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On the "Poisson boundaries" of the family of weighted Kolmogorov statistics

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Abstract: Berk and Jones (1979) introduced a goodness of fit test statistic R_n which is the supremum of pointwise likelihood ratio tests for testing $H_0 : F(x) = F_0(x)$ versus $H_1 : F(x) \neq F_0(x)$. They showed that their statistic does not always converge almost surely to a constant under alternatives F, and, in fact that there exists an alternative distribution function F such $R_n \to_d \sup_{t>0} \mathbb{N}(t)/t$ where $\mathbb{N}$ is a standard Poisson process on $[0, \infty)$. We call the particular distribution function F which leads to this limiting Poisson behavior the *Poisson boundary distribution function for* R_n. We investigate Poisson boundaries for weighted Kolmogorov statistics $D_n(\psi)$ for various weight functions ψ and comment briefly on the history of results concerning Bahadur efficiency of these statistics. One result of note is that the logarithmically weighted Kolmogorov statistic of Groeneboom and Shorack (1981) has the same Poisson boundary as the statistic of Berk and Jones (1979).

1. Introduction

Suppose that $X_1, \ldots, X_n$ are i.i.d. F on $\mathbb{R}$ and we want to test the null hypothesis

$$H : F(x) = F_0(x) \qquad \text{for all} \quad x \in \mathbb{R}$$

where F_0 is continuous, versus the alternative hypothesis

$$K : F(x) \neq F_0(x) \qquad \text{for some} \quad x \in \mathbb{R}.$$

As usual, we can reduce to the case when F_0 is the Uniform$(0, 1)$ distribution on $[0, 1]$; i.e. $F_0(x) = x$ for $0 \leq x \leq 1$.

Berk and Jones (1979) introduced the test statistic R_n, which is defined as

$$R_n = \sup_{-\infty < x < \infty} K\big(\mathbb{F}_n(x), F_0(x)\big), \tag{1.1}$$

where

$$K(x, y) = x \log \frac{x}{y} + (1 - x) \log \frac{1 - x}{1 - y}, \tag{1.2}$$

and $\mathbb{F}_n$ is the empirical distribution functions of the X_i's, given by

$$\mathbb{F}_n(x) = \frac{1}{n} \sum_{i=1}^{n} 1_{[X_i \leq x]}. \tag{1.3}$$

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Define

$$K^{+}(x,y) = \begin{cases} K(x,y), & 0 < y < x < 1, \\ 0, & 0 \le x \le y \le 1, \\ \infty, & \text{otherwise} \end{cases}, \tag{1.4}$$

and

$$K^{-}(x,y) = \begin{cases} K(x,y), & 0 < x < y < 1, \\ 0, & 0 \le y \le x \le 1, \\ \infty, & \text{otherwise.} \end{cases}$$

Berk and Jones also studied the one-sided statistics R_n^+ and R_n^- defined by

$$R_n^+ = \sup_x K^+\big(\mathbb{F}_n(x), x\big), \qquad R_n^- = \sup_x K^-\big(\mathbb{F}_n(x), x\big).$$

Berk and Jones (1979) discussed the optimality properties of the statistics R_n^+ and R_n. They showed, in particular, that they have greater Bahadur efficiency than the corresponding Kolmogorov statistics. Berk and Jones (1979) also extended this comparison to weighted Kolmogorov statistics via the results of Abrahamson (1967). In view of the results of Groeneboom and Shorack (1981), these comparisons are trivial for any weight funtion ψ of the form $\psi(x) = [x(1-x)]^{-b}$ for any positive b since Groeneboom and Shorack show that the limiting efficacy of the weighted Kolmogorov statistics with power function weighting is in fact zero for any alternative for which the efficacy makes sense. Moreover, as we show here the efficacies of the weighted Kolmogorov statistics are not well-defined (and the Bahadur efficiency comparison is not meaningful) for fixed alternatives at or beyond certain "Poisson boundaries" which we describe below. Thus it seems to us that the assertion by Owen (1995), at the end of his section 1, that the statistics of Berk and Jones (1979) have "increased efficiency over any weighted Kolmogorov–Smirnov method at any alternative distribution" is an over-interpretation of the results of Berk and Jones (1979).

Wellner and Koltchinskii (2003) present a proof of the limiting null distribution of the Berk-Jones statistic, and Owen (1995) computes exact quantiles under the null distribution for finite n; see also Owen (2001). Using these quantiles, Owen constructed confidence bands for F by inverting the Berk and Jones test, and then calculates the power associated with the Berk-Jones test statistic for fixed alternatives of the form $F(x) = F_0(x)^\alpha$. See Jager and Wellner (2004) for some corrections of the results of Owen (1995).

One of the interesting results for the statistic R_n proved in Berk and Jones (1979) is the following limit behavior under a rather extreme alternative distribution.

Theorem 1 (Berk and Jones (1979)). *Suppose that $X_1, \ldots, X_n$ are i.i.d. with distribution function F given by*

$$F(x) = \frac{1}{1+\log(1/x)}, \qquad 0 < x < 1 \qquad \textit{and} \qquad 0 < b < 1. \tag{1.5}$$

Then

$$R_n^+ \underset{d}{\rightarrow} \sup_{0<t<\infty} \frac{\mathbb{N}(t)}{t} \overset{d}{=} \frac{1}{U},$$

$$R_n \underset{d}{\rightarrow} \sup_{0<t<\infty} \frac{\mathbb{N}(t)}{t} \overset{d}{=} \frac{1}{U}$$

where $\mathbb{N}$ *is a standard Poisson process on* $[0,\infty)$ *and* U *is a Uniform*$[0,1]$ *random variable.*

Because of the Poisson nature of the limiting distribution in Theorem 1, we call the corresponding alternative distribution function F a "Poisson boundary" for the test statistic R_n. The fact that $\sup_{t>0}\mathbb{N}(t)/t \stackrel{d}{=} 1/U$ follows from results of Pyke (1959), page 571, and elementary manipulations, or, alternatively from the classical result of Daniels (1945) that

$$P\left(\sup_{0<t\le 1}\mathbb{G}_n(t)/t \ge x\right) = 1/x \qquad \text{for} \qquad x\ge 1$$

where $\mathbb{G}_n$ is the empirical distribution function of n i.i.d. Uniform$(0,1)$ random variables (see e.g. Shorack and Wellner (1986), page 404) together with the Poisson convergence results of Wellner (1977b).

For alternatives F that are "less extreme" than the F given in Theorem 1, Berk and Jones (1979) give sufficient conditions under which following more usual or "expected" behavior holds:

$$R_n^+ \underset{a.s.}{\to} \sup_x K^+\big(F(x),x\big), \qquad \text{and} \qquad R_n \underset{a.s.}{\to} \sup_x K\big(F(x),x\big).$$

Some questions related to this type of result are discussed further in Section 4.

Our main purpose here is to note that the phenomena of a Poisson boundary is not unique to the Berk–Jones statistic R_n, but that in fact this type of behavior holds for a general class of "weighted" type statistics. Indeed we will show that the Poisson boundary for the weighted Kolmogorov statistics is a much less extreme alternative than the Poisson boundary distribution function F (given in (1.5)) found by Berk and Jones (1979) for their statistic.

2. "Poisson boundaries" for weighted Kolmogorov statistics

Consider the family of weighted Kolmogorov–Smirnov statistics given by

$$D_n(b) \equiv \sup_{0<x<1}\frac{|\mathbb{F}_n(x)-x|}{(x(1-x))^b} \tag{2.6}$$

where $\mathbb{F}_n$ is the empirical distribution function of the X_i's and $0<b<1$. The asymptotic behavior of $D_n(b)$ under the null hypothesis H is well-known: for $0< b<1/2$

$$n^{1/2}D_n(b) \underset{d}{\to} \sup_{0<t<1}\frac{|\mathbb{U}(t)|}{(t(1-t))^b}$$

where $\mathbb{U}$ is a standard Brownian bridge process, while for $1/2<b\le 1$

$$n^{1-b}D_n(b) \underset{d}{\to} \max\left\{\sup_{0<t<\infty}\frac{|\mathbb{N}(t)-t|}{t^b},\ \sup_{0<t<\infty}\frac{|\tilde{\mathbb{N}}(t)-t|}{t^b}\right\}$$

where $\mathbb{N}$ and $\tilde{\mathbb{N}}$ are independent standard Poisson processes. The case $0<b<1/2$ follows from Chibisov (1964) and O'Reilly (1974); see e.g. Shorack and Wellner (1986), pages 461–466, or Csörgő and Horváth (1993), Theorem 3.2, page 217. The case $1/2<b<1$ follows from Mason (1983); see also Csörgő and Horváth (1993), Theorem 1.2, page 265. When $b=1/2$ the limit behavior is due to Jaeschke (1979) and Eicker (1979), which in turn rely on the classical results of Darling and Erdös (1956):

$$b_n n^{1/2} D_n(b) - c_n \underset{d}{\rightarrow} E_v^4$$

where $b_n = (2\log\log n)^{1/2}$, $c_n = 2\log\log n + (1/2)\log\log\log n - (1/2)\log(4\pi)$, and $P(E_v^4 \le x) = \exp(-4e^{-x})$; see e.g. Shorack and Wellner (1986), page 600.

Our goal here is to prove the following theorems concerning particular fixed alternative hypotheses.

Theorem 2. *Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. F where $F(x) = x^b$ for $0 \le x \le 1$. Then*

$$D_n(b) \underset{d}{\rightarrow} \sup_{0<t<\infty} \frac{\mathbb{N}(t)}{t} \overset{d}{=} \frac{1}{U} \tag{2.7}$$

where $U \sim Uniform(0,1)$.

Theorem 2 does not cover the interesting special case $b = 1$. For $b = 1$ we have the following (more special) result.

Theorem 2A. *Suppose that $c > 1$ and that $X_1, X_2, \ldots, X_n$ are i.i.d. F where*

$$F(x) = \begin{cases} 0, & -\infty < x < 0, \\ cx, & 0 \le x \le 1/c, \\ 1, & 1/c \le x < \infty. \end{cases}$$

Then

$$D_n(1) \underset{d}{\rightarrow} \left(c \sup_{0<t<\infty} \frac{\mathbb{N}(t)}{t} - 1 \right) \bigvee c \overset{d}{=} \left(c\frac{1}{U} - 1 \right) \bigvee c \equiv Y_c$$

where $U \sim Uniform(0,1)$ and

$$P(Y_c \le x) = \begin{cases} 0, & x < c, \\ 1 - c/(x+1), & x \ge c. \end{cases} \tag{2.8}$$

Theorems 2 and 2A do not cover the case of (very light) logarithmic weights which are of interest because of their connection to the results of Groeneboom and Shorack 1981. These authors showed that with $\psi = \psi_2$ where $\psi_2(x) \equiv -\log(x(1-x))$, the ψ-weighted Kolmogorov statistics

$$D_n(\psi) \equiv \sup_{0<x<1} |\mathbb{F}_n(x) - F(x)|\psi(x), \qquad D_n^+(\psi) \equiv \sup_{0<x<1} (\mathbb{F}_n(x) - F(x))\psi(x) \tag{2.9}$$

have non-trivial large deviation behavior under the null hypothesis and hence have non-trivial Bahadur slopes as long as

$$D_n(\psi) \rightarrow_{a.s.} d(\psi, F), \qquad D_n^+(\psi) \rightarrow_{a.s.} d^+(\psi, F) \tag{2.10}$$

respectively under the alternative hypothesis F. Thus it is of interest to determine under what conditions (for what F's) (2.10) holds. A step in this direction is to find the Poisson boundary for $D_n(\psi_2)$. As it turns out, $D_n(\psi_2)$ has the same Poisson boundary distribution function as the Berk-Jones statistic R_n.

Theorem 2B. *Let F be the distribution function given by* (1.5). *If $X_1, \ldots, X_n$ are i.i.d. F, then*

$$D_n^+(\psi_2) \underset{d}{\rightarrow} \sup_{0<t<\infty} \frac{\mathbb{N}(t)}{t} \stackrel{d}{=} \frac{1}{U},$$

$$D_n(\psi_2) \underset{d}{\rightarrow} \sup_{0<t<\infty} \frac{\mathbb{N}(t)}{t} \stackrel{d}{=} \frac{1}{U}$$

where $\mathbb{N}$ is a standard Poisson process and $U \sim$ Uniform$(0,1)$.

An alternative test statistic, $\tilde{R}_n$, which we have called the *reversed Berk–Jones statistic* in Jager and Wellner (2004), is defined by

$$\tilde{R}_n = \sup_{X_{(1)} \le x < X_{(n)}} K\big(F_0(x), \mathbb{F}_n(x)\big) \tag{2.11}$$

where $X_{(1)}$ and $X_{(n)}$ are the first and last order statistics, respectively.

The motivation behind this statistic comes from examination of the functions $K(F_0(x), F(x))$ and $K(F(x), F_0(x))$, for an alternative distribution function F. When F is stochastically smaller than F_0, we expect the Berk-Jones test to be more powerful than the reversed Berk-Jones statistic, since $\sup_x K(F(x), F_0(x)) > \sup_x K(F_0(x), F(x))$ in this case. However, in the case where F is stochastically larger than F_0, we have $\sup_x K(F(x), F_0(x)) < \sup_x K(F_0(x), F(x))$, and so we expect the reversed statistic to be more powerful.

We do not yet know if $\tilde{R}_n$ has a "Poisson boundary". The question is: does there exist an alternative distribution function F such that when sampling from F we have

$$\tilde{R}_n \underset{d}{\rightarrow} g(\mathbb{N})$$

for some functional g of a (standard?) Poisson process $\mathbb{N}$?

Before giving the proofs we state two results that will be used repeatedly in the proofs: the weighted Glivenko–Cantelli theorem of Lai (1974) (see also Wellner (1977a) and Shorack and Wellner (1986), page 410), and bounds for the sup of ratios given by Wellner (1978) and Berk and Jones (1979) (see also Shorack and Wellner (1986), Inequality 10.3.2, pages 415 and 416) that will be used several times in the proofs. Let $\mathbb{G}_n(t) = n^{-1}\sum_{i=1}^n 1_{[0,t]}(\xi_i)$ where $\xi_1, \ldots, \xi_n, \ldots$ are i.i.d. Uniform$(0,1)$ random variables, and let I be the identity function on $[0,1]$.

Theorem W-GC (Lai (1974); Wellner (1977a)). *Suppose that ψ is positive on $(0,1)$, decreasing on $(0,1/2]$, and symmetric about $1/2$. Then*

$$\limsup_{n\to\infty} \|(\mathbb{G}_n - I)\psi\| = \begin{cases} 0 & a.s. \\ \infty & a.s. \end{cases} \quad \text{according as} \quad \int_0^1 \psi(t)dt \begin{cases} < \infty \\ = \infty. \end{cases}$$

Theorem (Ratio bounds). (Wellner (1978), Berk and Jones (1979)). *For all $x \ge 1$ and $0 < \epsilon \le 1$*

$$P\left(\sup_{\epsilon \le t \le 1} \frac{\mathbb{G}_n(t)}{t} \ge x\right) \le \begin{cases} \exp(-n\epsilon h(x)) \\ \exp(-nK^+(\epsilon x, \epsilon)) \end{cases} \tag{2.12}$$

and

$$P\left(\sup_{\epsilon \le t \le 1} \frac{t}{\mathbb{G}_n(t)} \ge x\right) \le \begin{cases} \exp(-n\epsilon h(1/x)) \\ \exp(-nK^+(1-\epsilon/x, 1-\epsilon)) \end{cases} \tag{2.13}$$

where $h(x) \equiv x(\log x - 1) + 1$ and where K^+ is as defined in (1.4).

Now we provide proofs for Theorems 2, 2A, and 2B.

Proof of Theorem 2. Let $0 < \alpha < 1$. We write

$$
\begin{aligned}
D_n(b) &= \sup_{0<x<1} \frac{|\mathbb{F}_n(x) - x|}{(x(1-x))^b} \\
&= \sup_{x:F(x)<n^{-\alpha}} \frac{|\mathbb{F}_n(x) - x|}{(x(1-x))^b} \bigvee \sup_{x:F(x)\geq n^{-\alpha}} \frac{|\mathbb{F}_n(x) - x|}{(x(1-x))^b} \\
&= \sup_{x:F(x)<n^{-\alpha}} \frac{|\mathbb{F}_n(x) - x|}{F(x)(1-F(x)^{1/b})^b} \bigvee \sup_{x:F(x)\geq n^{-\alpha}} \frac{|\mathbb{F}_n(x) - x|}{F(x)(1-F(x)^{1/b})^b} \\
&\equiv D_n^{(1)}(b) \bigvee D_n^{(2)}(b).
\end{aligned}
$$

Now

$$
\begin{aligned}
&D_n^{(1)}(b) - \sup_{x:F(x)<n^{-\alpha}\frac{\mathbb{F}_n(x)}{F(x)}} \\
&= \sup_{x:F(x)<n^{-\alpha}} \frac{|\mathbb{F}_n(x) - x|}{F(x)(1-F(x)^{1/b})^b} - \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)}{F(x)} \\
&= \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x) - x}{F(x)(1-F(x)^{1/b})^b} \bigvee \sup_{x:F(x)<n^{-\alpha}} \frac{x - \mathbb{F}_n(x)}{F(x)(1-F(x)^{1/b})^b} \\
&\quad - \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)}{F(x)} \\
&\leq \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)}{F(x)(1-F(x)^{1/b})^b} - \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)}{F(x)} \\
&\quad + \sup_{x:F(x)<n^{-\alpha}} \frac{x}{F(x)(1-F(x)^{1/b})^b} \\
&\leq \left| \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)}{F(x)(1-F(x)^{1/b})^b} - \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)}{F(x)} \right| \\
&\quad + 2 \sup_{x:\, x<n^{-\alpha/b}} \frac{x}{x^b(1-x)^b} \\
&\leq \sup_{x:F(x)<n^{-\alpha}} \left| \frac{\mathbb{F}_n(x)}{F(x)(1-F(x)^{1/b})^b} - \frac{\mathbb{F}_n(x)}{F(x)} \right| + o(1) \\
&\leq \sup_{x:F(x)<n^{-\alpha}} \left| \frac{\mathbb{F}_n(x)}{F(x)} \left(\frac{1}{(1-x)^b} - 1 \right) \right| + o(1) \\
&\leq \sup_{x:F(x)<n^{-\alpha}} \left| \frac{\mathbb{F}_n(x)}{F(x)} \right| \sup_{x:F(x)<n^{-\alpha}} \left| \left(\frac{1}{(1-x)^b} - 1 \right) \right| + o(1) \\
&\leq O_p(1)o(1) + o(1) = o_p(1).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)}{F(x)} - D_n^{(1)}(b) \\
&\quad = \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)}{F(x)} - \sup_{x:F(x)<n^{-\alpha}} \frac{|\mathbb{F}_n(x) - x|}{F(x)(1-F(x)^{1/b})^b} \\
&\quad \leq \sup_{x:F(x)<n^{-\alpha}} \frac{x}{x^b(1-x)^b} = o(1)
\end{aligned}
$$

since

$$\begin{aligned}
&\sup_{x:F(x)<n^{-\alpha}} \frac{|\mathbb{F}_n(x)-x|}{F(x)(1-F(x)^{1/b})^b} \\
&\quad \geq \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)-x}{F(x)(1-F(x)^{1/b})^b} \\
&\quad \geq \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)}{F(x)(1-F(x)^{1/b})^b} - \sup_{x:F(x)<n^{-\alpha}} \frac{x}{x^b(1-x)^b} \\
&\quad \geq \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)}{F(x)} - o(1).
\end{aligned}$$

Concerning $D_n^{(2)}(b)$ we have

$$\begin{aligned}
D_n^{(2)}(b) &= \sup_{x:F(x)\geq n^{-\alpha}} \frac{|\mathbb{F}_n(x)-x|}{F(x)(1-F(x)^{1/b})^b} \\
&\leq \sup_{x:F(x)\geq n^{-\alpha}} \frac{|\mathbb{F}_n(x)-F(x)|}{F(x)(1-F(x)^{1/b})^b} \\
&\qquad + \sup_{x:F(x)\geq n^{-\alpha}} \frac{|F(x)-x|}{F(x)(1-F(x)^{1/b})^b} \\
&\leq \sup_{x:n^{-\alpha}\leq F(x)\leq 1/2} \frac{|\mathbb{F}_n(x)-F(x)|}{F(x)(1-F(x)^{1/b})^b} \\
&\qquad + \sup_{x:1/2\leq F(x)<1} \frac{|\mathbb{F}_n(x)-F(x)|}{F(x)(1-F(x)^{1/b})^b} + 1 \\
&\leq \frac{1}{(1-(1/2)^{1/b})^b} \sup_{x:n^{-\alpha}\leq F(x)\leq 1/2} \frac{|\mathbb{F}_n(x)-F(x)|}{F(x)} \\
&\qquad + 2 \sup_{x:1/2\leq F(x)<1} \frac{|\mathbb{F}_n(x)-F(x)|}{(1-F(x)^{1/b})^b} + 1 \\
&= o(1)+o(1)+1
\end{aligned}$$

almost surely by Lemma 4.3 of Berk and Jones (1979) for the first term, and by the weighted Glivenko–Cantelli Theorem W-GC for the second term since

$$\int_0^1 \frac{1}{(1-x^{1/b})^b}\,dx = \int_0^1 (1-u)^{-b} b u^{b-1}\,du = b\Gamma(1-b)\Gamma(b) < \infty$$

for $b \in (0,1)$. Hence it follows that $\limsup_{n\to\infty} D_n^{(2)}(b) \leq 1$ almost surely. Putting all this together with the fact that

$$\sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)}{F(x)} \to_d \sup_{0<t<\infty} \frac{\mathbb{N}(t)}{t} \stackrel{d}{=} 1/U$$

finishes the proof of Theorem 2. □

Proof of Theorem 2A. Since $\mathbb{F}_n \stackrel{d}{=} \mathbb{G}_n(F)$ where $\mathbb{G}_n$ is the empirical distribution function of i.i.d. Uniform$(0,1)$ random variables $\xi_1,\ldots,\xi_n$, we can write

$$\begin{aligned}
D_n(1) &\stackrel{d}{=} \sup_{0<x<1} \frac{|\mathbb{G}_n(F(x)) - x|}{x(1-x)} \\
&= \sup_{0<x\le 1/c} \frac{|\mathbb{G}_n(cx) - x|}{x(1-x)} \bigvee \sup_{1/c<x\le 1} \frac{|1-x|}{x(1-x)} \\
&= \sup_{0<t\le n} \frac{|n\mathbb{G}_n(t/n) - t/c|}{(t/c)(1-t/(cn))} \bigvee c \\
&\to_d \sup_{0<t<\infty} \frac{|\mathbb{N}(t) - t/c|}{t/c} \bigvee c \\
&= \left(c \sup_{0<t<\infty} \frac{\mathbb{N}(t)}{t} - 1 \right) \bigvee 1 \bigvee c \\
&\stackrel{d}{=} \left(c\frac{1}{U} - 1 \right) \bigvee c \equiv Y_c.
\end{aligned}$$

since $c > 1$ and since the process $\{n\mathbb{G}_n(t/n) : \ 0 < t \le n\}$ converges weakly to the standard Poisson process $\mathbb{N}$ in a topology that makes the weighted supremum functional in the last display continuous; see e.g. Wellner (1977b), Theorem 7, page 1007. Computation of the distribution of Y_c is straightforward. (Note that this distribution has a jump at c of height $1/(1+c)$.) □

Proof of Theorem 2B. Let $0 < \alpha < 1$. We write

$$\begin{aligned}
D_n(\psi_2) &= \sup_{0<x<1} \left|\mathbb{F}_n(x) - x\right|\psi_2(x) \\
&= \sup_{x:F(x)<n^{-\alpha}} \left|\mathbb{F}_n(x) - x\right|\psi_2(x) \bigvee \sup_{x:F(x)\ge n^{-\alpha}} \left|\mathbb{F}_n(x) - x\right|\psi_2(x) \\
&= \sup_{x:F(x)<n^{-\alpha}} \left|\mathbb{F}_n(x) - x\right|\psi_2(x) \bigvee \sup_{x:F(x)\ge n^{-\alpha}} \left|\mathbb{F}_n(x) - x\right|\psi_2(x) \\
&\equiv D_n^{(1)}(\psi_2) \bigvee D_n^{(2)}(\psi_2).
\end{aligned}$$

We first deal with $D_n^{(2)}(\psi_2)$. Note that

$$\begin{aligned}
D_n^{(2)}(\psi_2) &= \sup_{x:F(x)\ge n^{-\alpha}} \left|\mathbb{F}_n(x) - x\right|\psi_2(x) \\
&\le \sup_{x:F(x)\ge n^{-\alpha}} \left|\mathbb{F}_n(x) - F(x)\right|\psi_2(x) \\
&\qquad + \sup_{x:F(x)\ge n^{-\alpha}} \left|F(x) - x\right|\psi_2(x) \\
&\le \sup_{x:n^{-\alpha}\le F(x)\le 1/2} \frac{|\mathbb{F}_n(x) - F(x)|}{F(x)} F(x)\psi_2(x) \\
&\qquad + \sup_{x:1/2\le F(x)<1} \frac{|\mathbb{F}_n(x) - F(x)|}{(1-F(x))^{3/4}} (1-F(x))^{3/4}\psi_2(x) + 1 \\
&\le \sup_{x:n^{-\alpha}\le F(x)\le 1/2} \frac{|\mathbb{F}_n(x) - F(x)|}{F(x)} \\
&\qquad + \sup_{x:1/2\le F(x)<1} \frac{|\mathbb{F}_n(x) - F(x)|}{(1-F(x))^{3/4}} + 1 \\
&= o(1) + o(1) + 1
\end{aligned}$$

almost surely by Lemma 4.3 of Berk and Jones (1979) or Wellner (1978) for the first term, and Theorem W-GC for the second term. Here we also used $\psi_2(x)F(x) \le 1$ for $0 < x \le 1/2$, and $(1 - F(x))^{3/4}\psi_2(x) \le 1$ for $1/2 \le x < 1$.

To handle $D_n^{(1)}(\psi_2)$, note that

$$
\begin{aligned}
D_n^{(1)}(\psi_2) &- \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)}{F(x)} \\
&= \sup_{x:F(x)<n^{-\alpha}} \frac{|\mathbb{F}_n(x) - x|}{F(x)} F(x)\psi_2(x) - \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)}{F(x)} \\
&= \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x) - x}{F(x)} F(x)\psi_2(x) \bigvee \sup_{x:F(x)<n^{-\alpha}} \frac{x - \mathbb{F}_n(x)}{F(x)} F(x)\psi_2(x) \\
&\quad - \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)}{F(x)} \\
&\le \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)}{F(x)} F(x)\psi_2(x) - \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)}{F(x)} + \sup_{x:F(x)<n^{-\alpha}} x\psi_2(x) \\
&\le \left| \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)}{F(x)} F(x)\psi_2(x) - \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)}{F(x)} \right| + o(1) \\
&\le \sup_{x:F(x)<n^{-\alpha}} \left| \frac{\mathbb{F}_n(x)}{F(x)} \left(F(x)\psi_2(x) - 1\right) \right| + o(1) \\
&\le \sup_{x:F(x)<n^{-\alpha}} \left| \frac{\mathbb{F}_n(x)}{F(x)} \right| \sup_{x:F(x)<n^{-\alpha}} \left| F(x)\psi_2(x) - 1 \right| + o(1) \\
&\le O_p(1)o(1) + o(1) = o_p(1).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\sup_{x:F(x)<n^{-\alpha}} & \frac{\mathbb{F}_n(x)}{F(x)} - D_n^{(1)}(\psi_2) \\
&= \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)}{F(x)} - \sup_{x:F(x)<n^{-\alpha}} \frac{|\mathbb{F}_n(x) - x|}{F(x)} F(x)\psi_2(x) \\
&\le \sup_{x:F(x)<n^{-\alpha}} x\psi(x) = o(1)
\end{aligned}
$$

since

$$
\begin{aligned}
\sup_{x:F(x)<n^{-\alpha}} & \frac{|\mathbb{F}_n(x) - x|}{F(x)} F(x)\psi_2(x) \\
&\ge \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x) - x}{F(x)} F(x)\psi_2(x) \\
&\ge \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)}{F(x)} F(x)\psi_2(x) - \sup_{x:F(x)<n^{-\alpha}} x\psi_2(x) \\
&\ge \sup_{x:F(x)<n^{-\alpha}} \frac{\mathbb{F}_n(x)}{F(x)} (1 - o(1)) - o(1).
\end{aligned}
$$

Combining these pieces as in the proof of Theorem 2 completes the proof for $D_n(\psi_2)$. The proof for $D_n^+(\psi_2)$ is similar (and easier). □

3. A consistency result

Theorems 2, 2A, 2B suggest that we might expect classical behavior for the weighted Kolmogorov statistics under fixed alternatives F sufficiently "inside" their respective Poisson boundaries. Here are two of the expected consistency results. They are, in fact, corollaries the weighted Glivenko–Cantelli Theorem W-GC in Section 2, or of general Glivenko–Cantelli theory (see e.g. Dudley (1999) or Vaart and Wellner 1996).

Theorem 3. *Suppose that $X_1, X_2, \ldots$ are i.i.d. F on $[0,1]$ and $0 < b < 1$.*
(i) *If $E[(X(1-X))^{-b}] < \infty$, then*

$$D_n(b) \equiv \sup_{0<x<1} \frac{|\mathbb{F}_n(x) - x|}{(x(1-x))^b} \rightarrow_{a.s.} \sup_{0<x<1} \frac{|F(x) - x|}{(x(1-x))^b} \equiv d(b, F) < \infty.$$

(ii) *If $E[(X(1-X))^{-b}] = \infty$, then $\limsup_{n\to\infty} D_n(b) = +\infty$ a.s.*

Theorem 3B. *Suppose that $X_1, X_2, \ldots$ are i.i.d. F on $[0,1]$ and $\psi_2(x) \equiv -\log(x(1-x))$.*
(i) *If $E[\psi_2(X)] < \infty$, then*

$$D_n(\psi) \equiv \sup_{0<x<1} |\mathbb{F}_n(x) - x|\psi_2(x) \rightarrow_{a.s.} \sup_{0<x<1} |F(x) - x|\psi_2(x) \equiv d(\psi_2, F) < \infty.$$

(ii) *If $E[\psi_2(X)] = \infty$ then $\limsup_{n\to\infty} D_n(\psi_2) = +\infty$ almost surely.*

Proof of Theorem 3. Note that

$$\begin{aligned}
|D_n(b) - d(b,F)| &\leq \sup_{0<x<1} \frac{|\mathbb{F}_n(x) - F(x)|}{(x(1-x))^b} \\
&= \sup_{0<x<1} \frac{|\mathbb{G}_n(F(x)) - F(x)|}{(x(1-x))^b} \\
&= \sup_{0<u<1} \frac{|\mathbb{G}_n(u) - u|}{(F^{-1}(u)(1-F^{-1}(u)))^b} \\
&\underset{a.s.}{\rightarrow} 0
\end{aligned}$$

if

$$\int_0^1 \frac{1}{(F^{-1}(u)(1-F^{-1}(u)))^b}\, du < \infty \tag{3.14}$$

by Theorem W-GC, or by part A, of Wellner (1977a) and remark 1 on page 475. But (3.14) holds if and only if the stated hypothesis holds by the fact that $F^{-1}(U) \stackrel{d}{=} X \sim F$ for $U \sim U(0,1)$. $\square$

Remark 1. Note that for the "Poisson boundary" distribution function $F(x) = x^b$ for $D_n(b)$

$$E[(X(1-X))^{-b}] = \int_0^1 \frac{bx^{b-1}}{(x(1-x))^b}\, dx = b\int_0^1 \frac{1}{x(1-x)^b}\, dx = \infty,$$

so the hypothesis of Theorem 3 part (i) (just) fails. On the other hand, if $F(x) = x^c$ with $b < c < 1$, then

$$E[(X(1-X))^{-b}] = \int_0^1 \frac{cx^{c-1}}{(x(1-x))^b}\, dx = c\int_0^1 \frac{1}{x^{1+b-c}(1-x)^b}\, dx < \infty,$$

so the hypothesis of Theorem 3(i) holds and $D_n(b) \rightarrow_{a.s.} d(b, F)$.

Remark 2. Note that for the "Poisson boundary" distribution function $F(x) = (1+\log(1/x))^{-1}$ for the statistic $D_n(\psi_2)$,

$$E_F[\psi_2(X)] = \int_0^1 \log\left(\frac{1}{x(1-x)}\right) \frac{1}{x(1+\log(1/x))^2}\, dx = \infty$$

so the hypothesis of Theorem 3B part (i) (just) fails.

4. Further problems

Here is a partial list of open problems in connection with the statistics discussed here and in Jager and Wellner (2004).

Question 1. What are the theorems corresponding to Theorem 3 in the case of R_n and $\tilde{R}_n$? In other words, for exactly which alternative distribution functions F does it hold that

$$R_n \to_{a.s.} \sup_x K\big(F(x), F_0(x)\big) \equiv r(F, F_0)? \tag{4.15}$$

For exactly which alternative distribution functions F does it hold that

$$\tilde{R}_n \to_{a.s.} \sup_x K\big(F_0(x), F(x)\big) \equiv \tilde{r}(F, F_0)\,? \tag{4.16}$$

Question 2. For alternative distribution functions F such that (4.15) holds, can we obtain useful approximations to the power of R_n via limit theorems for

$$\sqrt{n}\big(R_n - r(F, F_0)\big)$$

along the lines of Raghavachari (1973)? Similarly for F's for which (4.16) holds for $\tilde{R}_n$?

Question 3. Donoho and Jin (2004) consider testing $H_0: \ F = N(0,1) = \Phi$ versus $H_1: \ F = (1-\epsilon)N(0,1) + \epsilon N(\mu, 1)$ where $\epsilon_n = n^{-\beta}$ and $\mu = \mu_n = \sqrt{2r\log n}$ for $\beta > 1/2$ and $r > 0$. They show that a natural "detection boundary" is given by

$$r^*(\beta) = \begin{cases} \beta - 1/2, & 1/2 < \beta \le 3/4 \\ (1-\sqrt{1-\beta})^2, & 3/4 < \beta < 1. \end{cases}$$

How do the statistics R_n, $\tilde{R}_n$, and $K_n(1/2)$ compare along the "detection boundary" of Donoho and Jin (2004) Note that Donoho and Jin (2004) find that $D_n(1/2)$ and R_n have quite comparable power behavior for their testing problem, but they show that $D_n(1/2)$ has better power in the region $r > r^*(\beta)$ and $3/4 < \beta < 1$.

Question 4. What is the limiting null distribution of $\tilde{R}_n$?

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