A Note on Bruss' Stopping Problem with Random Availability

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Abstract

Bruss (1987) has studied a continuous-time generalization of the so-called secretary problem, where options arise according to homogeneous Poisson processes with an unknown intensity of λ. In this note, the solution is extended to the case with random availability, that is, there exists a fixed known probability $p(0 < p \leq 1)$ of availability, and the number of offering chances allowed at most is $m(\geq 1)$. The case when the probability of availability depends on m is also studied.

Keywords: Apartment problem, secretary problem, Pascal process, optimality principle

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1 Introduction

Bruss (1987) has studied the following problem. A decision maker has been allowed a fixed time T in which to find an apartment. Opportunities to inspect apartments occur at the epochs of a homogeneous Poisson process of unknown intensity λ. The decision maker inspects each apartment immediately when the opportunity arises, and he must decide immediately whether to accept or not. At any epoch he is able to rank a given apartment among all those inspected to date, where all permutations of ranks are equally likely and independent of the Poisson process. The objective is to maximize the probability of selecting the best apartment from those (if any) available in the interval $(0, T]$. This is an extension of the problem studied by Cowan and Zabczyk (1976), who assume that the intensity λ of the process is known. Bruss (1987) has shown that if the prior density of the intensity of the Poisson process is an exponential with the rate parameter $a \geq 0$, then the optimal stopping rule is to accept the first relatively best option (if any) after time $s^* = (T + a)/e - a$. Sakaguchi (1989) has studied the full-information problem. These problems may be regarded as the extended problem of the classical secretary problem, whose history is reviewed in the papers of Ferguson (1989) and Samuels (1991).

This note extends Bruss' problem to the problem in which each owner of apartment can accept the offer proposed by apartment's searcher with a fixed known probability $p(0 < p \leq 1, q = 1-p)$, and the decision maker is allowed to make at most $m(\geq 1)$ offers, where m is a predetermined number. When an offer is made, even if an apartment is not available m drops to $m-1$. The case of $p = m = 1$ equals Bruss' problem. Bruss considered also the case of an inhomogeneous Poisson processes, but this case is not considered in this paper.

The secretary problem with random availability of each secretary is sometimes called the problem of uncertain employment. Smith (1975), Tamaki (1991), Sweet (1994) and Ano, Tamaki and Hu (1996) have studied the problem of uncertain employment. Viewing applications to real world problems, this setting of random availability is more attractive.

We show that the optimal stopping rule for the problem with a Poisson arrival at intensity $\lambda > 0$ having a prior exponential distribution with rate parameter $a \geq 0$ and a probability p availability when we can make m more offers is to make an offer to the first relatively best option (if any) after time $s_m^* = (T+a)\exp\{-C^{(m)}(q)\} - a$, where $C^{(m)}(q)$ is constant. For $a = 0$, it is interesting to compare the values $s_1^* = T\exp\{-1\}$, $s_2^* = T\exp\{-(1+q/2)\}$, $s_3^* = T\exp\{-(1+q/2+q^2/3+q^3/8)\}, \ldots$ with the values $s_1^* = n\exp\{-1\}$, $s_2^* = n\exp\{-(1+q/2)\}$, $s_3^* = n\exp\{-(1+q/2+q^2/3+q^3/8)\}, \ldots$ for large n in the no-information secretary problem with probability p of availability, which has been solved by Ano, Tamaki, and Hu(1996). They have studied the case of a fixed sample size n of apartments and shown that the optimal stopping rule is to give an offer the first relatively best option which appears at period s_m^* or after period s_m^*.

In Section 2, we formulate the problem. Section 3 gives the optimal stopping rule for the cases with $m = 1, 2$. Section 4 involves a consideration of the general case with $m \geq 3$. Section 5 considers the case when the probability of availability depends on m.

2 Formulation

Let $\tau_1, \tau_2, \ldots$ denote the arrival times of a Poisson process in chronological order, and let $\{N(t)\}_{t\geq 0}$ be the corresponding counting process. For the unknown intensity λ of the process, we suppose a prior gamma distribution with parameters a and l, i.e.,
$a^l\lambda^{l-1}\{\exp(-a\lambda)/(l-1)!\}I(\lambda > 0)$, where a is a known nonnegative parameter. The corresponding conditional density of λ given $\{\tau_i = s\}$ can be

straightforwardly computed and yields

$$f(\lambda|\tau_i = s) = \lambda^{l+i-1}(s+a)^{l+i}\{\exp(-\lambda(s+a)/(l+i-1)!)\}$$

The posterior distribution of N given $\{\tau_i = s\}$ is found in Bruss (1987) and turns out to be a Pascal distribution with parameters $(i+l)$ and $(s+a)/(T+a)$, i.e.,

$$(1)\quad P(N(T) = n|\tau_i = s) = \binom{n+l-1}{i+l-1}\left(\frac{s+a}{T+a}\right)^{i+l}\left(\frac{T-s}{T+a}\right)^{n-i}.$$

When $l = 1$, the prior gamma density equals an exponential density. Hereafter we focus on the case of $l = 1$, because then we can show that the one-step look-ahead function (defined later) is independent of i.

We define the state of the process as (i, m, s), when we observe that the ith option arriving at time s is the relatively best option, and we can offer more m options thereafter. Let $W_i^{(m)}(s)$ denote the maximum probability of obtaining the best option starting from state (i, m, s). Similarly, let $U_i^{(m)}(s)(V_i^{(m)}(s))$ be the corresponding probability when we make an offer(we don't make an offer) to the current relatively best option and proceed optimally thereafter. Then, by the principle of optimality, we have for $i, m \geq 1$,

$$(2)\qquad W_i^{(m)}(s) = \max\{U_i^{(m)}(s), V_i^{(m)}(s)\} \quad \text{for} \quad s \in (0, T]$$

with boundary conditions $W_i^{(m)}(T) = p$ for $i, m \geq 1$ and $W_i^{(0)}(s) = 0$ for all i and s.

Using (1), we can show $(l = 1)$

$$\begin{aligned} U_i^{(m)}(s) &= p\sum_{n\geq i}(i/n)P(N(T) = n|\tau_i = s) + qV_i^{(m-1)}(s) \\ (3)\qquad &= p\left(\frac{s+a}{T+a}\right) + qV_i^{(m-1)}(s). \end{aligned}$$

Let $p_{(i,s)}^{(k,\mu)}$ denote the one-step transition probability from state (i, s, m) to state $(i+k, s+\mu, m)$. We then have

$$(4)\qquad V_i^{(m)}(s) = \int_0^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,\mu)} W_{i+k}^{(m)}(s+\mu)d\mu,$$

and for $k \geq 1, \mu \in (0, T-s]$,

$$(5)\ p_{(i,s)}^{(k,\mu)} = \int_0^\infty \Big\{\frac{\lambda e^{-\lambda\mu}(\lambda\mu)^{k-1}}{(k-1)!} \times \frac{i}{(i+k-1)(i+k)} \frac{e^{-\lambda(s+a)}\lambda^i(s+a)^{i+1}}{i!}\Big\} d\lambda = \frac{s+a}{(s+a+\mu)^2}\binom{i+k-2}{k-1}\left(\frac{s+a}{s+a+\mu}\right)^i\left(\frac{\mu}{s+a+\mu}\right)^{k-1}.$$

(5) follows from $\int_0^\infty \lambda^{k+i}e^{-\lambda(s+a+\mu)}d\lambda = \Gamma(k+i+1)/(s+a+\mu)^{k+i+1}$.

Let B_m be the one-step look-ahead stopping region, that is, B_m is the set of state (i,s,m) for which making an immediate offer to the current relatively best option and to make an offer. Thus

$$B_m = \{(i,s,m) : U_i^{(m)}(s) \geq \int_0^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,\mu)} U_{i+k}^{(m)}(s+\mu)d\mu\}.$$

Let

$$g_i^{(m)}(s) = U_i^{(m)}(s) - \int_0^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,\mu)} U_{i+k}^{(m)}(s+\mu)d\mu.$$

We call $g_i^{(m)}(s)$ a *one-step look-ahead function*. Then $B_m = \{(i,s,m) : g_i^{(m)}(s) \geq 0\}$ and $g_i^{(m)}(s)$ can be written as follows from (3) and (4):

$$\begin{aligned} g_i^{(m)}(s) &= p\left(\frac{s+a}{T+a}\right) + qV_i^{(m-1)}(s) \\ &\quad - \int_0^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,\mu)} \left\{p\left(\frac{s+\mu+a}{T+a}\right) + qV_{i+k}^{(m-1)}(s+\mu)\right\} d\mu \\ &= p\left\{\left(\frac{s+a}{T+a}\right) - \int_0^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,\mu)} \left(\frac{s+\mu+a}{T+a}\right) d\mu\right\} \\ &\quad + q\int_0^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,\mu)} \{W_{i+k}^{(m-1)}(s+\mu) - V_{i+k}^{(m-1)}(s+\mu)\}d\mu \\ (6) \qquad &= p\left(\frac{s+a}{T+a}\right)\left\{1+\log\left(\frac{s+a}{T+a}\right)\right\} \\ &\quad + q\int_0^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,\mu)} \{W_{i+k}^{(m-1)}(s+\mu) - V_{i+k}^{(m-1)}(s+\mu)\}d\mu, \end{aligned}$$

where we use $\sum_{k\geq 1} p_{(i,s)}^{(k,\mu)} = (s+a)/(s+a+\mu)^2$, because $p_{(i,s)}^{(k,\mu)} = (s+a)/(s+a+\mu)^2 \times \{$ Pascal distribution with parameters $(k, \mu/(s+a+\mu))\}$. It is well-known that if B_m is closed, e.g., $B_m = \{(i,s,m) : \tau_i = s \geq s_m^*\}$ for some specified value s_m^*, then B_m gives the optimal stopping region.

Let $h_i^{(m)}(s) = p^{-1}((T+a)/(s+a))g_i^{(m)}(s)$. Then, $B_m = \{(i,s,m) : h_i^{(m)}(s) \geq 0\}$, so that we again call $h_i^{(m)}(s)$ a one-step look-ahead function.

3 The cases $m = 1, 2$

Theorem 3.1 ($m = 1$) *The optimal stopping rule for the problem with random arrivals on $(0,T]$ following a Poisson process at intensity $\lambda > 0$ having an exponential distribution with rate parameter $a \geq 0$ and availability probability p $(0 < p \leq 1)$ when we can make one more offer thereafter is to make an offer for the first relatively best option after time $s_1^* = (T+a)/e - a$.*

Remark: It is interesting to see that p has no influence on the optimal policy.

Proof. The one-step look-ahead stopping region for $m = 1$, B_1, can be written as $B_1 = \{(i,s,1) : h_i^{(1)}(s) \geq 0\} = \{(i,s,1) : 1 + \log((s+a)/(T+a))) \geq 0\} = \{(i,s,1) : \tau_i = s \geq s_1^*\}$, where $s_1^* = (T+a)/e - a$. Thus B_1 is closed and gives the optimal stopping region.

Theorem 3.2 ($m = 2$, Same conditions as in Theorem 3.1) *The optimal stopping rule is to make an offer for the first relatively best option after time $s_2^* = (T+a)\exp\{-(1+q/2)\} - a$.*

Proof. From Theorem 3.1, we have

$$\begin{aligned} W_{i+k}^{(1)}(s+\mu) &- V_{i+k}^{(1)}(s+\mu) \\ &= \begin{cases} U_{i+k}^{(1)}(s+\mu) - \int_0^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,\mu)} U_{i+k}^{(1)}(s+\mu)d\mu, \\ \qquad\qquad \text{for } s+\mu \geq s_1^* \\ V_{i+k}^{(1)}(s+\mu) - V_{i+k}^{(1)}(s+\mu), \\ \qquad\qquad \text{for } s+\mu < s_1^* \end{cases} \\ &= g_i^{(1)}(s+\mu)I(s+\mu \geq s_1^*) \\ &= p\left(\frac{s+\mu+a}{T+a}\right) h^{(1)}(s+\mu)I(s+\mu \geq s_1^*), \end{aligned} \tag{7}$$

where $I(A)$ is the indicator function of A.Let $h_i^{(m)}(s) = (T+a)/((s+a)p)g_i^{(m)}(s)$ and we write $h_i^{(1)}(s)$ as $h^{(1)}(s)$ because $h^{(1)}(s)$ is independent of i and $h^{(1)}(s) = 1 + \log((s+a)/(T+a))$. From (6) and (7),

$$h_i^{(2)}(s) = p^{-1}\left(\frac{T+a}{s+a}\right) g_i^{(2)}(s)$$

$$\begin{aligned} &= p^{-1}\left(\frac{T+a}{s+a}\right)\left\{p\left(\frac{s+a}{T+a}\right)\left(1+\log\left(\frac{s+a}{T+a}\right)\right)\right\} \\ &\quad +p^{-1}\left(\frac{T+a}{s+a}\right)q\int_0^{T-s}\Bigl(\sum_{k\geq 1} p_{(i,s)}^{(k,\mu)} p\left(\frac{s+\mu+a}{T+a}\right) \\ &\qquad \times h^{(1)}(s+\mu)I(s+\mu\geq s_1^*)\Bigr)d\mu \\ &= 1+\log\left(\frac{s+a}{T+a}\right) \\ &\quad +q\int_{(s_1^*-s)^+}^{T-s}\frac{1}{s+\mu+a}\left(1+\log\left(\frac{s+\mu+a}{T+a}\right)\right)d\mu. \end{aligned}$$

Then, for $0 < s \leq s_1^*$,

$$h_1^{(2)}(s) = \log\left(\frac{s+a}{T+a}\right) + C^{(2)}(q), \tag{8}$$

where the constant $C^{(2)}(q)$ is calculated by changing variable $(s+\mu+a)/(T+a)$ to v as follows.

$$\begin{aligned} C^{(2)}(q) &= 1+q\int_{(T+a)/e-a-s}^{T-s}\frac{1}{s+\mu+a}\left(1+\log\left(\frac{s+\mu+a}{T+a}\right)\right)d\mu \\ &= 1+q\int_{e^{-1}}^{1}\frac{1}{v}(1+\log v)dv = 1+q/2. \qquad (9) \end{aligned}$$

Therefore we have for $s \in (0, s_1^*]$,

$$h_i^{(2)}(s) = 1+\frac{q}{2}+\log\left(\frac{s+a}{T+a}\right) \ (\equiv h^{(2)}(s)),$$

which is nondecreasing in $s \in (0, s_1^*]$. For $s \in [s_1^*, T]$, $h^{(1)}(s)$ is nonnegative, because $h^{(1)}(s)$ in nonnegative in $s \in [s_1^*, T]$, Then we have $B_2 = \{(i,s,2) : h_i^{(2)}(s) \geq 0\} = \{(i,s,2) : \tau_i = s \geq s_2^*\}$, where $s_2^* = (T+a)\exp\{-(1+q/2)\} - a (\geq s_1^*)$. Thus B_2 is closed and gives the optimal stopping region.

4 The case $m \geq 3$

We extend the results of Section 3 to the general case with $m \geq 3$.

Theorem 4.1 ($m \geq 3$. Same conditions as in Theorem 3.1) *The optimal stopping rule is to make an offer for the first relatively best option after time $s_m^* = (T+a)\exp\{-C^{(m)}(q)\} - a$, where $C^{(m)}(q)$ is constant. s_m^* is nonincreasing in m.*

Proof. We carry out an induction on m. It is sufficient to show that B_m is closed and $h_i^{(m+1)}(s) \geq h_i^{(m)}(s)$ for any m. So we assume that (A1)

$h_i^{(m)}(s)$ is independent of i, is nondecreasing in $s \in (0, s^*_{m-1}]$, is nonnegative in $s \in [s^*_{m-1}, T]$, and can be written as

$$h^{(m)}(s) = C^{(m)}(q) + \log\left(\frac{s+a}{T+a}\right), \quad for \quad 0 < s \le s^*_{m-1}, \tag{10}$$

where

$$C^{(m)}(q) \;=\; 1 + q\int_{(s^*_m - s)^+}^{T-s} \frac{1}{s+\mu+a} h^{(m-1)}(s+\mu)d\mu \tag{11}$$

and (A2) $h^{(m+1)}(s) \ge h^{(m)}(s)$ for all $s \in (0, T]$ and $s^*_{m+1} \le s^*_m$.

Note that the hypotheses imply that B_m is closed and can be written as $B_m = \{(i, s, m) : h^{(m)}(s) \ge 0\} = \{(i, s, m) : \tau_i = s \ge s^*_m\}$,where $s^*_m = (T+a)\exp\{-C^{(m)}(q)\} - a \le s^*_{m-1}$.

When $m = 1$, the induction hypotheses are valid from Theorems 3.1 and 3.2. For the rest of the proof, we show that both (A1) and (A2) hold with m replaced by $m+1$.

From the hypotheses, we have

$$\begin{aligned} W_{i+k}^{(m)}(s+\mu) \;&-\; V_{i+k}^{(m)}(s+\mu) \\ &= g_i^{(m)}(s+\mu)I(s+\mu \ge s^*_m) \\ &= p\left(\frac{s+\mu+a}{T+a}\right)h^{(m)}(s+\mu)I(s+\mu \ge s^*_m). \end{aligned}$$

Then, from (6)

$$\begin{aligned} h_i^{(m+1)}(s) &= p^{-1}\left(\frac{T+a}{s+a}\right)g_i^{(m+1)}(s) \\ &= 1 + \log\left(\frac{s+a}{T+a}\right) \\ &\quad + q\left(\frac{T+a}{s+a}\right)\int_{(s^*_m - s)^+}^{T-s}\left\{\frac{s+a}{(s+\mu+a)^2}\left(\frac{s+\mu+a}{T+a}\right)\right. \\ &\qquad\qquad \left.\times h^{(m)}(s+\mu)\right\}d\mu \\ &= 1 + \log\left(\frac{s+a}{T+a}\right) + q\int_{(s^*_m - s)^+}^{T-s}\frac{1}{s+\mu+a}h^{(m)}(s+\mu)d\mu. \end{aligned}$$

Thus, $h^{(m+1)}(s)$ is nondecreasing in $s \in (0, s^*_m]$, and is nonnegative in $s \in [s^*_m, T]$, because $h^{(m)}(s)$ is nonnegative in $s \in [s^*_m, T]$. For $0 < s \le s^*_m$,

$$h^{(m+1)}(s) = \log\left(\frac{s+a}{T+a}\right) + C^{(m+1)}(q), \tag{12}$$

where

$$\begin{aligned} C^{(m+1)}(q) &= 1+q\int_{(T+a)\exp\{-C^{(m)}(q)\}+a-s}^{T-s} \frac{1}{s+\mu+a} h^{(m)}(s+\mu)d\mu \\ &= 1+q\int_{\exp\{-C^{(m)}(q)\}}^{1} \frac{1}{v} h^{(m)}((T+a)v-a)dv. \end{aligned} \tag{13}$$

Therefore (A1) holds with m replaced by $m+1$.

As follows, it can be easily shown than (A2) holds with m replaced by $m+1$. From (10) and (12), we have

$$\begin{aligned} h^{(m+2)}(s) &- h^{(m+1)}(s) \\ &= q\int_{(s^*_{m+1}-s)+}^{T-s} \frac{1}{s+\mu+a} h^{(m+1)}(s+\mu)d\mu \\ &\qquad -q\int_{(s^*_m-s)+}^{T-s} \frac{1}{s+\mu+a} h^{(m)}(s+\mu)d\mu \\ &\geq q\int_{(s^*_m-s)+}^{T-s} \frac{1}{s+\mu+a} \{h^{(m+1)}(s+\mu)-h^{(m)}(s+\mu)\}d\mu \\ &\geq 0. \end{aligned}$$

The first inequality follows from the second part of the hypothesis (A2), and the last one follows from the first part of the hypothesis (A2). The proof is complete.

The constant $C^{(3)}(q)$ is easily computed. From (10), we have

$$h^{(2)}(s) = \begin{cases} 1+\frac{q}{2}+\log\left(\frac{s+a}{T+a}\right), & 0 < s \leq s^*_1 \\ 1+(1-q)\log\left(\frac{s+a}{T+a}\right)-\frac{q}{2}\log^2\left(\frac{s+a}{T+a}\right), & s^*_1 \leq s \leq T. \end{cases}$$

We thus get

$$\begin{aligned} C^{(3)}(q) &= 1+q\int_{(s^*_2-s)+}^{T-s} \frac{1}{s+\mu+a} h^{(2)}(s+\mu)d\mu \\ &= 1+q\int_{e^{-C^{(2)}(q)}}^{1} \frac{1}{v} h^{(2)}((T+a)v-a)dv \\ &= 1+q\int_{e^{-(1+\frac{q}{2})}}^{e^{-1}} \frac{1}{v}(1+\frac{q}{2}+\log v)dv \\ &\qquad +q\int_{e^{-1}}^{1} \frac{1}{v}(1+(1-q)\log v-\frac{q}{2}\log^2 v)dv \\ &= 1+\frac{q}{2}+\frac{q^2}{3}+\frac{q^3}{8}. \end{aligned}$$

Then $s_3^* = (T+a)\exp\{-(1+q/2+q^2/3+q^3/8)\} - a$.

For $a = 0$, it is of interest to compare the values $s_1^* = T\exp\{-1\}$, $s_2^* = T\exp\{-(1+q/2)\}$, $s_3^* = T\exp\{-(1+q/2+q^2/3+q^3/8)\}$,$\cdots$, with the values for large n, $s_1^* = n\exp\{-1\}$, $s_2^* = n\exp\{-(1+q/2)\}$, $s_3^* = n\exp\{-(1+q/1+q^2/3+q^3/8)\}$,$\cdots$, of the no-information problem with random availability, which has been solved by Ano, Tamaki, and Hu (1996).

5 Availability probability depends on m.

We assume that $p_m q_{m+1}/p_{m+1} \geq p_{m-1}q_m/p_m$, ($q_m = 1 - p_m$) for $m = 2, 3, \ldots$. Under this assumption, we can see that the one-step look-ahead stopping rule for this problem is optimal. By the same method developed in Sections 2,3,and 4, we have the following one-step look-ahead function,

$$\begin{aligned} g_i^{(m)}(s) &= p_m\left(\frac{s+a}{T+a}\right)\left\{1+\log\left(\frac{s+a}{T+a}\right)\right\} \\ &\quad + q_m\int_0^{T-s}\sum_{k\geq 1} p_{(i,s)}^{(k,\mu)}\{W_{i+k}^{(m-1)}(s+\mu) - V_{i+k}^{(m-1)}(s+\mu)\}d\mu, \end{aligned}$$

Let $h_i^{(m)}(s) = p_m^{-1}((T+a)/(s+a))g_i^{(m)}(s)$, then for $m = 1$, $h^{(1)}(s) = 1 + \log((s+a)/(T+a))$, which is independent of i, and is nondecreasing in s. Therefore the one-step look-ahead stopping region for $m = 1$, B_1, is written as $B_1 = \{s : s \geq s_1^* = (T+a)/e - a\}$, is closed and gives the optimal stopping region for $m = 1$, where s_1^* is a unique root of the equation $h^{(1)}(s) = 0$.

For $m = 2$, we have

$$(14)\ h^{(2)}(s) = 1 + \log\left(\frac{s+a}{T+a}\right) + \frac{p_1q_2}{p_2}\int_{(s_1^*-s)^+}^{T-s}\frac{1}{s+\mu+a}h^{(1)}(s+\mu)d\mu.$$

For $s \in (0, s_1^*]$, $h^{(2)}(s) = 1 + \log((s+a)/(T+a)) + (p_1q_2)/(2p_2)$, which is increasing in $s \in (0, s_1^*]$. For $s \in [s_1^*, T]$, $h^{(2)}(s)$ is nonnegative, because $h^{(1)}(s)$ is nonnegative for $s \in [s_1^*, T]$. Therefore B_2 can be written as $B_2 = \{s : s \geq s_2^* = (T+a)\exp\{-(1+(p_1q_2)/(2p_2))\} - a\}$, is closed and gives the optimal stopping region for $m = 2$.

For $m \geq 3$, we have the following theorem. It is essentially the same approach employed in Section 4 to prove it, so we omit the proof.

Theorem 5.1 *Suppose that* $p_mq_{m+1}/p_{m+1} \geq p_{m-1}q_m/p_m$ *for* $m = 2, 3, \ldots$. *The optimal stopping rule for the problem with random arrivals on* $(0, T]$ *following a Poisson process at intensity* $\lambda > 0$ *having an exponential distribution with rate parameter* $a \geq 0$ *and availability probability* p_m $(0 < p_m \leq 1)$ *when we can make* m *more offers thereafter is to make an offer for the first* *relatively best option after time* $s_m^* = (T+a)\exp\{-C^{(m)}(p_1, \cdots, p_m)\} - a$, *where* $C^{(m)}(p_1, \cdots, p_m)$ *is constant.* s_m^* *is nonincreasing in* m.

The constant $C^{(m)}(p_1, \cdots, p_m)$ is given by

$$C^{(m)}(p_1. \cdots, p_m) = 1 + \frac{p_{m-1}q_m}{p_m} \int_{(s_m^*+a)/(T+a)}^{1} \frac{1}{v} h^{(m-1)}((T+a)v - a)dv,$$

where the one-step look-ahead function, $h^{(m)}(s)$, for this problem can be written as

$$\begin{aligned} h^{(m)}(s) &= 1 + \log\left(\frac{s+a}{T+a}\right) \\ &\quad + \frac{p_{m-1}q_m}{p_m} \int_{(s_{m-1}^*-s)^+}^{T-s} \frac{1}{s+\mu+a} h^{(m-1)}(s+\mu+a)d\mu. \end{aligned}$$

Monotonicity of s_m^* can be shown using the same induction on m as the proof of Theorem 4.1 and the assumption on p_m as follows.

$$\begin{aligned} h^{(m+1)}(s) &- h^{(m)}(s) \\ &= \frac{p_m q_{m+1}}{p_{m+1}} \int_{(s_m^*-s)^+}^{T-s} \frac{1}{s+\mu+a} h^{(m+1)}(s+\mu)d\mu \\ &\quad - \frac{p_{m-1}q_m}{p_m} \int_{(s_{m-1}^*-s)^+}^{T-s} \frac{1}{s+\mu+a} h^{(m)}(s+\mu)d\mu \\ &\geq \left(\frac{p_m q_{m+1}}{p_{m+1}} - \frac{p_{m-1}q_m}{p_m}\right) \int_{(s_{m-1}^*-s)^+}^{T-s} \frac{1}{s+\mu+a} h^{(m)}(s+\mu)d\mu \\ &\geq 0. \end{aligned}$$

Using $h^{(2)}(s) = \log((s+a)/(T+a)) + 1 + (p_1q_2)/(2p_2)$ for $s \in (0, s_1^*]$ and $h^{(2)}(s) = \log((s+a)/(T+a)) + 1 - (p_1q_2)/(2p_2)\{\log((s+a)/(T+a)) + (1/2)$ $\log^2((s+a)/(T+a)\}$ for $s \in [s_1^*, T]$, we have

$$\begin{aligned} C^{(3)}(p_1, p_2, p_3) &= 1 + \frac{p_2q_3}{p_3} \int_{e^{-(1+p_1q_2/p_2)}}^{e^{-1}} \frac{1}{v}\left(\log v + 1 + \frac{p_1q_2}{p_2}\right) dv \\ &\quad + \frac{p_2q_3}{p_3} \int_{e^{-1}}^{1} \frac{1}{v}\Big\{1 + \left(1 - \frac{p_1q_2}{p_2}\right)\log\left(\frac{s+a}{T+a}\right) \\ &\quad - \frac{p_1q_2}{2p_2}\log^2 v\Big\}dv \\ &= 1 + \frac{p_2q_3}{p_3}\left(\frac{1}{2} + \frac{p_1q_2}{3p_2} + \frac{p_1^2q_2^2}{8p_2^2}\right), \end{aligned}$$

and then $s_3^* = (T+a)/\exp\{1 + (p_2q_3)/(2p_3) + (p_1p_2q_2q_3)/(3p_2p_3) + (p_1^2p_2q_2^2$ $q_3)/(8p_2^2p_3)\}$ $-a$. When $p_1 = p_2 = p_3, (q_1 = q_2 = q_3 = q)$, the values, s_1^*, s_2^*, s_3^*, coincide with the values, s_1^*, s_2^*, s_3^*, in Section 4.

6 Outlook for further research

The full-information version of our problem, i.e., extension of Sakaguchi (1989) remains to be solved.

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