^{-1}\left(x^{-m}-1\right)=1$, which is well-known in the full-information secretary problem (see [2], [4] and [6]). If $\gamma=0$ (i.e. bivariate independent uniform distribution), (15) reduces to $\bar{a}\sum_{m=1}^{j} m^{-1}\left(x^{-m}-1\right)=1$.

It is well known that if one-step stopping region is realizable and "closed", it becomes the optimal stopping region. From Lemma 3.1 we thus get

**Theorem 3** *The optimal stopping rule for the optimality equation* (13), *for* $|\gamma| \le 1$, *is: Stop at the earliest* $(X_i, Y_i)$ *that satisfies* $Y_i \ge a$ *and* $X_i = \max_{1\le t\le i} X_t > d_{n-i}$, *where each* $d_j$ *is given by a unique root in* $[0,1]$ *of the equation* (15).

Table 3 shows the values of $d_j$, for $j = 1(1)10, 15, 20$, $\gamma = -1.0(0.5)1.0$, and $a = 0.6, 0.7$ and gives the following example:

Example 1. For $n=5$, $\gamma=0.5$ and $a=0.6$, the optimal stopping rule is:

| $a$ | 0.6 | | | | |
|---|---|---|---|---|---|
| $n$ $\gamma$ | -1.0 | -0.5 | 0.0 | 0.5 | 1.0 |
| 1 | 0.2533 | 0.2689 | 0.2857 | 0.3038 | 0.3232 |
| 2 | 0.4313 | 0.4661 | 0.5000 | 0.5326 | 0.5636 |
| 3 | 0.5383 | 0.5805 | 0.6185 | 0.6523 | 0.6822 |
| 4 | 0.6102 | 0.6546 | 0.6921 | 0.7239 | 0.7508 |
| 5 | 0.6622 | 0.7064 | 0.7421 | 0.7713 | 0.7953 |
| 6 | 0.7016 | 0.7447 | 0.7782 | 0.8049 | 0.8264 |
| 7 | 0.7326 | 0.7741 | 0.8055 | 0.8299 | 0.8493 |
| 8 | 0.7576 | 0.7975 | 0.8268 | 0.8493 | 0.8669 |
| 9 | 0.7783 | 0.8164 | 0.8439 | 0.8647 | 0.8808 |
| 10 | 0.7957 | 0.8321 | 0.8580 | 0.8772 | 0.8921 |
| 15 | 0.8531 | 0.8824 | 0.9021 | 0.9162 | 0.9268 |
| 20 | 0.8852 | 0.9095 | 0.9253 | 0.9364 | 0.9446 |
| $a$ | 0.7 | | | | |
| $n$ $\gamma$ | -1.0 | -0.5 | 0.0 | 0.5 | 1.0 |
| 1 | 0.2045 | 0.2170 | 0.2308 | 0.2457 | 0.2620 |
| 2 | 0.3724 | 0.4071 | 0.4413 | 0.4746 | 0.5068 |
| 3 | 0.4793 | 0.5249 | 0.5660 | 0.6029 | 0.6359 |
| 4 | 0.5535 | 0.6038 | 0.6461 | 0.6820 | 0.7126 |
| 5 | 0.6084 | 0.6602 | 0.7015 | 0.7351 | 0.7629 |
| 6 | 0.6508 | 0.7025 | 0.7420 | 0.7732 | 0.7983 |
| 7 | 0.6846 | 0.7355 | 0.7729 | 0.8018 | 0.8246 |
| 8 | 0.7123 | 0.7619 | 0.7973 | 0.8240 | 0.8449 |
| 9 | 0.7354 | 0.7834 | 0.8169 | 0.8417 | 0.8609 |
| 10 | 0.7550 | 0.8014 | 0.8330 | 0.8562 | 0.8740 |
| 15 | 0.8209 | 0.8597 | 0.8842 | 0.9015 | 0.9143 |
| 20 | 0.8586 | 0.8915 | 0.9114 | 0.9251 | 0.9351 |

Table 3: Bivariate uniform version of Problem (2°). Values of $d_j$ in (15) for $a = 0.6$ and $a = 0.7$

$$\text{If}\begin{cases} Y_1 \geq 0.6 \ \& \ X_1 \geq 0.7239 \\ \text{otherwise} \end{cases}, \text{ then}\begin{cases} \text{Stop.} \\ \text{observe } (X_2, Y_2) \end{cases}$$

$$\rightarrow\text{If}\begin{cases} Y_2 \geq 0.6 \ \ \& \ \ X_2 \geq X_1 \vee 0.6523 \\ \text{otherwise} \end{cases}, \text{ then}\begin{cases} \text{Stop.} \\ \text{observe } (X_3, Y_3) \end{cases}$$

$$\rightarrow\text{If}\begin{cases} Y_3 \geq 0.6 \ \ \& \ \ X_3 \geq X_1 \vee X_2 \vee 0.5326 \\ \text{otherwise} \end{cases}, \text{ then}\begin{cases} \text{Stop.} \\ \text{observe } (X_4, Y_4) \end{cases}$$

$$\rightarrow\text{If}\begin{cases} Y_4 \geq 0.6 \ \& \ X_4 \geq X_1 \vee X_2 \vee X_3 \vee 0.3038 \\ \text{otherwise} \end{cases},$$

$$\text{then}\begin{cases} \text{Stop.} \\ \text{observe } (X_5, Y_5) \ \ \& \ \ \text{stop.} \end{cases}$$

The reward at the earliest [2nd earliest,$\cdots$, latest] stop is $I(Y_1 \geq 0.6 \ \& \ X_1 \geq \max_{2\leq t\leq 5} X_t)$ $[I(Y_2 \geq 0.6 \ \ \& \ \ X_2 \geq \max_{3\leq t\leq 5} X_t), \ \cdots,$ $I(Y_5 \geq 0.6 \ \& \ \ X_5 = \max_{1\leq t\leq 5} X_t)]$.

## 3.2 Bivariate normal distribution with pdf (7)

Definition of state $(x|n,i)$ and value $v_{n,i}(x)$ value are the same as in the previous Subsection 3.1. The Optimality Equation is now

$$\text{(16)} \quad v_{n,i}(x) = \max\Bigg[(\Phi(x))^{n-i}, \sum_{j=i+1}^{n} (\Phi(x))^{j-i-1} \int_x^{\infty} \phi(z)\, \bar{\Phi}\left(\frac{a-\rho z}{\sqrt{1-\rho^2}}\right) v_{n,j}(z)dz\Bigg]$$

$$(i = 1, 2, \cdots, n; -\infty \leq x \leq \infty; v_{n,n}(x) \equiv 1)$$

since$\int_a^{\infty} p(z,y)dy = \phi(z)\, \bar{\Phi}\left(\frac{a-\rho z}{\sqrt{1-\rho^2}}\right)$, by (7).

The one-step stopping region corresponding to this optimality equation is

$$B \equiv \Bigg\{(x|n,i) \,\Bigg|\, (\Phi(x))^{n-i} \geq \sum_{j=i+1}^{n} (\Phi(x))^{j-i-1} \times \int_x^{\infty} \phi(z)\, \bar{\Phi}\left(\frac{a-\rho z}{\sqrt{1-\rho^2}}\right) (\Phi(z))^{n-j}\, dz\Bigg\},$$

which becomes

$$B = \left\{ (x|j) \middle| 1 \geq \sum_{m=1}^{j} (\Phi(x))^{-m} \int_x^{\infty} \bar{\Phi} \left( \frac{a - \rho z}{\sqrt{1-\rho^2}} \right) \phi(z) \left(\Phi(z)\right)^{m-1} dz \right\}, \tag{17}$$

where we have again set $j = n - i$, as was done in Subsection 3.1.

From (17) we obtain, for any $|\rho| \leq 1$,

$$B = \left\{ (x|j) \middle| 1 \geq \sum_{m=1}^{j} m^{-1} \left[ (\Phi(x))^{-m} - 1 \right] \right\}, \; if \; a \downarrow -\infty, \tag{18}$$

which is the optimal stopping region for the univariate normal distribution. This is an evident result, since if $X \sim \mathbf{N}(0,1)$, then $\Phi(X) \sim \mathbf{U}_{[0,1]}$.

**Theorem 4** *The optimal stopping rule for the Optimality Equation* (??) *for bivariate normal distribution* (7) *is to: Stop at the earliest* $(X_i, Y_i)$ *that satisfies* $Y_i \geq a$ *and* $X_i = \max_{1 \leq t \leq i} X_t > f_{n-i}$, *where each* $f_j$ *is given by a unique root in* $(-\infty, \infty)$ *of the equation*

$$\sum_{m=1}^{j} (\Phi(x))^{-m} \int_x^{\infty} \bar{\Phi} \left( \frac{a - \rho z}{\sqrt{1-\rho^2}} \right) \phi(z) \left(\Phi(z)\right)^{m-1} dz = 1. \tag{19}$$

Proof. Let the l.h.s. of (19) be $G_j(x)$. Then $G_j(x)$ is decreasing in $(-\infty, \infty)$, with $\lim_{x \to -\infty} G_j(x) = +\infty$ and $\lim_{x \to \infty} G_j(x) = 0$. Hence $G_j(x) = 1$ has a unique root $f_j$ in $(-\infty, \infty)$ and $G_j(x) \leq 1$, if and only if $x \geq f_j$. It is easy to see that the region $B$ given by (17) is "closed".□

The values of $\{f_j\}$, when $\rho = 0$, are given for $j = 1(1)10, 15, 20$ and $a = 0.52, 0.60, 0.70, 0.84, 1.00, 1.28, 1.65$ in Table 4. That is, $f_j$, here, is the unique root of the equation

$$\sum_{m=1}^{j} m^{-1} \left[ (\Phi(x))^{-m} - 1 \right] = \left( \bar{\Phi}\,(a) \right)^{-1} \tag{20}$$

Example 2. For $n = 5$, $\rho = 0$ and $a = 0.84$, the optimal stopping rule is:

| $a$ | 0.52 | 0.60 | 0.70 | 0.84 | 1.00 | 1.28 | 1.65 |
|---|---|---|---|---|---|---|---|
| $j \setminus \bar{\Phi}(a)$ | 0.6985 | 0.7257 | 0.7580 | 0.7995 | 0.8413 | 0.8997 | 0.9505 |
| 1 | -0.733 | -0.788 | -0.860 | -0.966 | -1.094 | -1.334 | -1.673 |
| 2 | -0.145 | -0.193 | -0.254 | -0.345 | -0.453 | -0.653 | -0.929 |
| 3 | 0.168 | 0.125 | 0.069 | -0.014 | -0.112 | -0.292 | -0.538 |
| 4 | 0.377 | 0.336 | 0.283 | 0.206 | 0.115 | -0.053 | -0.280 |
| 5 | 0.531 | 0.492 | 0.442 | 0.369 | 0.282 | 0.123 | -0.090 |
| 6 | 0.652 | 0.614 | 0.566 | 0.496 | 0.412 | 0.261 | 0.058 |
| 7 | 0.751 | 0.714 | 0.668 | 0.600 | 0.519 | 0.374 | 0.178 |
| 8 | 0.834 | 0.799 | 0.753 | 0.688 | 0.609 | 0.468 | 0.279 |
| 9 | 0.905 | 0.871 | 0.827 | 0.763 | 0.687 | 0.549 | 0.366 |
| 10 | 0.968 | 0.935 | 0.891 | 0.829 | 0.754 | 0.620 | 0.442 |
| 15 | 1.198 | 1.167 | 1.127 | 1.070 | 1.001 | 0.879 | 0.717 |
| 20 | 1.351 | 1.322 | 1.284 | 1.230 | 1.165 | 1.050 | 0.898 |

Table 4: Bivariate normal version of Problem (2°). Values of $f_j$ in (20) for various $a$

$$\text{If} \left\{ \begin{array}{l} Y_1 \geq 0.84 \ \& \ X_1 \geq 0.206 \\ \text{otherwise} \end{array} \right\}, \text{ then} \left\{ \begin{array}{l} \text{Stop.} \\ \text{observe } (X_2, Y_2) \end{array} \right.$$

$$\rightarrow \text{If} \left\{ \begin{array}{l} Y_2 \geq 0.84 \ \& \ X_2 \geq X_1 \vee (-0.014) \\ \text{otherwise} \end{array} \right\}, \text{ then} \left\{ \begin{array}{l} \text{Stop.} \\ \text{observe } (X_3, Y_3) \end{array} \right.$$

$$\rightarrow \text{If} \left\{ \begin{array}{l} Y_3 \geq 0.84 \ \& \ X_3 \geq X_1 \vee X_2 \vee (-0.345) \\ \text{otherwise} \end{array} \right\}, \text{ then} \left\{ \begin{array}{l} \text{Stop.} \\ \text{observe } (X_4, Y_4) \end{array} \right.$$

$$\rightarrow \text{If} \left\{ \begin{array}{l} Y_4 \geq 0.84 \ \& \ X_4 \geq X_1 \vee X_2 \vee X_3 \vee (-0.966) \\ \text{otherwise} \end{array} \right\},$$

$$\text{then} \left\{ \begin{array}{l} \text{Stop.} \\ \text{observe } (X_5, Y_5) \ \& \text{ stop.} \end{array} \right.$$

The stop reward at each stage is identical as in Example 1, with $Y_5 \geq 0.6$ replaced by $Y_5 \geq 0.84$.

# 4 Final remarks.

1) The expected reward obtained by employing the optimal strategy, in Theorem 3 as well as in Theorem 4, is $W_n = \max_\tau \Pr\{\max_{1 \leq i \leq n} X_i = X_\tau \textit{ and } Y_\tau \geq a\}$, and the explicit expression of $W_n$ is presently not known.

2) We note that $d_j$ in Theorem 3 and $f_j$ in Eq.(20) following Theorem 4

have interesting limiting expression as $j \to \infty$. Writing $d_j$ in Theorem 3 as $d_j = (1+\delta/j)^{-1}$, $j = 0, 1, 2, \cdots$, substituting it into (15), and letting $j \to \infty$, we obtain

$$\int_0^{\delta} t^{-1}\left(e^t - 1\right) dt = \left\{\bar{a}\,(1+\gamma a)\right\}^{-1}, \tag{21}$$

since we have, as $j \to \infty$, $j \log(1+\delta/j) \to \delta$ and $\sum_{m=1}^{j} m^{-1}\left(d_j^{-m} - 1\right) \to \int_0^1 t^{-1}\left(e^{\delta t} - 1\right) dt$. Equation (21) has a unique root $\delta > 0$, since $|\gamma a| \le 1$. If $\gamma = 0$ and $a = 0.6$, we get $\delta = 1.5763$ by finding the root of $\int_0^{\delta}\left(e^t - 1\right)/t dt = \sum_{i=1}^{\infty} \frac{\delta^i}{i \cdot i!} = 2.5$.

Similarly writing $f_j$ in Eq.(20) as $\Phi(f_j) = (1+\varepsilon/j)^{-1}$, we get, as $j \to \infty$, the equation

$$\int_0^{\varepsilon} t^{-1}\left(e^t - 1\right) dt = \left(\bar{\Phi}\,(a)\right)^{-1} \tag{22}$$

which has a unique root in $\varepsilon > 0$. Hence if $\rho = 0$ and $a = 0.84$ , we get $\varepsilon = 2.3613$ by finding the root of $\int_0^{\varepsilon}\left(e^t - 1\right)/t dt = \sum_{i=1}^{\infty} \frac{\varepsilon^i}{i \cdot i!} = 5$.

3) We note that various similar problems like (1°) and (2°) stated in Section 1 arise around the works done in the present paper, but they are more difficult to derive explicit solutions. We show an example.

Let the objective be to derive the stopping rule $\tau$ that maximizes $\mathrm{E}\left[X_\tau I\left(Y_\tau = \max_{1\le t\le n} Y_t\right)\right]$. Define the state $(x, y|n, i)$ to mean that no stop has yet been made, and we face the $i$-th r.v. with $X_i = x$ and $Y_i = \max(Y_1, Y_2, \cdots, Y_i) = y$. Denote by $v_{n,i}(x, y)$ the expected reward obtained by employing the optimal rule for the $n$-object problem at state $(x, y|n, i)$. Let the common distribution of $(X_i, Y_i)'s$ be bivariate uniform given by (4). Then the Optimality Equation is $v_{n,i}(x, y) = \max\left(x, A_{n,i}(y)\right)$, where

$$A_{n,i}(y) \equiv \sum_{j=i+1}^{n} y^{j-i-1} \int_0^1 dx' \int_y^1 p(x', y') v_{n,j}(x', y') dy'$$

$$\left(i = 1, 2, \cdots, n;\ (x, y) \in [0,1]^2;\ v_{n,n}(x, y) = x\right)$$

To derive the optimal stopping rule we have to solve the above recursion downward.

$$\begin{aligned}
A_{n,n-1}(y) &= \int_0^1 x' dx' \int_y^1 p(x', y') dy' = \bar{y}\left(\frac{1}{2} + \frac{1}{6}\gamma y\right), \\
v_{n,n-1}(x, y) &= x \vee A_{n,n-1}(y) \\
A_{n,n-2}(y) &= \int_0^1 dx' \int_y^1 p(x', y')\left(x' \vee A_{n,n-1}(y')\right) dy' + y A_{n,n-1}(y) \\
v_{n,n-2}(x, y) &= x \vee A_{n,n-2}(y)
\end{aligned}$$

and so on. But the calculation until reaching $v_{n,1}(x,y)$ is a tediusly lengthy job even for a small $n$.

4) However, the present work on the best-choice problem where the sequence of bivariate random variables is concerned is one of the other approaches than that was tried in Sakaguchi [5].

5) Similar problems are investigated in Sakaguchi and Szajowski [6]. In this work, $(X_i, Y_i)$ is a bivariate independent r.v., $Y_i s$ are in the condition of secretary problem, *i.e.*, $\Pr(Y_i = j) = i^{-1}$ $(j = 1, 2, \cdots, i)$. The problem of maximizing $E[X_\tau I(Y_\tau = 1 \;\&\; Y_{\tau+1}, \cdots, Y_n \geq 2)]$, and others, are studied.

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† 3-26-4 Midorigaoka, Toyonaka, Osaka 560-0002 , JAPAN

‡ Department of Management Sciences, Kobe University of Commerce
8-2-1 Gakuen-nishimachi, Nishi-ku, Kobe 651-2197 JAPAN