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Non-linear filtering with Gaussian martingale noise: Kalman filter with fBm noise

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Abstract: We consider non-linear filtering problem with Gaussian martingales as a noise process, and obtain iterative equations for the optimal filter. We apply that result in the case of fractional Browian motion noise process and derive Kalman type equations in the linear case.

1. Introduction

The study of filtering of a stochastic process with a general Gaussian noise was initiated in [8]. In case the system satisfies a stochastic differential equation, we derived an iterative form for the optimal filter given by the Zakai equation ([3]). It was shown in [2] that in the case of a Gaussian noise, one can derive the FKK equation from which one can obtain the Kalman filtering equation. However in order to obtain Kalman's equation in the case of fractional Brownian motion (fBm) noise, we had to assume in [3] the form of the observation process, which was not intuitive. Using the ideas in [5], we are able to study the problem with a natural form of the observation process as in the classical work. In order to get such a result from the general theory we have to study the Bayes formula for Gaussian martingale noise and use the work in [5]. This is accomplished in Section 1. In Section 2, we obtain iterative equations for the optimal filter and in Section 3 we apply them to the case of fBm noise.

The problem of filtering with system and observation processes driven by fBm was considered in [1]. However, even the form of the Bayes formula in this case is complicated and no iterative equations for the filter can be obtained. The Bayes formula in [8] is applicable to any system process and observation process with Gaussian noise. In order to get iterative equations in non-linear case we assume that the system process is a solution of a martingale problem. This allows us to obtain an analogue of the Zakai and FKK equations. As a consequence, we easily derive the Kalman equations in the linear case. If the data about the "signal" are sent to the server and transmitted to AWACS, the resulting process has bursts [6]. We assume a particular form for this observation process (see equation (3)). In most cases, signal (missile trajectory, e.g.) is Markovian.

The work completed by D. Fisk under the guidence of Professor Herman Rubin has found applications in deriving filtering equations in the classical case [4].

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2. Bayes formula with Gaussian martingale noise

Let us consider the filtering problem with a signal or system process $\{X_t, 0 \leq t \leq T\}$, which is unobservable. Information about X_t is obtained by observing another process Y_t, which is a function of X_t, and is corrupted by noise, i.e.

$$Y_t = \beta(t, X) + N_t, \quad 0 \leq t \leq T,$$

where $\beta(t, \cdot)$ is measurable with respect to the σ–field $\mathcal{F}_t^X$, generated by the signal process $\{X_s,\ 0 \leq s \leq t\}$, and the noise $\{N_t,\ 0 \leq t \leq T\}$ is independent of $\{X_t,\ 0 \leq t \leq T\}$. The observation σ–field $\mathcal{F}_t^Y = \sigma\{Y_s,\ 0 \leq s \leq t\}$ contains all the available information about the signal X_t. The primary aim of filtering theory is to get an estimate for X_t based on the σ–field $\mathcal{F}_t^Y$. This is given by the conditional distribution $\hat{\Pi}_t$ of X_t given $\mathcal{F}_t^Y$ or, equivalently, by the conditional expectation $E\left(f(X_t)\,\middle|\mathcal{F}_t^Y\right)$ for a rich enough class of functions f. Since this estimate minimizes the squared error loss, $\hat{\Pi}_t$ is called the optimal filter.

In [8] an expression for an optimal filter was given for $\{N_t,\ 0 \leq t \leq T\}$, a Gaussian process and $\beta(\cdot, X) \in H(R)$, the reproducing kernel Hilbert space (RKHS) of the covariance R of the process N_t ([8]). Throughout we assume, without loss of generality, that $E(N_t) = 0$.

Let us assume that $N_t = M_t$, a continuous Gaussian martingale with the covariance function R_M. We shall first compute the form of $H(R_M)$. As we shall be using this notation exclusively for the martingale M_t, we will drop the subscript M from now on and denote the RKHS of R by $H(R)$. Let us also denote by $m(t)$ the expectation EM_t^2. Note that $m(t)$ is a non–decreasing function on $[0, T]$ and, abusing the notation, we will denote by m the associated measure on the Borel subsets $\mathcal{B}([0, T])$. With this convention, we can write

$$H(R) = \left\{ g :\ g(t) = \int_0^t g^*(u)\, dm(u),\ \ 0 \leq t \leq T,\ g^* \in L^2(m) \right\}.$$

The scalar product in $H(R)$ is given by $(g_1, g_2)_{H(R)} =< g_1^*, g_2^* >_{L^2(m)}$. If we denote by $H(R\,:\,t)$ the RKHS of $R|_{[0,t]\times[0,t]}$, then it follows from the above that

$$H(R\,:\,t) = \left\{ g :\ g(s) = \int_0^s g^*(u)\, dm(u),\ \ 0 \leq s \leq t,\ g^* \in L^2(m) \right\}.$$

It is well known (see [8], Section 2), that there exists an isometry π between $H(R)$ and $\overline{sp}^{L^2}\{M_t,\ \ 0 \leq t \leq T\}$, which, in case M is a martingale, is given by

$$\pi(g) = \int_0^T g^*(u)\, dM_u,$$

where the RHS denotes the stochastic integral of the deterministic function g^* with respect to M. The isometry

$$\pi_t(g) : H(R\,:\,t) \to \overline{sp}^{L^2}\{M_s,\ \ 0 \leq s \leq t\}$$

is given by $\pi_t(g) = \int_0^t g^*(u) dM_u$.

Suppose now

$$Y_t = \int_0^t h(s, X) dm(s) + M_t,$$

where $h(s,X)$ is $\mathcal{F}_s^X$-measurable and $h(\cdot,X) \in L^2(m)$. Then using Theorem 3.2 of [8] we get the Bayes formula for an $\mathcal{F}_T^X$-measurable and integrable function $g(T,X)$

$$E\left(g(T,X)\,\middle|\mathcal{F}_t^Y\right) = \frac{\int g(T,\mathbf{x}) e^{\int_0^t h(s,\mathbf{x}) dY_s - \frac{1}{2}\int_0^t h^2(s,\mathbf{x})\,dm(s)}\,dP\circ X^{-1}}{\int e^{\int_0^t h(s,\mathbf{x})\,dY_s - \frac{1}{2}\int_0^t h^2(s,\mathbf{x})\,dm(s)} dP\circ X^{-1}}. \tag{1}$$

3. Equations for non-linear filter with martingale noise

In this section we derive the Zakai equation for the so-called "unconditional" measure-valued process. We follow the techniques developed in [2]. We assume that $\{X_t,\ \ 0 \le t \le T\}$ is a solution of the martingale problem. Let $C_c^2(\mathbb{R}^n)$ be the space of twice continuously differentiable functions with compact support. Let

$$(L_t f)(x) = \sum_{j=1}^n b_j(t,x)\frac{\partial f}{\partial x_j}(x) + 1/2\sum_{i,j=1}^n \sigma_{i,j}(t,x)\frac{\partial^2 f}{\partial x_i x_j}(x),$$

for $f \in C_c^2(\mathbb{R}^n)$, with $b_j(t,x)$ and $\sigma_{i,j}(t,x)$ bounded and continuous. We assume that X_t is a solution to the martingale problem, i.e., for $f \in C_c(\mathbb{R}^n)$,

$$f(X_t) - \int_0^t (L_u f)(X_u)\,du$$

is an $\mathcal{F}_t^X$–martingale with respect to the measure P. Consider the probability space $(\Omega\times\Omega, \mathcal{F}\otimes\mathcal{F}, P\otimes P')$, where P' is a probability measure given by

$$dP' = \exp\left(-\int_0^t h(s,X)dY_s + \frac{1}{2}\int_0^t h^2(s,X)dm(s)\right)\,dP.$$

Then under the measure P', the process Y_t has the same distribution as M_t and is independent of X_t. In addition, $P\circ X^{-1} = P'\circ X^{-1}$. This follows from Theorem 3.1 in [8]. Define

$$\alpha_t(\omega',\omega) = \exp\left(\int_0^t h\left(s,X(\omega')\right)\,dY_s(\omega) - 1/2\int_0^t h^2\left(s,X(\omega')\right)\,dm(s)\right).$$

Then, with a notation $g(\omega') = g(T, X(\omega'))$, equation (1) can be written as

$$E\left(g(T,X)\,\middle|\mathcal{F}_t^Y\right) = \frac{\int g(\omega')\alpha_t(\omega',\omega)\,dP\circ X^{-1}(\omega')}{\int \alpha_t(\omega',\omega)\,dP\circ X^{-1}(\omega')}.$$

For a function $f \in C_c^2(\mathbb{R}^n)$, denote

$$\hat\sigma_t(f,Y)(\omega) = \int f\left(X_t(\omega')\right)\alpha_t(\omega',\omega)\,dP(\omega').$$

Then we get the following analogue of the Zakai equation. We assume here that m is mutually absolutely continuous with respect to the Lebesgue measure.

Theorem. *The quantity $\hat\sigma_t(f,Y)$ defined above satisfies the equation*

$$d\hat\sigma_t\left(f(\cdot),Y\right) = \hat\sigma_t\left(L_t f(\cdot),Y\right)\,dt + \hat\sigma_t\left(h(t,\cdot)f(\cdot),Y\right)\,dY_t.$$

Proof. We follow the argument as in [2]. Consider $g_t(\omega') = f(X_T(\omega')) - \int_t^T (L_s f)(X_s(\omega'))\, ds$, with $f \in C_c^2(\mathbb{R}^n)$. Then

$$E_P\left(g_t \,\middle|\, \mathcal{F}_t^X\right) = f(X_t), \quad 0 \le t \le T.$$

We can represent $\hat{\sigma}_t(f, Y)$ as

$$\begin{aligned}
\hat{\sigma}_t(f,Y) &= \int f\left(X_t\left(\omega'\right)\right) \alpha_t(\omega',\omega)\, dP(\omega') \\
&= \int E_P\left(g_t(\omega')\alpha_t(\omega',\omega) \,\middle|\, \mathcal{F}_t^X\right)\, dP(\omega') \\
&= E_P\left(g_t(\omega')\alpha_t(\omega',\omega)\right) \\
&=: \sigma_t'\left(g_t, Y\right).
\end{aligned}$$

By definition of g_t,

$$dg_t = \left(L_t f\right)\left(X_t'\right)\, dt,$$

with X_t' an independent copy of X_t as a function of ω'. Using Itô's formula,

$$d\alpha_t = \alpha_t h\left(t, X'\right)\, dY_t.$$

Since $\sigma_t'\left(g_t, Y\right) = E_P(g_t \alpha_t)$, utilizing the Fubini theorem and Theorem 5.14 in [7], we rewrite the latter as

$$\begin{aligned}
E_P\left(g_t \alpha_t\right) &= E_P g_0 + \int_0^t \hat{\sigma}_s\left(L_s f, Y\right)\, ds \\
&\quad + \int \int_0^t g_s(\omega')\alpha_s(\omega',\omega) h\left(s, X(\omega')\right)\, dY_s(\omega)\, dP(\omega') \\
&= E_P g_0 + \int_0^t \hat{\sigma}_s\left(L_s f, Y\right)\, ds \\
&\quad + \int_0^t \hat{\sigma}_s\left(h\left(s, X(\omega')\right) f\left(X_s(\omega')\right), Y\right)\, dY_s.
\end{aligned}$$

It should be noted the application of Theorem 5.14 above is valid due to the fact that the martingale M_t is a time changed Brownian motion with non–singular time. □

Now we note that the optimal filter is given by

$$\hat{\Pi}_t(f) = E\left(f(X_t) \,\middle|\, \mathcal{F}_t^Y\right) = \frac{\hat{\sigma}(f,Y)}{\hat{\sigma}(1,Y)}.$$

Under our construction, Y_t is a continuous Gaussian martingale with the increasing process $m(t)$. Using Itô's formula we obtain

$$d\hat{\Pi}_t(f) = \hat{\Pi}_t(L_t f)dt + \left[\hat{\Pi}_t(hf) - \hat{\Pi}_t(f)\hat{\Pi}_t(h)\right] d\nu_t, \tag{2}$$

where $\nu_t = Y_t - \int_0^t \hat{\Pi}_s(h)\, dm(s)$.

4. Filtering equations in case of fractional Brownian motion noise

Let us start with the definition of fractional Brownian motion (fBm). We say that a Gaussian process $\{W_t^H,\ 0 \le t \le T\}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, with continuous sample paths is a fractional Brownian motion if $W_0^H = 0$, $EW_t^H = 0$, and for $0 < H < 1$,

$$EW_s^H W_t^H = \frac{1}{2}[s^{2H} + t^{2H} - |s-t|^{2H}], \quad 0 \le s,t \le T.$$

Let us set up some notation following [5].

$$k_H(t,s) = \kappa_H^{-1} s^{1/2-H}(t-s)^{1/2-H}, \quad \text{where } \kappa_H = 2H\Gamma(3/2-H)\Gamma(H+1/2),$$
$$w_t^H = \lambda_H^{-1} t^{2-2H}, \quad \text{with } \lambda_H = \frac{2H\Gamma(3-2H)\Gamma(H+1/2)}{\Gamma(3/2-H)},$$
$$M_t^H = \int_0^t k_H(t,s) dW_s^H.$$

The integral with respect to fBm W_t^H is described in [9]. The process M_t^H is a Gaussian martingale. Define

$$Q_H^c(t) = \frac{d}{dw_t^H} \int_0^t k_H(t,s) C(s)\, ds,$$

where $C(t)$ is an $\mathcal{F}_t$–adapted process and the derivative is understood to be in terms of absolute continuity. Then the following result can be derived from [5].

Let $Y_t = \int_0^t C(s,X)\, ds + W_t^H$. Then

$$Z_t = \int_0^t Q_H^c(s)\, dw_s^H + M_t^H$$

is an $\mathcal{F}_t^Y$ semi–martingale and $\mathcal{F}_t^Y = \mathcal{F}_t^Z$. Let us now consider the filtering problem as in Section 1, with the noise $N_t = W_t^H$, and the observation process

$$Y_t = \int_0^t C(s,X)\, ds + W_t^H. \tag{3}$$

Then the equivalent filtering problem is given by the system process X_t and the observation process

$$Z_t = \int_0^t Q_H^c(s,X)\, dw_s^H + M_t^H.$$

Using results of Section 2, and assuming that X_t is a solution to the martingale problem, equation (2) reduces to

$$d\hat{\Pi}_t(f) = \hat{\Pi}_t(L_t f)dt + \left[\hat{\Pi}_t(Q_H^c f) - \hat{\Pi}_t(f)\hat{\Pi}_t(Q_H^c)\right] d\nu_t,$$

where $\nu_t = Z(t) - \int_0^t \hat{\Pi}_s(Q_H^c)\, dw_s^H$. By Theorem 2 in [5] we get that ν_t is a continuous Gaussian $\mathcal{F}_t^Y$–martingale with variance w_t^H.

Let us now assume that the system process and observation processes are given by

$$\begin{aligned} X_t &= \int_0^t b(u)X_u\, du + \int_0^t \sigma(u)\, dW_u \\ Y_t &= \int_0^t c(u)X_u\, du + W_t^H, \end{aligned}$$

where the processes W_t and W_t^H are independent. Because (X_t, Z_t) is jointly Gaussian we get

$$\begin{aligned} &\hat{\Pi}_t(X_t X_s) - \hat{\Pi}_t(X_t)\hat{\Pi}_t(X_s) \\ &\quad = E\left\{(X_t - \hat{\Pi}_t(X_t))(X_s - \hat{\Pi}_t(X_s)) \,\middle|\, \mathcal{F}_t^Y\right\} \\ &\quad = E\left\{(X_t - \hat{\Pi}_t(X_t))(X_s - \hat{\Pi}_t(X_s))\right\} \\ &\quad = \Gamma(t,s). \end{aligned}$$

We obtain that

$$d\hat{\Pi}_t(X_t) = b(t)\hat{\Pi}_t(X_t)dt + \int_0^t k_H(t,s)\Gamma(t,s)\, ds\, d\nu_t. \tag{4}$$

Denote by $\gamma(t) = EX_t^2$, and $F(t) = E\left(\hat{\Pi}_t^2(X_t)\right)$. Then by the Itô formula for $f(x) = x^2$ and by taking the expectation, we get

$$\begin{aligned} d\gamma(t) &= 2b(t)\gamma(t)dt + \sigma^2(t)dt \\ \text{and} \quad dF(t) &= 2b(t)F(t)dt + \left(\int_0^t k_H(t,s)\Gamma(t,s)\, ds\right)^2 dw_t^H. \end{aligned}$$

Let us consider

$$\begin{aligned} \Gamma(t,t) &= E(X_t - \hat{\Pi}(X_t))^2 \\ &= E(X_t^2) - E(\hat{\Pi}_t^2(X_t)) \\ &= \gamma(t) - F(t). \end{aligned}$$

Then we arrive at

$$d\Gamma(t,t) = 2b(t)\Gamma(t,t)dt + \sigma^2(t)dt - \left(\int_0^t k_H(t,s)\Gamma(t,s)\, ds\right)^2 dw_s^H. \tag{5}$$

For $H = \frac{1}{2}$ this reduces to the Kalman equation.

Equations (4) and (5) give the Kalman filtering equations in the linear case.

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