CHAPTER 6

Topological Groups and Invariant Measures

The language of vector spaces has been used in the previous chapters to describe a variety of properties of random vectors and their distributions. Apart from the discussion in Chapter 4, not much has been said concerning the structure of parametric probability models for distributions of random vectors. Groups of transformations acting on spaces provide a very useful framework in which to generate and describe many parametric statistical models. Furthermore, the derivation of induced distributions of a variety of functions of random vectors is often simplified and clarified using the existence and uniqueness of invariant measures on locally compact topological groups. The ideas and techniques presented in this chapter permeate the remainder of this book.

Most of the groups occurring in multivariate analysis are groups of nonsingular linear transformations or related groups of affine transformations. Examples of matrix groups are given in Section 6.1 to illustrate the definition of a group. Also, examples of quotient spaces that arise naturally in multivariate analysis are discussed.

In Section 6.2, locally compact topological groups are defined. The existence and uniqueness theorem concerning invariant measures (integrals) on these groups is stated and the matrix groups introduced in Section 6.1 are used as examples. Continuous homomorphisms and their relation to relatively invariant measures are described with matrix groups again serving as examples. Some of the material in this section and the next is modeled after Nachbin (1965). Rather than repeat the proofs given in Nachbin (1965), we have chosen to illustrate the theory with numerous examples.

Section 6.3 is concerned with the existence and uniqueness of relatively invariant measures on spaces that are acted on transitively by groups of transformations. In fact, this situation is probably more relevant to statistical problems than that discussed in Section 6.2. Of course, the examples are selected with statistical applications in mind.

6.1. GROUPS

We begin with the definition of a group and then give examples of matrix groups.

Definition 6.1. A group $(G, \circ)$ is a set G together with a binary operation $\circ$ such that the following properties hold for all elements in G:

(i) $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$.

(ii) There is a unique element of G, denoted by e, such that $g \circ e = e \circ g = g$ for all $g \in G$. The element e is the *identity* in G.

(iii) For each $g \in G$, there is a unique element in G, denoted by g^{-1}, such that $g \circ g^{-1} = g^{-1} \circ g = e$. The element g^{-1} is the *inverse* of g.

Henceforth, the binary operation is ordinarily deleted and we write $g_1 g_2$ for $g_1 \circ g_2$. Also, parentheses are usually not used in expressions involving more than two group elements as these expressions are unambiguously defined in (i). A group G is called *commutative* if $g_1 g_2 = g_2 g_1$ for all $g_1, g_2 \in G$. It is clear that a vector space V is a commutative group where the group operation is addition, the identity element is $0 \in V$, and the inverse of x is $-x$.

◆ **Example 6.1.** If $(V, (\cdot, \cdot))$ is a finite dimensional inner product space, it has been shown that the set of all orthogonal transformations $\mathcal{O}(V)$ is a group. The group operation is the composition of linear transformations, the identity element is the identity linear transformation, and if $\Gamma \in \mathcal{O}(V)$, the inverse of Γ is Γ'. When V is the coordinate space R^n, $\mathcal{O}(V)$ is denoted by $\mathcal{O}_n$, which is just the group of $n \times n$ orthogonal matrices. ◆

◆ **Example 6.2.** Consider the coordinate space R^p and let G_T^+ be the set of all $p \times p$ lower triangular matrices with positive diagonal elements. The group operation in G_T^+ is taken to be matrix multiplication. It has been verified in Chapter 5 that G_T^+ is a group, the identity in G_T^+ is the $p \times p$ identity matrix, and if $T \in G_T^+$, T^{-1} is just the matrix inverse of T. Similarly, the set of $p \times p$ upper triangular matrices with positive diagonal elements G_U^+ is a group with the group operation of matrix multiplication. ◆

◆ **Example 6.3.** Let V be an n-dimensional vector space and let $Gl(V)$ be the set of all nonsingular linear transformations of V onto V. The group operation in $Gl(V)$ is defined to be composition of linear transformations. With this operation, it is easy to verify that $Gl(V)$ is a group, the identity in $Gl(V)$ is the identity linear transformation, and if $g \in Gl(V)$, g^{-1} is the inverse linear transformation of g. The group $Gl(V)$ is often called the *general linear group* of V. When V is the coordinate space R^n, $Gl(V)$ is denoted by Gl_n. Clearly, Gl_n is just the set of $n \times n$ nonsingular matrices and the group operation is matrix multiplication. ◆

It should be noted that $\mathcal{O}(V)$ is a subset of $Gl(V)$ and the group operation in $\mathcal{O}(V)$ is that of $Gl(V)$. Further, G_T^+ and G_U^+ are subsets of Gl_n with the inherited group operations. This observation leads to the definition of a subgroup.

Definition 6.2. If $(G, \circ)$ is a group and H is a subset of G such that $(H, \circ)$ is also a group, then $(H, \circ)$ is a *subgroup* of $(G, \circ)$.

In all of the above examples, each element of the group is also a one-to-one function defined on a set. Further, the group operation is in fact function composition. To isolate the essential features of this situation, we define the following.

Definition 6.3. Let $(G, \circ)$ be a group and let $\mathcal{X}$ be a set. The group $(G, \circ)$ *acts on the left* of $\mathcal{X}$ if to each pair $(g, x) \in G \times \mathcal{X}$, there corresponds a unique element of $\mathcal{X}$, denoted by gx, such that

(i) $g_1(g_2x) = (g_1 \circ g_2)x$.
(ii) $ex = x$.

The content of Definition 6.3 is that there is a function on $G \times \mathcal{X}$ to $\mathcal{X}$ whose value at (g, x) is denoted by gx and under this mapping, (g_1, g_2x) and $(g_1 \circ g_2, x)$ are sent into the same element. Furthermore, (e, x) is mapped to x. Thus each $g \in G$ can be thought of as a one-to-one onto function from $\mathcal{X}$ to $\mathcal{X}$ and the group operation in G is function composition. To make this claim precise, for each $g \in G$, define t_g on $\mathcal{X}$ to $\mathcal{X}$ by $t_g(x) = gx$.

Proposition 6.1. Suppose G acts on the left of $\mathfrak{X}$. Then each t_g is a one-to-one onto function from $\mathfrak{X}$ to $\mathfrak{X}$ and:

(i) $t_{g_1}t_{g_2} = t_{g_1 \circ g_2}$.
(ii) $t_g^{-1} = t_{g^{-1}}$.

Proof. To show t_g is onto, consider $x \in \mathfrak{X}$. Then $t_g(g^{-1}x) = g(g^{-1}x) = (g \circ g^{-1})x = ex = x$ where (i) and (ii) of Definition 6.3 have been used. Thus t_g is onto. If $t_g(x_1) = t_g(x_2)$, then $gx_1 = gx_2$ so

$$x_1 = ex_1 = (g^{-1} \circ g)x_1 = g^{-1}(gx_1) = g^{-1}(gx_2)$$
$$= (g^{-1} \circ g)x_2 = ex_2 = x_2.$$

Thus t_g is one-to-one. Assertion (i) follows immediately from (i) of Definition 6.3. Since t_e is the identity function on $\mathfrak{X}$ and (i) implies that

$$t_g t_{g^{-1}} = t_{g^{-1}} t_g = t_e,$$

we have $t_{g^{-1}} = t_g^{-1}$. □

Henceforth, we dispense with t_g and simply regard each g as a function on $\mathfrak{X}$ to $\mathfrak{X}$ where function composition is group composition and e is the identity function on $\mathfrak{X}$. All of the examples considered thus far are groups of functions on a vector space to itself and the group operation is defined to be function composition. In particular, $Gl(V)$ is the set of all one-to-one onto linear transformations of V to V and the group operation is function composition. In the next example, the motivation for the definition of the group operation is provided by thinking of each group element as a function.

◆ **Example 6.4.** Let V be an n-dimensional vector space and consider the set $Al(V)$ that is the collection of all pairs (A, x) with $A \in Gl(V)$ and $x \in V$. Each pair (A, x) defines a one-to-one onto function from V to V by

$$(A, x)v = Av + x, \qquad v \in V.$$

The composition of (A_1, x_1) and (A_2, x_2) is

$$(A_1, x_1)(A_2, x_2)v = (A_1, x_1)(A_2v + x_2) = A_1A_2v + A_1x_2 + x_1$$
$$= (A_1A_2, A_1x_2 + x_1)v.$$

Also, $(I, 0) \in Al(V)$ is the identity function on V and the inverse of (A, x) is $(A^{-1}, -A^{-1}x)$. It is now an easy matter to verify that $Al(V)$ is a group where the group operation in $Al(V)$ is

$$(A_1, x_1)(A_2, x_2) \equiv (A_1A_2, A_1x_2 + x_1).$$

This group $Al(V)$ is called the *affine group* of V. When V is the coordinate space R^n, $Al(V)$ is denoted by Al_n. ◆

An interesting and useful subgroup of $Al(V)$ is given in the next example.

◆ **Example 6.5.** Suppose V is a finite dimensional vector space and let M be a subspace of V. Let H be the collection of all pairs (A, x) where $x \in M$, $A(M) \subseteq M$, and $(A, x) \in Al(V)$. The group operation in H is that inherited from $Al(V)$. It is a routine calculation to show that H is a subgroup of $Al(V)$. As a particular case, suppose that V is R^n and M is the m-dimensional subspace of R^n consisting of those vectors $x \in R^n$ whose last $n - m$ coordinates are zero. An $n \times n$ matrix $A \in Gl_n$ satisfies $AM \subseteq M$ iff

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

where A_{11} is $m \times m$ and nonsingular, A_{12} is $m \times (n - m)$, and A_{22} is $(n - m) \times (n - m)$ and nonsingular. Thus H consists of all pairs (A, x) where $A \in Gl_n$ has the above form and x has its last $n - m$ coordinates zero. ◆

◆ **Example 6.6.** In this example, we consider two finite groups that arise naturally in statistical problems. Consider the space R^n and let P be an $n \times n$ matrix that permutes the coordinates of a vector $x \in R^n$. Thus in each row and in each column of P, there is a single element that is one and the remaining elements are zero. Conversely, any such matrix permutes the coordinates of vectors in R^n. The set $\mathcal{P}_n$ of all such matrices is called the group of *permutation matrices*. It is clear that $\mathcal{P}_n$ is a group under matrix multiplication and $\mathcal{P}_n$ has $n!$ elements. Also, let $\mathcal{D}_n$ be the set of all $n \times n$ diagonal matrices whose diagonal elements are plus or minus one. Obviously, $\mathcal{D}_n$ is a group under matrix multiplication and $\mathcal{D}_n$ has 2^n elements. The group $\mathcal{D}_n$ is called the group of *sign changes* on R^n. A bit of reflection shows that both $\mathcal{P}_n$ and $\mathcal{D}_n$ are subgroups of $\mathcal{O}_n$. Now, let

H be the set

$$H = \{PD \mid P \in \mathcal{P}_n,\ D \in \mathcal{D}_n\}.$$

The claim is that H is a group under matrix multiplication. To see this, first note that for $P \in \mathcal{P}_n$ and $D \in \mathcal{D}_n$, PDP' is an element of $\mathcal{D}_n$. Thus if P_1D_1 and P_2D_2 are in H, then

$$P_1D_1P_2D_2 = P_1P_2P_2'D_1P_2D_2 = P_3D_3 \in H$$

where $P_3 = P_1P_2$ and $D_3 = P_2'D_1P_2D_2$. Also,

$$(PD)^{-1} = DP' = P'PDP' \in H.$$

Therefore H is a group and clearly has $2^n n!$ elements. ◆

Suppose that G is a group and H is a subgroup of G. The quotient space G/H, to be defined next, is often a useful representation of spaces that arise in later considerations. The subgroup H of G defines an equivalence relation in G by $g_1 \simeq g_2$ iff $g_2^{-1}g_1 \in H$. That $\simeq$ is an equivalence relation is easily verified using the assumption that H is a subgroup of G. Also, it is not difficult to show that $g_1 \simeq g_2$ iff the set $g_1H = \{g_1h \mid h \in H\}$ is equal to the set g_2H. Thus the set of points in G equivalent to g_1 is the set g_1H.

Definition 6.4. If H is a subgroup of G, the quotient space G/H is defined to be the set whose elements are gH for $g \in G$.

The quotient space G/H is obviously the set of equivalence classes (defined by H) of elements of G. Under certain conditions on H, the quotient space G/H is in fact a group under a natural definition of a group operation.

Definition 6.5. A subgroup H of G is called a *normal subgroup* if $g^{-1}Hg = H$ for all $g \in G$.

When H is a normal subgroup of G, and $g_iH \in G/H$ for $i = 1, 2$, then

$$g_1Hg_2H \equiv \{g \mid g = g_1h_1g_2h_2;\ h_1, h_2 \in H\}$$

$$= g_1g_2g_2^{-1}Hg_2H = g_1g_2HH = g_1g_2H$$

since $HH = H$.

Proposition 6.2. When H is a normal subgroup of G, the quotient space G/H is a group under the operation

$$(g_1H)(g_2H) \equiv g_1g_2H.$$

Proof. This is a routine calculation and is left to the reader. □

◆ **Example 6.7.** Let $Al(V)$ be the affine group of the vector space V. Then

$$H \equiv \{(I, x)|x \in V\}$$

is easily shown to be a subgroup of G, since $(I, x_1)(I, x_2) = (I, x_1 + x_2)$. To show H is normal in $Al(V)$, consider $(A, x) \in Al(V)$ and $(I, x_0) \in H$. Then

$$\begin{aligned}(A, x)^{-1}(I, x_0)(A, x) &= (A^{-1}, -A^{-1}x)(A, x + x_0)\\ &= \left(I, A^{-1}x + A^{-1}x_0 - A^{-1}x\right)\\ &= \left(I, A^{-1}x_0\right),\end{aligned}$$

which is an element of H. Thus $g^{-1}Hg \subseteq H$ for all $g \in Al(V)$. But if $(I, x_0) \in H$ and $(A, x) \in Al(V)$, then

$$(A, x)^{-1}(I, Ax_0)(A, x) = (I, x_0)$$

so $g^{-1}Hg = H$, for $g \in Al(V)$. Therefore, H is normal in $Al(V)$. To describe the group $Al(V)/H$, we characterize the equivalence relation defined by H. For $(A_i, x_i) \in Al(V)$, $i = 1, 2$,

$$\begin{aligned}(A_1, x_1)^{-1}(A_2, x_2) &= \left(A_1^{-1}, -A_1^{-1}x_1\right)(A_2, x_2)\\ &= \left(A_1^{-1}A_2, A_1^{-1}x_2 - A_1^{-1}x_1\right)\end{aligned}$$

is an element of H iff $A_1^{-1}A_2 = I$ or $A_1 = A_2$. Thus (A_1, x_1) is equivalent to (A_2, x_2) iff $A_1 = A_2$. From each equivalence class, select the element $(A, 0)$. Then it is clear that the quotient group $Al(V)/H$ can be identified with the group

$$K = \{(A, 0)|A \in Gl(V)\}$$

where the group operation is

$$(A_1, 0)(A_2, 0) = (A_1 A_2, 0). \qquad \blacklozenge$$

Now, suppose the group G acts on the left of the set $\mathcal{X}$. We say G acts *transitively* on $\mathcal{X}$ if, for each x_1 and x_2 in $\mathcal{X}$, there exists a $g \in G$ such that $gx_1 = x_2$. When G acts transitively on $\mathcal{X}$, we want to show that there is a natural one-to-one correspondence between $\mathcal{X}$ and a certain quotient space. Fix an element $x_0 \in \mathcal{X}$ and let

$$H = \{h | hx_0 = x_0, h \in G\}.$$

The subgroup H of G is called the *isotropy subgroup* of x_0. Now, define the function τ on G/H to $\mathcal{X}$ by $\tau(gH) = gx_0$.

Proposition 6.3. The function τ is one-to-one and onto. Further,

$$\tau(g_1 gH) = g_1 \tau(gH).$$

Proof. The definition of τ clearly makes sense as $ghx_0 = gx_0$ for all $h \in H$. Also, τ is an onto function since G acts transitively on $\mathcal{X}$. If $\tau(g_1 H) = \tau(g_2 H)$, then $g_1 x_0 = g_2 x_0$ so $g_2^{-1} g_1 \in H$. Therefore, $g_1 H = g_2 H$ so τ is one-to-one. The rest is obvious. □

If H is any subgroup of G, then the group G acts transitively on $\mathcal{X} \equiv G/H$ where the group action is

$$g_1(gH) \equiv g_1 gH.$$

Thus we have a complete description of the spaces $\mathcal{X}$ that are acted on transitively by G. Namely, these spaces are simply relabelings of the quotient spaces G/H where H is a subgroup of G. Further, the action of g on $\mathcal{X}$ corresponds to the action of G on the quotient space described in Proposition 6.3. A few examples illustrate these ideas.

◆ **Example 6.8.** Take the set $\mathcal{X}$ to be $\mathfrak{F}_{p,n}$—the set of $n \times p$ real matrices Ψ that satisfy $\Psi'\Psi = I_p$, $1 \leqslant p \leqslant n$. The group $G = \mathcal{O}_n$ of all $n \times n$ orthogonal matrices acts on $\mathfrak{F}_{p,n}$ by matrix multiplication. That is, if $\Gamma \in \mathcal{O}_n$ and $\Psi \in \mathfrak{F}_{p,n}$, then $\Gamma\Psi$ is the matrix product of Γ and Ψ. To show that this group action is transitive, consider Ψ_1 and Ψ_2 in $\mathfrak{F}_{p,n}$. Then, the columns of Ψ_1 form a set of p orthonormal vectors in R^n as do the columns of Ψ_2. By Proposition 1.30, there exists an $n \times n$ orthogonal matrix Γ that maps the columns of Ψ_1 into the columns of Ψ_2. Thus $\Gamma\Psi_1 = \Psi_2$ so $\mathcal{O}_n$ is transitive on $\mathcal{F}_{p,n}$. A convenient choice of $x_0 \in \mathcal{F}_{p,n}$ to define the map τ is

$$x_0 = \begin{pmatrix} I_p \\ 0 \end{pmatrix}$$

where 0 is a block of $(n-p) \times p$ zeroes. It is not difficult to show that the subgroup $H = \{\Gamma | \Gamma x_0 = x_0, \Gamma \in \mathcal{O}_n\}$ is

$$H = \left\{ \Gamma | \Gamma = \begin{pmatrix} I_p & 0 \\ 0 & \Gamma_{22} \end{pmatrix}, \Gamma_{22} \in \mathcal{O}_{(n-p)} \right\}.$$

The function τ is

$$\tau(\Gamma H) = \Gamma x_0 = \Gamma \begin{pmatrix} I_p \\ 0 \end{pmatrix},$$

which is the $n \times p$ matrix consisting of the first p columns of Γ. This gives an obvious representation of $\mathcal{F}_{p,n}$. ◆

◆ **Example 6.9.** Let $\mathcal{X}$ be the set of all $p \times p$ positive definite matrices and let $G = Gl_p$. The transitive group action is given by $A(x) = AxA'$ where A is a $p \times p$ nonsingular matrix, $x \in \mathcal{X}$, and A' is the transpose of A. Choose $x_0 \in \mathcal{X}$ to be I_p. Obviously, $H = \mathcal{O}_p$ and the map τ is given by

$$\tau(AH) = A(x_0) = AA'.$$

The reader should compare this example with the assertion of Proposition 1.31. ◆

◆ **Example 6.10.** In this example, take $\mathcal{X}$ to be the set of all $n \times p$ real matrices of rank p, $p \leqslant n$. Consider the group G defined by

$$G = \{g | g = \Gamma \otimes T, \Gamma \in \mathcal{O}_n, T \in G_T^+\}$$

where G_T^+ is the group of all $p \times p$ lower triangular matrices with positive diagonal elements. Of course, $\otimes$ denotes the Kronecker product and group composition is

$$(\Gamma_1 \otimes T_1)(\Gamma_2 \otimes T_2) = (\Gamma_1\Gamma_2) \otimes (T_1 T_2).$$

The action of G on $\mathcal{X}$ is

$$(\Gamma \otimes T)X = \Gamma X T', \qquad X \in \mathcal{X}.$$

To show G acts transitively on $\mathcal{X}$, consider X_1, $X_2 \in \mathcal{X}$ and write $X_i = \Psi_i U_i$, where $\Psi_i \in \mathcal{F}_{p,n}$ and $U_i \in G_U^+$, $i = 1, 2$ (see Proposition 5.2). From Example 6.8, there is a $\Gamma \in \mathcal{O}_n$ such that $\Gamma\Psi_1 = \Psi_2$. Let $T' = U_1^{-1}U_2$ so

$$\Gamma X_1 T' = \Gamma\Psi_1 U_1 U_1^{-1} U_2 = \Psi_2 U_2 = X_2.$$

Choose $X_0 \in \mathcal{X}$ to be

$$X_0 = \begin{pmatrix} I_p \\ 0 \end{pmatrix}$$

as in Example 6.8. Then the equation $(\Gamma \otimes T)X_0 = X_0$ implies that

$$I_p = X_0'X_0 = ((\Gamma \otimes T)X_0)'(\Gamma \otimes T)X_0 = TX_0'\Gamma'\Gamma X_0 T' = TT'$$

so $T = I_p$ by Proposition 5.4. Then the equation $(\Gamma \otimes I_p)X_0 = X_0$ is exactly the equation occurring in Example 6.8 for elements of the subgroup H. Thus for this example,

$$H = \left\{ \Gamma \otimes I_p | \Gamma = \begin{pmatrix} I_p & 0 \\ 0 & \Gamma_{22} \end{pmatrix}, \Gamma_{22} \in \mathcal{O}_{n-p} \right\}.$$

Therefore,

$$\tau((\Gamma \otimes T)H) = (\Gamma \otimes T)X_0 = \Gamma \begin{pmatrix} I_p \\ 0 \end{pmatrix} T'$$

is the representation for elements of $\mathcal{X}$. Obviously,

$$\Gamma \begin{pmatrix} I_p \\ 0 \end{pmatrix} \equiv \Psi \in \mathcal{F}_{p,n}$$

and the representation of elements of $\mathcal{X}$ via the map τ is precisely the representation established in Proposition 5.2. This representation of $\mathcal{X}$ is used on a number of occasions. ◆

6.2. INVARIANT MEASURES AND INTEGRALS

Before beginning a discussion of invariant integrals on locally compact topological groups, we first outline the basic results of integration theory on locally compact topological spaces. Consider a set $\mathcal{X}$ and let $\mathcal{J}$ be a Hausdorff topology for $\mathcal{X}$.

Definition 6.6. The topological space $(\mathcal{X}, \mathcal{J})$ is a *locally compact* space if for each $x \in \mathcal{X}$, there exists a compact neighborhood of x.

Most of the groups introduced in the examples of the previous section are subsets of the space R^m, for some m, and when these groups are given the topology of R^m, they are locally compact spaces. The verification of this is not difficult and is left to the reader. If $(\mathcal{X}, \mathcal{J})$ is a locally compact space, $\mathcal{K}(\mathcal{X})$ denotes the set of all continuous real-valued functions that have compact support. Thus $f \in \mathcal{K}(\mathcal{X})$ if f is a continuous and there is a compact set K such that $f(x) = 0$ if $x \notin K$. It is clear that $\mathcal{K}(\mathcal{X})$ is a real vector space with addition and scalar multiplication being defined in the obvious way.

Definition 6.7. A real-valued function J defined on $\mathcal{K}(\mathcal{X})$ is called an *integral* if:

(i) $J(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 J(f_1) + \alpha_2 J(f_2)$ for $\alpha_1, \alpha_2 \in R$ and $f_1, f_2 \in \mathcal{K}(\mathcal{X})$.
(ii) $J(f) \geqslant 0$ if $f \geqslant 0, f \in \mathcal{K}(\mathcal{X})$.

An integral J is simply a linear function on $\mathcal{K}(\mathcal{X})$ that has the additional property that $J(f)$ is nonnegative when $f \geqslant 0$. Let $\mathcal{B}(\mathcal{X})$ be the σ-algebra generated by the compact subsets of $\mathcal{X}$. If μ is a measure on $\mathcal{B}(\mathcal{X})$ such that $\mu(K) < +\infty$ for each compact set K, it is clear that

$$J(f) \equiv \int_{\mathcal{X}} f(x)\mu(dx)$$

defines an integral on $\mathcal{K}(\mathcal{X})$. Such measures μ are called *Radon measures*. Conversely, given an integral J, there is a measure μ on $\mathcal{B}(\mathcal{X})$ such that $\mu(K) < +\infty$ for all compact sets K and

$$J(f) = \int_{\mathcal{X}} f(x)\mu(dx)$$

for $f \in \mathcal{K}(\mathcal{X})$. For a proof of this result, see Segal and Kunze (1978, Chapter 5). In the special case when $(\mathcal{X}, \mathcal{J})$ is a σ-compact space—that is, $\mathcal{X} = \cup_1^\infty K_i$ where K_i is compact—then the correspondence between integrals J and measures μ that satisfy $\mu(K) < +\infty$ for K compact is one-to-one (see Segal and Kunze, 1978). All of the examples considered here are σ-compact spaces and we freely identify integrals with Radon measures and vice versa.

Now, assume $(\mathcal{X}, \mathcal{J})$ is a σ-compact space. If an integral J on $\mathcal{K}(\mathcal{X})$ corresponds to a Radon measure μ on $\mathcal{B}(\mathcal{X})$, then J has a natural extension to the class of all $\mathcal{B}(\mathcal{X})$-measurable and μ-integrable functions. Namely, J is extended by the equation

$$J(f) = \int_{\mathcal{X}} f(x)\mu(dx)$$

for all f for which the right-hand side is defined. Obviously, the extension of J is unique and is determined by the values of J on $\mathcal{K}(\mathcal{X})$. In many of the examples in this chapter, we use J to denote both an integral on $\mathcal{K}(\mathcal{X})$ and its extension. With this convention, J is defined for any $\mathcal{B}(\mathcal{X})$ measurable function that is μ-integrable where μ corresponds to J.

Suppose G is a group and $\mathcal{J}$ is a topology on G.

Definition 6.8. Given the topology $\mathcal{J}$ on G, $(G, \mathcal{J})$ is a *topological group* if the mapping $(x, y) \to xy^{-1}$ is continuous from $G \times G$ to G. If $(G, \mathcal{J})$ is a topological group and $(G, \mathcal{J})$ is a locally compact topological space, $(G, \mathcal{J})$ is called a *locally compact topological group*

In what follows, all groups under consideration are locally compact topological groups. Examples of such groups include the vector space R^n, the general linear group Gl_n, the affine group Al_n, and G_T^+. The verification that these groups are locally compact topological groups with the Euclidean space topology is left to the reader.

If $(G, \mathcal{J})$ is a locally compact topological group, $\mathcal{K}(G)$ denotes the real vector space of all continuous functions on G that have compact support. For $s \in G$ and $f \in \mathcal{K}(G)$, the *left translate* of f by s, denoted by sf, is defined by $(sf)(x) \equiv f(s^{-1}x)$, $x \in G$. Clearly, $sf \in \mathcal{K}(G)$ for all $s \in G$. Similarly, the *right translate* of $f \in \mathcal{K}(G)$, denoted by fs, is $(fs)(x) \equiv f(xs^{-1})$ and $fs \in \mathcal{K}(G)$.

Definition 6.9. An integral $J \neq 0$ on $\mathcal{K}(G)$ is *left invariant* if $J(sf) = J(f)$ for all $f \in \mathcal{K}(G)$ and $s \in G$. An integral $J \neq 0$ on $\mathcal{K}(G)$ is *right invariant* if $J(fs) = J(f)$ for all $f \in \mathcal{K}(G)$ and $s \in G$.

The basic properties of left and right invariant integrals are summarized in the following two results.

Theorem 6.1. If G is a locally compact topological group, then there exist left and right invariant integrals on $\mathcal{K}(G)$. If J_1 and J_2 are left (right) invariant integrals on $\mathcal{K}(G)$, then $J_2 = cJ_1$ for some positive constant c.

Proof. See Nachbin (1965, Section 4, Chapter 2).

Theorem 6.2. Suppose that

$$J(f) \equiv \int f(x)\mu(dx)$$

is a left invariant integral on $\mathcal{K}(G)$. Then there exists a unique continuous function Δ_r mapping G into $(0, \infty)$ such that

$$\int f(xs^{-1})\mu(dx) = \Delta_r(s)\int f(x)\mu(dx)$$

for all $s \in G$ and $f \in \mathcal{K}(G)$. The function Δ_r, called the *right-hand modulus* of G, also satisfies:

(i) $\Delta_r(st) = \Delta_r(s)\Delta_r(t)$, $s, t \in G$.
(ii) $\int f(x^{-1})\mu(dx) = \int f(x)\Delta_r(x^{-1})\mu(dx)$.

Further, the integral

$$J_1(f) = \int f(x)\Delta_r(x^{-1})\mu(dx)$$

is right invariant.

Proof. See Nachbin (1965, Section 5, Chapter 2).

The two results above establish the existence and uniqueness of right and left invariant integrals and show how to construct right invariant integrals from left invariant integrals via the right-hand modulus Δ_r. The right-hand modulus is a continuous homomorphism from G into $(0, \infty)$—that is, Δ_r is continuous and satisfies $\Delta_r(st) = \Delta_r(s)\Delta_r(t)$, for $s, t \in G$. (The definition of a homomorphism from one group to another group is given shortly.)

Before presenting examples of invariant integrals, it is convenient to introduce relatively left (and right) invariant integrals. Proposition 6.4, given below, provides a useful method for constructing invariant integrals from relatively invariant integrals.

Definition 6.10. A nonzero integral J on $\mathcal{K}(G)$ given by

$$J(f) = \int f(x)m(dx), \qquad f \in \mathcal{K}(G),$$

is called *relatively left invariant* if there exists a function χ on G to $(0, \infty)$ such that

$$\int f(s^{-1}x)m(dx) = \chi(s)\int f(x)m(dx)$$

for all $s \in G$ and $f \in \mathcal{K}(G)$. The function χ is the *multiplier* for J.

It can be shown that any multiplier χ is continuous (see Nachbin, 1965). Further, if J is relatively left invariant with multiplier χ, then for $s, t \in G$ and $f \in \mathcal{K}(G)$,

$$\begin{aligned}\chi(st)\int f(x)m(dx) &= \int f\big((st)^{-1}x\big)m(dx) = \int (tf)(s^{-1}x)m(dx)\\ &= \chi(s)\int (tf)(x)m(dx) = \chi(s)\int f(t^{-1}x)m(dx)\\ &= \chi(s)\chi(t)\int f(x)m(dx).\end{aligned}$$

Thus $\chi(st) = \chi(s)\chi(t)$. Hence all multipliers are continuous and are homomorphisms from G into $(0, \infty)$. For any such homomorphism χ, it is clear that $\chi(e) = 1$ and $\chi(s^{-1}) = 1/\chi(s)$. Also, $\chi(G) = \{\chi(s) | s \in G\}$ is a subgroup of the group $(0, \infty)$ with multiplication as the group operation.

Proposition 6.4. Let χ be a continuous homomorphism on G to $(0, \infty)$.

(i) If $J(f) = \int f(x)\mu(dx)$ is left invariant on $\mathcal{K}(G)$, then

$$J_1(f) \equiv \int f(x)\chi(x)\mu(dx)$$

is a relatively left invariant integral on $\mathcal{K}(G)$ with multiplier χ.

(ii) If $J_1(f) = \int f(x)m(dx)$ is relatively left invariant with multiplier χ, then

$$J(f) \equiv \int f(x)\chi(x^{-1})m(dx)$$

is a left invariant integral.

Proof. The proof is a calculation. For (i),

$$J_1(sf) = \int (sf)(x)\chi(x)\mu(dx) = \int f(s^{-1}x)\chi(ss^{-1}x)\mu(dx)$$

$$= \chi(s)\int f(s^{-1}x)\chi(s^{-1}x)\mu(dx) = \chi(s)\int f(x)\chi(x)\mu(dx)$$

$$= \chi(s)J_1(f).$$

Thus J_1 is relatively left invariant with multiplier χ. For (ii),

$$J(sf) = \int f(s^{-1}x)\chi(x^{-1})m(dx) = \int f(s^{-1}x)\chi(s^{-1}sx^{-1})m(dx)$$

$$= \chi(s^{-1})\int f(s^{-1}x)\chi\left((s^{-1}x)^{-1}\right)m(dx)$$

$$= \chi(s^{-1})\chi(s)\int f(x)\chi(x^{-1})m(dx)$$

$$= \int f(x)\chi(x^{-1})m(dx) = J(f).$$

Thus J is a left invariant integral and the proof is complete. □

If J is a relatively left invariant integral with multiplier χ, say

$$J(x) = \int f(x)m(dx),$$

the measure m is also called relatively left invariant with multiplier χ. A nonzero integral J_1 on $\mathcal{K}(G)$ is *relatively right invariant* with multiplier χ if $J_1(fs) = \chi(s)J_1(f)$. Using the results given above, if J_1 is relatively right invariant with multiplier χ, then J_1 is relatively left invariant with multiplier

χ/Δ_r where Δ_r is the right-hand modulus of G. Thus all relatively right and left invariant integrals can be constructed from a given relatively left (or right) invariant integral once all the continuous homomorphisms are known. Also, if a relatively left invariant measure m can be found and its multiplier χ calculated, then a left invariant measure is given by m/χ according to Proposition 6.4. This observation is used in the examples below.

◆ **Example 6.11.** Consider the group Gl_n of all nonsingular $n \times n$ matrices. Let ds denote Lebesgue measure on Gl_n. Since $Gl_n = \{s|\det(s) \neq 0\}$, Gl_n is a nonempty open subset of n^2-dimensional Euclidean space and hence has positive Lebesgue measure. For $f \in \mathcal{K}(Gl_n)$, let

$$J(f) = \int f(t)\, dt.$$

To find a left invariant measure on Gl_n, it is now shown that $J(sf) = |\det(s)|^n J(f)$ so J is relatively left invariant with multiplier $\chi(s) = |\det(s)|^n$. From Proposition 5.10, the Jacobian of the transformation $g(t) = st$, $s \in Gl_n$, is $|\det(s)|^n$. Thus

$$J(sf) = \int f(s^{-1}t)\, dt = |\det(s)|^n \int f(t)\, dt = |\det(s)|^n J(f).$$

From Proposition 6.4, it follows that the measure

$$\mu(dt) = \frac{dt}{|\det(t)|^n}$$

is a left invariant measure on Gl_n. A similar Jacobian argument shows that μ is also right invariant, so the right-hand modulus of Gl_n is $\Delta_r \equiv 1$. To construct all of the relatively invariant measures on Gl_n, it is necessary that the continuous homomorphisms χ be characterized. For each $\alpha \in R$, let

$$\chi_\alpha(s) = |\det(s)|^\alpha, \qquad s \in Gl_n.$$

Obviously, each χ_α is a continuous homomorphism. However, it can be shown (see the problems at the end of this chapter) that if χ is a continuous homomorphism of Gl_n into $(0, \infty)$, then $\chi = \chi_\alpha$ for some $\alpha \in R$. Hence every relatively invariant measure on Gl_n is given by

$$m(dt) = c\chi_\alpha(t)\frac{dt}{\chi_n(t)}$$

where c is a positive constant and $\alpha \in R$. ◆

A group G for which $\Delta_r = 1$ is called *unimodular*. Clearly, all commutative groups are unimodular as a left invariant integral is also right invariant. In the following example, we consider the group G_T^+, which is not unimodular, but G_T^+ is a subgroup of the unimodular group Gl_n.

◆ **Example 6.12.** Let G_T^+ be the group of all $n \times n$ lower triangular matrices with positive diagonal elements. Thus G_T^+ is a nonempty open subset of $[n(n+1)/2]$-dimensional Euclidean space so G_T^+ has positive Lebesgue measure. Let dt denote $[n(n+1)/2]$-dimensional Lebesgue measure restricted to G_T^+. Consider the integral

$$J(f) \equiv \int f(t)\, dt$$

defined on $\mathcal{K}(G_T^+)$. The Jacobian of the transformation $g(t) = st$, $s \in G_T^+$, is equal to

$$\chi_0(s) \equiv \prod_{i=1}^{n} s_{ii}^{i}$$

where s has diagonal elements $s_{11}, \ldots, s_{nn}$ (see Proposition 5.13). Thus

$$J(sf) = \int f(s^{-1}t)\, dt = \chi_0(s) \int f(t)\, dt = \chi_0(s) J(f).$$

Hence J is relatively left invariant with multiplier χ_0 so the measure

$$\mu(dt) \equiv \frac{dt}{\prod_{i=1}^{n} t_{ii}^{i}} = \frac{dt}{\chi_0(t)}$$

is left invariant. To compute the right-hand modulus Δ_r for G_T^+, let

$$J_1(f) = \int f(t) \mu(dt)$$

so J_1 is left invariant. Then

$$J_1(fs) = \int f(ts^{-1}) \mu(dt) = \int f(ts^{-1}) \frac{dt}{\prod_1^n t_{ii}^{i}} = \int f(ts^{-1}) \frac{dt}{\chi_0(t)}$$

$$= \int f(ts^{-1}) \frac{\chi_0(s^{-1})}{\chi_0(ts^{-1})} dt = \chi_0(s^{-1}) \int \frac{f(ts^{-1})}{\chi_0(ts^{-1})} dt.$$

By Proposition 5.14, the Jacobian of the transform $g(t) = ts$ is

$$\chi_1(s) = \prod_{i=1}^{n} s_{ii}^{n-i+1}.$$

Therefore,

$$J_1(fs) = \chi_0(s^{-1}) \int \frac{f(ts^{-1})}{\chi_0(ts^{-1})} dt = \chi_0(s^{-1})\chi_1(s) \int f(t) \frac{dt}{\chi_0(t)}$$

$$= \frac{\chi_1(s)}{\chi_0(s)} J_1(f).$$

By Theorem 6.2,

$$\Delta_r(s) = \frac{\chi_1(s)}{\chi_0(s)} = \prod_{i=1}^{n} s_{ii}^{n-2i+1}$$

is the right-hand modulus for G_T^+. Therefore, the measure

$$\nu(dt) \equiv \frac{\mu(dt)}{\Delta_r(t)} = \frac{dt}{\chi_0(t)\Delta_r(t)} = \frac{dt}{\prod_{i=1}^{n} t_{ii}^{n-i+1}}$$

is right invariant. As in the previous example, a description of the relatively left invariant measures is simply a matter of describing all the continuous homomorphisms on G_T^+. For each vector $c \in R^n$ with coordinates $c_1, \ldots, c_n$, let

$$\chi_c(t) \equiv \prod_{i=1}^{n} (t_{ii})^{c_i}$$

where $t \in G_T^+$ has diagonal elements $t_{11}, \ldots, t_{nn}$. It is easy to verify that χ_c is a continuous homomorphism on G_T^+. It is known that if χ is a continuous homomorphism on G_T^+, then χ is given by χ_c for some $c \in R^n$ (see Problems 6.4 and 6.9). Thus every relatively left invariant measure on G_T^+ has the form

$$m(dt) = k\chi_c(t)\frac{dt}{\chi_0(t)}$$

for some positive constant k and some vector $c \in R^n$. ◆

The following two examples deal with the affine group and a subgroup of Gl_n related to the group introduced in Example 6.5.

◆ **Example 6.13.** Consider the group Al_n of all affine transformations on R^n. An element of Al_n is a pair (s, x) where $s \in Gl_n$ and $x \in R^n$. Recall that the group operation in Al_n is

$$(s_1, x_1)(s_2, x_2) = (s_1 s_2, s_1 x_2 + x_1)$$

so

$$(s, x)^{-1} = (s^{-1}, -s^{-1}x).$$

Let $ds\, dx$ denote Lebesgue measure restricted to Al_n. In order to construct a left invariant measure on Al_n, it is shown that the integral

$$J(f) \equiv \int f(t, y)\, dt\, dy$$

is relatively left invariant with multiplier

$$\chi_0(s, x) = |\det(s)|^{n+1}.$$

For $(s, x) \in Al_n$,

$$\begin{aligned} J((s, x)f) &= \int f\big((s, x)^{-1}(t, y)\big)\, dt\, dy \\ &= \int f\big((s^{-1}, -s^{-1}x)(t, y)\big)\, dt\, dy \\ &= \int f(s^{-1}t, s^{-1}y - s^{-1}x)\, dt\, dy \\ &= |\det(s)| \int f(s^{-1}t, u)\, dt\, du. \end{aligned}$$

The last equality follows from the change of variable $u = s^{-1}y - sx$, which has a Jacobian $|\det(s)|$. As in Example 6.11,

$$\int_{Gl_n} f(s^{-1}t, u)\, dt = |\det(s)|^n \int_{Gl_n} f(t, u)\, dt$$

for each fixed $u \in R^n$. Thus

$$\begin{aligned} J((s, x)f) &= |\det(s)|^{n+1} \int f(t, u)\, dt\, du = |\det(s)|^{n+1} J(f) \\ &= \chi_0(s, x) J(f) \end{aligned}$$

so J is relatively left invariant with multiplier χ_0. Hence the measure

$$\mu(ds, du) \equiv \frac{ds\,du}{\chi_0(s,u)} = \frac{ds\,du}{|\det(s)|^{n+1}}$$

is left invariant. To find the right-hand modulus of Al_n, let

$$J_1(f) = \int f(t,u)\frac{dt\,du}{\chi_0(t,u)}$$

be a left invariant integral. Then using an argument similar to that above, we have

$$\begin{aligned}
J_1(f(s,x)) &= \int f\big((t,u)(s,x)^{-1}\big)\frac{dt\,du}{\chi_0(t,u)}\\
&= \int f\big((t,u)(s^{-1}, -sx)\big)\frac{dt\,du}{\chi_0(t,u)}\\
&= \int f(ts^{-1}, u - ts^{-1}x)\frac{dt\,du}{|\det(t)|^{n+1}}\\
&= \int f(ts^{-1}, u)\frac{dt\,du}{|\det(t)|^{n+1}}\\
&= |\det(s^{-1})|^{n+1}\int f(ts^{-1}, u)\frac{dt\,du}{|\det(ts^{-1})|^{n+1}}\\
&= |\det(s^{-1})|^{n+1}|\det(s)|^{n}\int f(t,u)\frac{dt\,du}{|\det(t)|^{n+1}}\\
&= |\det(s)|^{-1}J_1(f).
\end{aligned}$$

Thus $\Delta_r(s,x) = |\det(s)|^{-1}$ so a right invariant measure on Al_n is

$$\nu(ds, du) = \frac{1}{\Delta_r(s,u)}\mu(ds, du) = \frac{ds\,du}{|\det(s)|^{n}}.$$

Now, suppose that χ is a continuous homomorphism on Al_n. Since

$$(s,x) = (s,0)(e, s^{-1}x) = (e,x)(s,0)$$

where e is the $n \times n$ identity matrix, χ must satisfy the equation

$$\chi(s, x) = \chi(s, 0)\chi(e, s^{-1}x) = \chi(s, 0)\chi(e, x)$$

Thus for all $s \in Gl_n$,

$$\chi(e, x) = \chi(e, s^{-1}x).$$

Letting s^{-1} converge to the zero matrix, the continuity of χ implies that

$$\chi(e, x) = \chi(e, 0) = 1$$

since $(e, 0)$ is the identity in Al_n. Therefore,

$$\chi(s, x) = \chi(s, 0), \qquad s \in Gl_n.$$

However,

$$\chi((s_1, 0)(s_2, 0)) = \chi((s_1 s_2), 0) = \chi(s_1, 0)\chi(s_2, 0)$$

so χ is a continuous homomorphism on Gl_n. But every continuous homomorphism on Gl_n is given by $s \to |\det(s)|^\alpha$ for some real α. In summary, χ is a continuous homomorphism on Al_n iff

$$\chi(s, x) = |\det(s)|^\alpha$$

for some real number α. Thus we have a complete description of all the relatively invariant integrals on Al_n. ◆

◆ **Example 6.14.** In this example, the group G consists of all the $n \times n$ nonsingular matrices s that have the form

$$s = \begin{pmatrix} s_{11} & s_{12} \\ 0 & s_{22} \end{pmatrix}; \qquad s_{11} \in Gl_p, \qquad s_{22} \in Gl_q$$

where $p + q = n$. Let M be the subspace of R^n consisting of those vectors whose last q coordinates are zero. Then G is the subgroup of Gl_n consisting of those elements s that satisfy $s(M) \subseteq M$. Let $ds_{11}\, ds_{12}\, ds_{22}$ denote Lebesgue measure restricted to G when G is regarded as a subset of $(p^2 + q^2 + pq)$-dimensional Euclidean space. Since G is a nonempty open subset of this space, G has positive Lebesgue measure. As in previous examples, it is shown that the integral

$$J(f) \equiv \int f(t)\, dt_{11}\, dt_{12}\, dt_{22}$$

is relatively left invariant. For $s \in G$,

$$J(sf) = \int f(s^{-1}t)\, dt_{11}\, dt_{12}\, dt_{22}.$$

A bit of calculation shows that

$$\begin{pmatrix} s_{11} & s_{12} \\ 0 & s_{22} \end{pmatrix}^{-1} = \begin{pmatrix} s_{11}^{-1} & -s_{11}^{-1}s_{12}s_{22}^{-1} \\ 0 & s_{22}^{-1} \end{pmatrix}$$

and

$$s^{-1}t = \begin{pmatrix} s_{11}^{-1}t_{11} & s_{11}^{-1}t_{12} - s_{11}^{-1}s_{12}s_{22}^{-1}t_{22} \\ 0 & s_{22}^{-1}t_{22} \end{pmatrix}.$$

Let

$$u_{11} = s_{11}^{-1}t_{11}, \qquad u_{22} = s_{22}^{-1}t_{22}$$

$$u_{12} = s_{11}^{-1}t_{12} - s_{11}^{-1}s_{12}s_{22}^{-1}t_{22}.$$

The Jacobian of this transformation is

$$\chi_0(s) \equiv |\det(s_{11})|^p |\det(s_{22})|^q |\det(s_{11})|^q = |\det(s_{11})|^n |\det(s_{22})|^q.$$

Therefore,

$$J(sf) = \chi_0(s)J(f)$$

so the measure

$$\mu(dt_{11}, dt_{12}, dt_{22}) \equiv \frac{dt_{11}\, dt_{12}\, dt_{22}}{|\det(t_{11})|^n |\det(t_{22})|^q}$$

is left invariant. Setting

$$J_1(f) \equiv \int f(t)\mu(dt_{11}, dt_{12}, dt_{22}),$$

a calculation similar to that above yields

$$J_1(fs) = \Delta_r(s) J_1(f)$$

where

$$\Delta_r(s) = |\det s_{11}|^{-q} |\det s_{22}|^{p}.$$

Thus Δ_r is the right-hand modulus of G and the measure

$$\nu(dt_{11}, dt_{12}, dt_{22}) \equiv \frac{\mu(dt_{11}, dt_{12}, dt_{22})}{\Delta_r(t)} = \frac{dt_{11}\,dt_{12}\,dt_{22}}{|\det(t_{11})|^{p} |\det(t_{22})|^{n}}$$

is right invariant. For $\alpha, \beta \in R$, let

$$\chi_{\alpha\beta}(s) \equiv |\det(s_{11})|^{\alpha} |\det(s_{22})|^{\beta}.$$

Clearly, $\chi_{\alpha\beta}$ is a continuous homomorphism of G into $(0, \infty)$. Conversely, it is not too difficult to show that every continuous homomorphism of G into $(0, \infty)$ is equal to $\chi_{\alpha\beta}$ for some $\alpha, \beta \in R$. Again, this gives a complete description of all the relatively invariant integrals on G. ◆

In the four examples above, the same argument was used to derive the left and right invariant measures, the modular function, and all of the relatively invariant measures. Namely, the group G had positive Lebesgue measure when regarded as a subset of an obvious Euclidean space. The integral on $\mathcal{K}(G)$ defined by Lebesgue measure was relatively left invariant with a multiplier that we calculated. Thus a left invariant measure on G was simply Lebesgue measure divided by the multiplier. From this, the right-hand modulus and a right invariant measure were easily derived. The characterization of the relatively invariant integrals amounted to finding all the solutions to the functional equation $\chi(st) = \chi(s)\chi(t)$ where χ is a continuous function on G to $(0, \infty)$. Of course, the above technique can be applied to many other matrix groups—for example, the matrix group considered in Example 6.5. However, there are important matrix groups for which this argument is not available because the group has Lebesgue measure zero in the "natural" Euclidean space of which the group is a subset. For example, consider the group of $n \times n$ orthogonal matrices $\mathcal{O}_n$. When regarded as a subset of n^2-dimensional Euclidean space, $\mathcal{O}_n$ has Lebesgue measure zero. But, without a fairly complicated parameterization of $\mathcal{O}_n$, it is not possible to regard $\mathcal{O}_n$ as a set of positive Lebesgue measure of some Euclidean space.

For this reason, we do not demonstrate directly the existence of an invariant measure on $\mathcal{O}_n$ in this chapter. In the following chapter, a probabilistic proof of the existence of an invariant measure on $\mathcal{O}_n$ is given.

The group $\mathcal{O}_n$, as well as other groups to be considered later, are in fact compact topological groups. A basic property of such groups is given next.

Proposition 6.5. Suppose G is a locally compact topological group. Then G is compact iff there exists a left invariant probability measure on G.

Proof. See Nachbin (1965, Section 5, Chapter 2). □

The following result shows that when G is compact, left invariant measures are right invariant measures and all relatively invariant measures are in fact invariant.

Proposition 6.6. If G is compact and χ is a continuous homomorphism on G to $(0, \infty)$, then $\chi(s) = 1$ for all $s \in G$.

Proof. Since χ is continuous and G is compact, $\chi(G) = \{\chi(s) | s \in G\}$ is a compact subset of $(0, \infty)$. Since χ is a homomorphism, $\chi(G)$ is a subgroup of $(0, \infty)$. However, the only compact subgroup of $(0, \infty)$ is $\{1\}$. Thus $\chi(s) = 1$ for all $s \in G$. □

The nonexistence of nontrivial continuous homomorphisms on compact groups shows that all compact groups are unimodular. Further, all relatively invariant measures are invariant. Whenever G is compact, the invariant measure on G is always taken to be a probability measure.

6.3. INVARIANT MEASURES ON QUOTIENT SPACES

In this section, we consider the existence and uniqueness of invariant integrals on spaces that are acted on transitively by a group. Throughout this section, $\mathcal{X}$ is a locally compact Hausdorff space and $\mathcal{K}(\mathcal{X})$ denotes the set of continuous functions on $\mathcal{X}$ that have compact support. Also, G is a locally compact topological group that acts on the left of $\mathcal{X}$.

Definition 6.11. The group G acts *topologically* on $\mathcal{X}$ if the function from $G \times \mathcal{X}$ to $\mathcal{X}$ given by $(g, x) \to gx$ is continuous. When G acts topologically on $\mathcal{X}$, $\mathcal{X}$ is a *left homogeneous space* if for each $x \in \mathcal{X}$, the function π_x on G to $\mathcal{X}$ defined by $\pi_x(g) = gx$ is continuous, open, and onto $\mathcal{X}$.

The assumption that each π_x is an onto function is just another way to say that G acts transitively on $\mathcal{X}$. Also, it is not difficult to show that if, for one $x \in \mathcal{X}$, π_x is continuous, open, and onto $\mathcal{X}$, then for all x, π_x is continuous, open, and onto $\mathcal{X}$. To describe the structure of left homogeneous spaces $\mathcal{X}$, fix an element $x_0 \in \mathcal{X}$ and let

$$H_0 = \{g | gx_0 = x_0, g \in G\}.$$

That H_0 is a closed subgroup of G is easily verified. Further, the function τ considered in Proposition 6.3 is now one-to-one, onto, and τ and τ^{-1} are both continuous. Thus we have a one-to-one, onto, bicontinuous mapping between $\mathcal{X}$ and the quotient space G/H_0 endowed with the quotient topology. Conversely, let H be a closed subgroup of G and take $\mathcal{X} = G/H$ with the quotient topology. The group G acts on G/H in the obvious way $(g(g_1H) = gg_1H)$ and it is easily verified that G/H is a left homogeneous space (see Nachbin 1965, Section 3, Chapter 3). Thus we have a complete description of the left homogeneous spaces (up to relabelings by τ) as quotient spaces G/H where H is a closed subgroup of G.

In the notation above, let $\mathcal{X}$ be a left homogeneous space.

Definition 6.12. A nonzero integral J on $\mathcal{K}(\mathcal{X})$

$$J(f) = \int f(x)m(dx), \qquad f \in \mathcal{K}(\mathcal{X})$$

is *relatively invariant* with multiplier χ if, for each $s \in G$,

$$\int f(s^{-1}x)m(dx) = \chi(s)\int f(x)m(dx)$$

for all $f \in \mathcal{K}(\mathcal{X})$.

For $f \in \mathcal{K}(\mathcal{X})$, the function sf given by $(sf)(x) = f(s^{-1}x)$ is the *left translate* of f by $s \in G$. Thus an integral J on $\mathcal{K}(\mathcal{X})$ is relatively invariant with multiplier χ if $J(sf) = \chi(s)J(f)$. For such an integral,

$$\chi(st)J(f) = J((st)f) = J(s(tf)) = \chi(s)J(tf) = \chi(s)\chi(t)J(f)$$

so $\chi(st) = \chi(s)\chi(t)$. Also, any multiplier χ is continuous, which implies that a multiplier is a continuous homomorphism of G into the multiplicative group $(0, \infty)$.

◆ **Example 6.15.** Let $\mathfrak{X}$ be the set of all $p \times p$ positive definite matrices. The group $G = Gl_p$ acts transitively on $\mathfrak{X}$ as shown in Example 6.9. That $\mathfrak{X}$ is a left homogeneous space is easily verified. For $\alpha \in R$, define the measure m_α by

$$m_\alpha(dx) = (\det(x))^{\alpha/2} \frac{dx}{(\det(x))^{(p+1)/2}}$$

where dx is Lebesgue measure on $\mathfrak{X}$. Let $J_\alpha(f) \equiv \int f(x) m_\alpha(dx)$. For $s \in Gl_p$, $s(x) = sxs'$ is the group action on $\mathfrak{X}$. Therefore,

$$\begin{aligned}
J_\alpha(sf) &= \int f(s^{-1}(x)) m_\alpha(dx) \\
&= \int f(s^{-1}xs'^{-1})(\det(x))^{\alpha/2} \frac{dx}{(\det(x))^{(p+1)/2}} \\
&= |\det(s)|^\alpha \int f(s^{-1}xs'^{-1}) \det(s^{-1}xs'^{-1})^{\alpha/2} \frac{dx}{(\det(x))^{(p+1)/2}} \\
&= |\det(s)|^\alpha \int f(x)(\det(x))^{\alpha/2} \frac{dx}{(\det(x))^{(p+1)/2}}.
\end{aligned}$$

The last equality follows from the change of variable $x = sys'$, which has a Jacobian equal to $|\det(s)|^{p+1}$ (see Proposition 5.11). Hence

$$J_\alpha(sf) = |\det(s)|^\alpha J(f)$$

for all $s \in Gl_p$, $f \in \mathcal{K}(\mathfrak{X})$, and J_α is relatively invariant with multiplier $\chi_\alpha(s) = |\det(s)|^\alpha$. For this example, it has been shown that for every continuous homomorphism χ on G, there is a relatively invariant integral with multiplier χ. That this is not the case in general is demonstrated in future examples. ◆

The problem of the existence and uniqueness of relatively invariant integrals on left homogeneous spaces $\mathfrak{X}$ is completely solved in the following result due to Weil (see Nachbin, 1965, Section 4, Chapter 3). Recall that x_0 is a fixed element of $\mathfrak{X}$ and

$$H_0 = \{g | gx_0 = x_0, g \in G\}$$

is a closed subgroup of G. Let Δ_r denote the right-hand modulus of G and let Δ_r^0 denote the right-hand modulus of H_0.

Theorem 6.3. In the notation above:

(i) If $J(f) = \int f(x)m(dx)$ is relatively invariant with multiplier χ, then

$$\Delta_r^0(h) = \chi(h)\Delta_r(h) \quad \text{for all } h \in H_0.$$

(ii) If χ is a continuous homomorphism of G to $(0, \infty)$ that satisfies $\Delta_r^0(h) = \chi(h)\Delta_r(h)$, $h \in H_0$, then a relatively invariant integral with multiplier χ exists.

(iii) If J_1 and J_2 are relatively invariant with the same multiplier, then there exists a constant $c > 0$ such that $J_2 = cJ_1$.

Before turning to applications of Theorem 6.3, a few general comments are in order. If the subgroup H_0 is compact, then $\Delta_r^0(h) = 1$ for all $h \in H_0$. Since the restrictions of χ and of Δ_r to H_0 are both continuous homomorphisms on H_0, $\Delta_r(h) = \chi(h) = 1$ for all $h \in H_0$ as H_0 is compact. Thus when H_0 is compact, any continuous homomorphism χ is a multiplier for a relatively invariant integral and the description of all the relatively invariant integrals reduces to finding all the continuous homomorphisms of G. Further, when G is compact, then only an invariant integral on $\mathcal{K}(X)$ can exist as $\chi \equiv 1$ is the only continuous homomorphism. When G and H are not compact, the situation is a bit more complicated. Both Δ_r and Δ_r^0 must be calculated and then, the continuous homomorphisms χ on G to $(0, \infty)$ that satisfy (ii) of Theorem 6.3 must be found. Only then do we have a description of the relatively invariant integrals on $\mathcal{K}(X)$. Of course, the condition for the existence of an invariant integral ($\chi \equiv 1$) is that $\Delta_r^0(h) = \Delta_r(h)$ for all $h \in H_0$.

If J is a relatively invariant integral (with multiplier χ) given by

$$J(f) = \int f(x)m(dx), \qquad f \in \mathcal{K}(\mathfrak{X}),$$

then the measure m is called relatively invariant with multiplier χ. In Example 6.15, it was shown that for each $\alpha \in R$, the measure m_α was relatively invariant under Gl_p with multiplier χ_α. Theorem 6.3 implies that any relatively invariant measure on the space of $p \times p$ positive definite matrices is equal to a positive constant times an m_α for some $\alpha \in R$. We now proceed with further examples.

◆ **Example 6.16.** Let $\mathfrak{X} = \mathfrak{F}_{p,n}$ and let $G = \mathcal{O}_n$. It was shown in Example 6.8 that $\mathcal{O}_n$ acts transitively on $\mathfrak{F}_{p,n}$. The verification that

$\mathfrak{F}_{p,n}$ is a left homogeneous space is left to the reader. Since $\mathcal{O}_n$ is compact, Theorem 6.3 implies that there is a unique probability measure μ on $\mathfrak{F}_{p,n}$ that is invariant under the action of $\mathcal{O}_n$ on $\mathfrak{F}_{p,n}$. Also, any relatively invariant measure on $\mathfrak{F}_{p,n}$ will be equal to a positive constant times μ. The distribution μ is sometimes called the *uniform distribution* on $\mathfrak{F}_{p,n}$. When $p = 1$, then

$$\mathfrak{F}_{1,n} = \{x | x \in R^n, \|x\| = 1\},$$

which is the rim of the unit sphere in R^n. The uniform distribution on $\mathfrak{F}_{1,n}$ is just surface Lebesgue measure normalized so that it is a probability measure. When $p = n$, then $\mathfrak{F}_{n,n} = \mathcal{O}_n$ and μ is the uniform distribution on the orthogonal group. A different argument, probabilistic in nature, is given in the next chapter, which also establishes the existence of the uniform distribution on $\mathfrak{F}_{p,n}$. ◆

◆ **Example 6.17.** Take $\mathfrak{X} = R^p - \{0\}$ and let $G = Gl_p$. The action of Gl_p on $\mathfrak{X}$ is that of a matrix acting on a vector and this action is obviously transitive. The verification that $\mathfrak{X}$ is a left homogeneous space is routine. Consider the integral

$$J(f) = \int f(x)\,dx, \qquad f \in \mathcal{K}(\mathfrak{X})$$

where dx is Lebesgue measure on $\mathfrak{X}$. For $s \in Gl_p$, it is clear that $J(sf) = |\det(s)| J(f)$ so J is relatively invariant with multiplier $\chi_1(s) = |\det(s)|$. We now show that J is the only relatively invariant integral on $\mathcal{K}(\mathfrak{X})$. This is done by proving that χ_1 is the only possible multiplier for relatively invariant integrals on $\mathcal{K}(X)$. A convenient choice of $x_0 \in \mathfrak{X}$ is $x_0 = \varepsilon_1$ where $\varepsilon_1' = (1, 0, \ldots, 0)$. Then

$$H_0 = \{h | h\varepsilon_1 = \varepsilon_1, h \in Gl_p\}.$$

A bit of reflection shows that $h \in H_0$ iff

$$h = \begin{pmatrix} 1 & h_{12} \\ 0 & h_{22} \end{pmatrix}$$

where $h_{22} \in Gl_{(p-1)}$ and h_{12} is $1 \times (p-1)$. A calculation similar to that in Example 6.14 yields

$$\mu(dh_{12}, dh_{22}) = \frac{dh_{12}\, dh_{22}}{|\det(h_{22})|^{p-1}}$$

as a left invariant measure on H_0. Then the integral

$$J_1(f) \equiv \int f(h)\mu(dh_{12}, dh_{22})$$

is left invariant on $\mathcal{K}(H_0)$ and a standard Jacobian argument yields

$$J_1(fh) = \Delta_r^0(h) J_1(f), \qquad f \in \mathcal{K}(H_0)$$

where

$$\Delta_r^0(h) = |\det(h_{22})|, \qquad h \in H_0.$$

Every continuous homomorphism on Gl_p has the form $\chi_\alpha(s) = |\det(s)|^\alpha$ for some $\alpha \in R$. Since $\Delta_r = 1$ for Gl_p, χ_α can be a multiplier for an invariant integral iff

$$\Delta_r^0(h) = \chi_\alpha(h), \qquad h \in H_0.$$

But $\Delta_r^0(h) = |\det(h_{22})|$ and for $h \in H_0$, $\chi_\alpha(h) = |\det(h_{22})|^\alpha$ so the only value for α for which χ_α can be a multiplier is $\alpha = 1$. Further, the integral J is relatively invariant with multiplier χ_1. Thus Lebesgue measure on $\mathfrak{X}$ is the only (up to a positive constant) relatively invariant measure on $\mathfrak{X}$ under the action of Gl_p. ◆

Before turning to the next example, it is convenient to introduce the direct product of two groups. If G_1 and G_2 are groups, the *direct product* of G_1 and G_2, denoted by $G \equiv G_1 \times G_2$, is the group consisting of all pairs (g_1, g_2) with $g_i \in G_i$, $i = 1, 2$, and group operation

$$(g_1, g_2)(h_1, h_2) \equiv (g_1 h_1, g_2 h_2).$$

If e_i is the identity in G_i, $i = 1, 2$, then (e_1, e_2) is the identity in G and $(g_1, g_2)^{-1} = (g_1^{-1}, g_2^{-1})$. When G_1 and G_2 are locally compact topological groups, then $G_1 \times G_2$ is a locally compact topological group when endowed with the product topology. The next two results describe all the continuous homomorphisms and relatively left invariant measures on $G_1 \times G_2$ in terms of continuous homomorphisms and relatively left invariant measures on G_1 and G_2.

Proposition 6.7. Suppose G_1 and G_2 are locally compact topological groups. Then χ is a continuous homomorphism on $G_1 \times G_2$ iff $\chi((g_1, g_2)) = \chi_1(g_1)\chi_2(g_2)$, $(g_1, g_2) \in G_1 \times G_2$, where χ_i is a continuous homomorphism on G_i, $i = 1, 2$.

Proof. If $\chi((g_1, g_2)) = \chi_1(g_1)\chi_2(g_2)$, clearly χ is a continuous homomorphism on $G_1 \times G_2$. Conversely, since $(g_1, g_2) = (g_1, e_2)(e_1, g_2)$, if χ is a continuous homomorphism on $G_1 \times G_2$, then

$$\chi((g_1, g_2)) = \chi(g_1, e_2)\chi(e_1, g_2).$$

Setting $\chi_1(g_1) = \chi(g_1, e_2)$ and $\chi_2(g_2) = \chi(e_1, g_2)$, the desired result follows. □

Proposition 6.8. Suppose χ is a continuous homomorphism on $G_1 \times G_2$ with $\chi(g_1, g_2) = \chi_1(g_1)\chi_2(g_2)$ where χ_i is a continuous homomorphism on G_i, $i = 1, 2$. If m is a relatively left invariant measure with multiplier χ, then there exist relatively left invariant measures m_i on G_i with multipliers χ_i, $i = 1, 2$, and m is product measure $m_1 \times m_2$. Conversely, if m_i is a relatively left invariant measure on G_i with multiplier χ_i, $i = 1, 2$, then $m_1 \times m_2$ is a relatively left invariant measure on $G_1 \times G_2$ with multiplier χ, which satisfies $\chi(g_1, g_2) = \chi_1(g_1)\chi_2(g_2)$.

Proof. This result is a direct consequence of Fubini's Theorem and the existence and uniqueness of relatively left invariant integrals. □

The following example illustrates many of the results presented in this chapter and has a number of applications in multivariate analysis. For example, one of the derivations of the Wishart distribution is quite easy given the results of this example.

◆ **Example 6.18.** As in Example 6.10, $\mathfrak{X}$ is the set of all $n \times p$ matrices with rank p and G is the direct product group $\mathcal{O}_n \times G_T^+$. The action of $(\Gamma, T) \in \mathcal{O}_n \times G_T^+$ on $\mathfrak{X}$ is

$$(\Gamma, T)X \equiv (\Gamma \otimes T)X = \Gamma X T', \qquad X \in \mathfrak{X}.$$

Since $\mathfrak{X} = \{X | X \in \mathcal{L}_{p,n}, \det(X'X) > 0\}$, $\mathfrak{X}$ is a nonempty open subset of $\mathcal{L}_{p,n}$. Let dX be Lebesgue measure on $\mathcal{X}$ and define a measure on $\mathcal{X}$ by

$$m(dX) = \frac{dX}{(\det(X'X))^{n/2}}.$$

Using Proposition 5.10, it is an easy calculation to show that the integral

$$J(f) \equiv \int f(X)m(dX)$$

is invariant—that is, $J((\Gamma, T)f) = J(f)$ for $(\Gamma, T) \in \mathcal{O}_n \times G_T^+$ and $f \in \mathcal{K}(\mathcal{X})$. However, it takes a bit more work to characterize all the relatively invariant measures on $\mathcal{X}$. First, it was shown in Example 6.10 that, if X_0 is

$$X_0 = \begin{pmatrix} I_p \\ 0 \end{pmatrix} \in \mathcal{X},$$

then $H_0 = \{(\Gamma, T)|(\Gamma, T)X_0 = X_0\}$ is a closed subgroup of $\mathcal{O}_n$ and hence is compact. By Theorem 6.3, every continuous homomorphism on $\mathcal{O}_n \times G_T^+$ is the multiplier for a relatively invariant integral. But every continuous homomorphism χ on $\mathcal{O}_n \times G_T^+$ has the form $\chi(\Gamma, T) = \chi_1(\Gamma)\chi_2(T)$ where χ_1 and χ_2 are continuous homomorphisms on $\mathcal{O}_n$ and G_T^+. Since $\mathcal{O}_n$ is compact, $\chi_1 = 1$. From Example 6.12,

$$\chi_2(T) = \prod_{i=1}^{p} (t_{ii})^{c_i} \equiv \chi_c(T)$$

where $c \in R^p$ has coordinates $c_1, \dots, c_p$. Now that all the possible multipliers have been described, we want to exhibit the relatively invariant integrals on $\mathcal{K}(\mathcal{X})$. To this end, consider the space $\mathcal{Y} = \mathcal{F}_{p,n} \times G_U^+$ so points in $\mathcal{Y}$ are (Ψ, U) where Ψ is an $n \times p$ linear isometry and U is a $p \times p$ upper triangular matrix in G_U^+. The group $\mathcal{O}_n \times G_T^+$ acts transitively on $\mathcal{Y}$ under the group action

$$(\Gamma, T)(\Psi, U) \equiv (\Gamma\Psi, UT').$$

Let μ_0 be the unique probability measure on $\mathcal{F}_{p,n}$ that is $\mathcal{O}_n$-invariant and let ν_r be the particular right invariant measure on the group G_U^+

given by

$$\nu_r(dU) = \frac{dU}{\prod_{i=1}^{p} u_{ii}^{i}}.$$

Obviously, the integral

$$J_1(f) \equiv \int\int f(\Psi, U)\mu_0(d\Psi)\nu_r(dU)$$

is invariant under the action of $\mathcal{O}_n \times G_T^+$ on $\mathcal{F}_{p,n} \times G_U^+$, $f \in \mathcal{K}(\mathcal{F}_{p,n} \times G_U^+)$. Consider the integral

$$J_2(f) \equiv \int\int f(\Psi, U)\chi_c(U')\mu_0(d\Psi)\nu_r(dU)$$

defined on $\mathcal{K}(\mathcal{F}_{p,n} \times G_U^+)$ where χ_c is a continuous homomorphism on G_T^+. The claim is that $J_2((\Gamma, T)f) = \chi_c(T)J_2(f)$ so J_2 is relatively invariant with multiplier χ_c. To see this, compute as follows:

$$\begin{aligned}
J_2((\Gamma, T)f) &= \int\int f\big((\Gamma, T)^{-1}(\Psi, U)\big)\chi_c(U')\mu_0(d\Psi)\nu_r(dU) \\
&= \int\int f(\Gamma'\Psi, UT'^{-1})\chi_c(TT^{-1}U')\mu_0(d\Psi)\nu_r(dU) \\
&= \chi_c(T)\int\int f(\Gamma'\Psi, UT'^{-1})\chi_c\big((UT'^{-1})'\big)\mu_0(d\Psi)\nu_r(dU) \\
&= \chi_c(T)J_2(f).
\end{aligned}$$

The last equality follows from the invariance of μ_0 and ν_r. Thus all the relatively invariant integrals on $\mathcal{K}(\mathcal{F}_{p,n} \times G_U^+)$ have been explicitly described. To do the same for $\mathcal{K}(\mathcal{X})$, the basic idea is to move the integral J_2 over to $\mathcal{K}(\mathcal{X})$. It was mentioned earlier that the map ϕ_0 on $\mathcal{F}_{p,n} \times G_U^+$ to $\mathcal{X}$ given by

$$\phi_0(\Psi, U) = \Psi U \in \mathcal{X}$$

is one-to-one, onto, and satisfies

$$\phi_0((\Gamma, T)(\Psi, U)) = (\Gamma, T)\phi_0(\Psi, U),$$

for group elements (Γ, T). For $f \in \mathcal{K}(\mathcal{X})$, consider the integral

$$J_3(f) \equiv \int\int f(\phi_0(\Psi, U))\mu_0(d\Psi)\nu_r(dU).$$

Then for $(\Gamma, T) \in \mathcal{O}_n \times G_T^+$,

$$J_3((\Gamma, T)f) = \int\int f\big((\Gamma, T)^{-1}\phi_0(\Psi, U)\big)\mu_0(d\Psi)\nu_r(dU)$$

$$= \int\int f\big((\Gamma', T^{-1})\phi_0(\mu, U)\big)\mu_0(d\Psi)\nu_r(dU)$$

$$= \int\int f\big(\phi_0(\Gamma'\Psi, UT'^{-1})\big)\mu_0(d\Psi)\nu_r(dU) = J_3(f)$$

since μ_0 and ν_r are invariant. Therefore, J_3 is an invariant integral on $\mathcal{K}(\mathcal{X})$. Since J is also an invariant integral on $\mathcal{K}(\mathcal{X})$, Theorem 6.3 shows that there is a positive constant k such that

$$J(f) = kJ_3(f), \qquad f \in \mathcal{K}(\mathcal{X}).$$

More explicitly, we have the equation

$$\int f(X)\frac{dX}{|X'X|^{n/2}} = k\int\int f(\Psi U)\mu_0(d\Psi)\nu_r(dU)$$

for all $f \in \mathcal{K}(\mathcal{X})$. This equation is a formal way to state the very nontrivial fact that the measure m on $\mathcal{X}$ gets transformed into the measure $k(\mu_0 \times \nu_r)$ on $\mathcal{F}_{p,n} \times G_U^+$ under the mapping ϕ_0^{-1}. To evaluate the constant k, it is sufficient to find one particular function so that both sides of the above equality can be evaluated. Consider

$$f_0(X) = |X'X|^{n/2}(2\pi)^{-np/2}\exp\left[-\tfrac{1}{2}\operatorname{tr}(X'X)\right].$$

Clearly,

$$\int f_0(X)\frac{dX}{|X'X|^{n/2}} = 1$$

so

$$\frac{1}{k} = \int\int f_0(\Psi U)\mu_0(d\Psi)\nu_r(dU)$$

$$= (2\pi)^{-np/2}\int |U'U|^{n/2}\exp\left[-\frac{1}{2}\operatorname{tr} U'U\right]\nu_r(dU)$$

$$= (2\pi)^{-np/2}\int \prod_1^p u_{ii}^{n-i}\exp\left[-\frac{1}{2}\sum_{i\leqslant j} u_{ij}^2\right] dU$$

$$= (2\pi)^{-np/2}2^{-p}c(n, p).$$

The last equality follows from the result in Example 5.1, where $c(n, p)$ is defined. Therefore,

$$(6.1)\quad \int f(X)\frac{dX}{|X'X|^{n/2}} = \frac{(2\pi)^{np/2}2^p}{c(n, p)}\int\int f(\Psi U)\mu_0(d\Psi)\nu_r(dU).$$

It is now an easy matter to derive all the relatively invariant integrals on $\mathcal{K}(\mathcal{X})$. Let χ_c be a given continuous homomorphism on G_T^+. For each $X \in \mathcal{X}$, let $U(X)$ be the unique element in G_U^+ such that $X = \Psi U(X)$ for some $\Psi \in \mathcal{F}_{p,n}$ (see Proposition 5.2). It is clear that $U(\Gamma X T') = U(X)T'$ for $\Gamma \in \mathcal{O}_n$ and $T \in G_T^+$. We have shown that

$$J_2(f) = \int\int f(\Psi, U)\chi_c(U')\mu_0(d\Psi)\nu_r(dU)$$

is relatively invariant with multiplier χ_c on $\mathcal{K}(\mathcal{F}_{p,n} \times G_U^+)$. For $h \in \mathcal{K}(X)$, define an integral J_4 by

$$J_4(h) = \int\int h(\Psi U)\chi_c(U')\mu_0(d\Psi)\nu_r(dU).$$

Clearly, J_4 is relatively invariant with multiplier χ_c since $J_4(h) = J_2(\tilde{h})$ where $\tilde{h}(\Psi, U) \equiv h(\Psi U)$. Now, we move J_4 over to $\mathcal{X}$ by (6.1). In (6.1), take $f(X) = h(X)\chi_c(U'(X))$ so $f(\Psi U) = h(\Psi U)\chi_c(U')$. Thus the integral

$$J_5(h) = \int h(X)\chi_c(U'(X))\frac{dX}{|X'X|^{n/2}}$$

is relatively invariant with multiplier χ_c. Of course, any relatively invariant integral with multiplier χ_c on $\mathcal{K}(\mathcal{X})$ is equal to a positive constant times J_5. ◆

6.4. TRANSFORMATIONS AND FACTORIZATIONS OF MEASURES

The results of Example 6.18 describe how an invariant measure on the set of $n \times p$ matrices is transformed into an invariant measure on $\mathcal{F}_{p,n} \times G_U^+$ under a particular mapping. The first problem to be discussed in this section is an abstraction of this situation. The notion of a group homomorphism plays a role in what follows.

Definition 6.13. Let G and H be groups. A function η from G onto H is a *homomorphism* if:

(i) $\eta(g_1 g_2) = \eta(g_1)\eta(g_2)$, $g_1, g_2 \in G$.
(ii) $\eta(g) = (\eta(g))^{-1}$, $g \in G$.

When there is a homomorphism from G to H, H is called a *homomorphic image* of G.

For notational convenience, a homomorphic image of G is often denoted by $\bar{G}$ and the value of the homomorphism at g is $\bar{g}$. In this case, $\overline{g_1 g_2} = \bar{g}_1 \bar{g}_2$ and $\overline{g^{-1}} = \bar{g}^{-1}$. Also, if e is the identity in G, then $\bar{e}$ is the identity in $\bar{G}$.

Suppose $\mathcal{X}$ and $\mathcal{Y}$ are locally compact spaces, and G and $\bar{G}$ are locally compact topological groups that act topologically on $\mathcal{X}$ and $\mathcal{Y}$, respectively. It is assumed that $\bar{G}$ is a homomorphic image of G.

Definition 6.14. A measurable function ϕ from $\mathcal{X}$ onto $\mathcal{Y}$ is called *equivariant* if $\phi(gx) = \bar{g}\phi(x)$ for all $g \in G$ and $x \in \mathcal{X}$.

Now, consider an integral

$$J(f) = \int f(x)\mu(dx), \qquad f \in \mathcal{K}(\mathcal{X}),$$

which is invariant under the action of G on $\mathcal{X}$, that is

$$J(gf) \equiv \int f(g^{-1}x)\mu(dx) = \int f(x)\mu(dx) = J(f)$$

for $g \in G$ and $f \in \mathcal{K}(\mathcal{X})$. Given an equivariant function ϕ from $\mathcal{X}$ to $\mathcal{Y}$, there is a natural measure ν induced on $\mathcal{Y}$. Namely, if B is a measurable subset of $\mathcal{Y}$, $\nu(B) \equiv \mu(\phi^{-1}(B))$. The result below shows that under a regularity condition on ϕ, the measure ν defines an invariant (under $\overline{G}$) integral on $\mathcal{K}(\mathcal{Y})$.

Proposition 6.9. If ϕ is an equivariant function from $\mathcal{X}$ onto $\mathcal{Y}$ that satisfies $\mu(\phi^{-1}(K)) < +\infty$ for all compact sets $K \subseteq \mathcal{Y}$, then the integral

$$J_1(f) \equiv \int f(y)\nu(dy), \qquad f \in \mathcal{K}(\mathcal{Y})$$

is invariant under $\overline{G}$.

Proof. First note that J_1 is well defined and finite since $\mu(\phi^{-1}(K)) < +\infty$ for all compact sets $K \subseteq \mathcal{Y}$. From the definition of the measure ν, it follows immediately that

$$J_1(f) = \int f(y)\nu(dy) = \int f(\phi(x))\mu(dx), \qquad f \in \mathcal{K}(\mathcal{Y}).$$

Using the equivariance of ϕ and the invariance of μ, we have

$$J_1(\bar{g}f) = \int f(\bar{g}^{-1}y)\nu(dy) = \int f(\bar{g}^{-1}\phi(x))\mu(dx)$$

$$= \int f(\phi(g^{-1}x))\mu(dx) = \int f(\phi(x))\mu(dx) = J_1(f)$$

so J_1 is invariant under $\overline{G}$. □

Before presenting some applications of Proposition 6.9, a few remarks are in order. The groups G and $\overline{G}$ are not assumed to act transitively on $\mathcal{X}$ and $\mathcal{Y}$, respectively. However, if $\overline{G}$ does act transitively on $\mathcal{Y}$ and if $\mathcal{Y}$ is a left homogeneous space, then the measure ν is uniquely determined up to a positive constant. Thus if we happen to know an invariant measure on $\mathcal{Y}$, the identity

$$\int f(y)\nu(dy) = \int f(\phi(x))\mu(dx), \qquad f \in \mathcal{K}(\mathcal{Y})$$

relates the G-invariant measure μ to the $\overline{G}$-invariant measure ν. It was this line of reasoning that led to (6.1) in Example 6.18. We now consider some further examples.

◆ **Example 6.19.** As in Example 6.18, let $\mathcal{X}$ be the set of all $n \times p$ matrices of rank p, and let $\mathcal{Y}$ be the space $\mathbb{S}_p^+$ of $p \times p$ positive definite matrices. Consider the map ϕ on $\mathcal{X}$ to $\mathbb{S}_p^+$ defined by

$$\phi(X) = X'X, \qquad X \in \mathcal{X}.$$

The group $\mathcal{O}_n \times Gl_p$ acts on $\mathcal{X}$ by

$$(\Gamma, A)X = (\Gamma \otimes A)X = \Gamma X A'$$

and the measure

$$\mu(dX) = \frac{dX}{|X'X|^{n/2}}$$

is invariant under $\mathcal{O}_n \times Gl_p$. Further,

$$\phi((\Gamma, A)X) = AX'XA' = A\phi(X)A',$$

and this defines an action of Gl_p on $\mathbb{S}_p^+$. It is routine to check that the mapping

$$(\Gamma, A) \to A \equiv \overline{(\Gamma, A)}$$

is a homomorphism. Obviously,

$$\phi((\Gamma, A)X) = \overline{(\Gamma, A)}\phi(X)$$

since the action of Gl_p on $\mathbb{S}_p^+$ is

$$A(S) = ASA'; \qquad S \in \mathbb{S}_p^+, \qquad A \in Gl_p.$$

Since Gl_p acts transitively on $\mathbb{S}_p^+$, the invariant measure

$$\nu_1(dS) = \frac{dS}{|S|^{(p+1)/2}}$$

is unique up to a positive constant. The remaining assumption to verify in order to apply Proposition 6.9 is that $\phi^{-1}(K)$ has finite μ measure for compact sets $K \subseteq \mathbb{S}_p^+$. To do this, we show that

$\phi^{-1}(K)$ is compact in $\mathcal{X}$. Recall that the mapping h on $\mathcal{F}_{p,n} \times S_p^+$ onto $\mathcal{X}$ given by

$$h(\Psi, S) = \Psi S \in \mathcal{X}$$

is one-to-one and is obviously continuous. Given the compact set $K \subseteq \mathcal{S}_p^+$, let

$$K_1 = \{S | S \in \mathcal{S}_p^+, S^2 \in K\}.$$

Then K_1 is compact so $\mathcal{F}_{p,n} \times K_1$ is a compact subset of $\mathcal{F}_{p,n} \times \mathcal{S}_p^+$. It is now routine to show that

$$\phi^{-1}(K) = \{X | X'X \in K\} = h(\mathcal{F}_{p,n} \times K_1),$$

which is compact since h is continuous and the continuous image of a compact set is compact. By Proposition 6.9, we conclude that the measure $\nu = \mu \circ \phi^{-1}$ is invariant under Gl_p and satisfies

$$\int_{\mathcal{X}} f(X'X) \frac{dX}{|X'X|^{n/2}} = \int_{\mathcal{S}_p^+} f(S) \nu(dS),$$

for all $f \in \mathcal{K}(\mathcal{S}_p^+)$. Since ν is invariant under Gl_p, $\nu = c\nu_1$ where c is a positive constant. Thus we have the identity

$$\int f(X'X) \frac{dX}{|X'X|^{n/2}} = c \int f(S) \frac{dS}{|S|^{(p+1)/2}}. \tag{6.2}$$

To find the constant c, it is sufficient to evaluate both sides of (6.2) for a particular function f_0. For f_0, take the function

$$f_0(S) = (\sqrt{2\pi})^{-np} |S|^{n/2} \exp[-\tfrac{1}{2} \operatorname{tr} S],$$

so

$$f_0(X'X) = (\sqrt{2\pi})^{-np} |X'X|^{n/2} \exp[-\tfrac{1}{2} \operatorname{tr} X'X].$$

Clearly, the left-hand side of (6.2) integrates to one and this yields the equation

$$c \int (\sqrt{2\pi})^{-np} |S|^{(n-p-1)/2} \exp[-\tfrac{1}{2} \operatorname{tr} S] \, dS = 1.$$

The result of Example 5.1 gives

$$c(\sqrt{2\pi})^{-np}c(n, p) = 1$$

so

$$c = \frac{(\sqrt{2\pi})^{np}}{c(n, p)} = (\sqrt{2\pi})^{np}\omega(n, p).$$

In conclusion, the identity

$$(6.3)\quad \int_{\mathfrak{X}} f(X'X)\frac{dX}{|X'X|^{n/2}} = (\sqrt{2\pi})^{np}\omega(n, p)\int_{\mathbb{S}_p^+} f(S)\frac{dS}{|S|^{(p+1)/2}}$$

has been established for all $f \in \mathcal{K}(\mathbb{S}_p^+)$, and thus for all measurable f for which either side exists. ◆

◆ **Example 6.20.** Again let $\mathfrak{X}$ be the set of $n \times p$ matrices of rank p so the group $\mathcal{O}_n \times G_T^+$ acts on $\mathfrak{X}$ by

$$(\Gamma, T)X \equiv (\Gamma \otimes T)X = \Gamma X T'.$$

Each element $X \in \mathfrak{X}$ has a unique representation $X = \Psi U$ where $\Psi \in \mathfrak{F}_{p,n}$ and $U \in G_U^+$. Define ϕ on $\mathfrak{X}$ onto G_U^+ by defining $\phi(X)$ to be the unique element $U \in G_U^+$ such that $X = \Psi U$ for some $\Psi \in \mathfrak{F}_{p,n}$. If $\phi(X) = U$, then $\phi((\Gamma, T)X) = UT'$, since when $X = \Psi U$, $(\Gamma, T)X = \Gamma\Psi UT'$. This implies that UT' is the unique element in G_U^+ such that $X = (\Gamma\Psi)UT'$ as $\Gamma\Psi \in \mathfrak{F}_{p,n}$. The mapping $(\Gamma, T) \to T \equiv \overline{(\Gamma, T)}$ is clearly a homomorphism of (Γ, T) onto G_T^+ and the action of G_T^+ on G_U^+ is

$$T(U) \equiv UT'; \qquad U \in G_U^+, T \in G_T^+.$$

Therefore, $\phi((\Gamma, T)X) = \overline{(\Gamma, T)}\phi(X)$ so ϕ is equivariant. The measure

$$\mu(dX) = \frac{dX}{|X'X|^{n/2}}$$

is $\mathcal{O}_n \times G_T^+$ invariant. To show that $\phi^{-1}(K)$ has finite μ measure when $K \subseteq G_U^+$ is compact, note that $h(\Psi, U) \equiv \Psi U$ is a continuous function on $\mathfrak{F}_{p,n} \times G_U^+$ onto $\mathfrak{X}$. It is easily verified that

$$\phi^{-1}(K) = h(\mathfrak{F}_{p,n} \times K).$$

But $\mathcal{F}_{p,n} \times K$ is compact, which shows that $\phi^{-1}(K)$ is compact since h is continuous. Thus $\mu(\phi^{-1}(K)) < +\infty$. Proposition 6.9 shows that $\nu \equiv \mu \circ \phi^{-1}$ is a G_T^+-invariant measure on G_U^+ and we have the identity

$$\int_{\mathfrak{X}} f(\phi(X)) \frac{dX}{|X'X|^{n/2}} = \int_{G_U^+} f(U) \nu(dU)$$

for all $f \in \mathcal{K}(G_U^+)$. However, the measure

$$\nu_1(dU) \equiv \frac{dU}{\prod_{i=1}^p u_{ii}^i}$$

is a right invariant measure on G_U^+, and therefore, ν_1 is invariant under the transitive action of G_T^+ on G_U^+. The uniqueness of invariant measures implies that $\nu = c\nu_1$ for some positive constant c and

$$\int_{\mathfrak{X}} f(\phi(X)) \frac{dX}{|X'X|^{n/2}} = c\int_{G_U^+} f(U) \frac{dU}{\prod_1^p u_{ii}^i}.$$

The constant c is evaluated by choosing f to be

$$f(U) = (\sqrt{2\pi})^{-np} |U'U|^{n/2} \exp[-\tfrac{1}{2} \operatorname{tr} U'U].$$

Since $(\phi(X))'\phi(X) = X'X$,

$$f(\phi(X)) = (\sqrt{2\pi})^{-np} |X'X|^{n/2} \exp[-\tfrac{1}{2} \operatorname{tr} X'X]$$

and

$$\int f(\phi(X)) \frac{dX}{|X'X|^{n/2}} = 1.$$

Therefore,

$$\begin{aligned} 1 &= c(\sqrt{2\pi})^{-np} \int_{G_U^+} |U'U|^{n/2} \exp\left[-\frac{1}{2} \operatorname{tr} U'U\right] \frac{dU}{\prod_1^p u_{ii}^i} \\ &= c(\sqrt{2\pi})^{-np} \int_{G_U^+} \prod_1^p u_{ii}^{n-i} \exp\left[-\frac{1}{2} \operatorname{tr} U'U\right] dU \\ &= c(\sqrt{2\pi})^{-np} 2^{-p} c(n, p) \end{aligned}$$

where $c(n, p)$ is defined in Example 5.1. This yields the identity

$$\int f(\phi(X)) \frac{dX}{|X'X|^{n/2}} = 2^p (\sqrt{2\pi})^{np} \omega(n, p) \int f(U) \frac{dU}{\prod_1^p u_{ii}^i}$$

for all $f \in \mathcal{K}(G_U^+)$. In particular, when $f(U) = f_1(U'U)$, we have

$$(6.4) \quad \int f_1(X'X) \frac{dX}{|X'X|^{n/2}} = 2^p (\sqrt{2\pi})^{np} \omega(n, p) \int f_1(U'U) \frac{dU}{\prod_1^p u_{ii}^i}$$

whenever either integral exists. Combining this with (6.3) yields the identity

$$(6.5) \qquad \int_{\mathbb{S}_p^+} f(S) \frac{dS}{|S|^{(p+1)/2}} = 2^p \int_{G_p^+} f(U'U) \frac{dU}{\prod_1^p u_{ii}^i}$$

for all measurable f for which either integral exists. Setting $T = U'$ in (6.5) yields the assertion of Proposition 5.18. ◆

The final topic in this chapter has to do with the factorization of a Radon measure on a product space. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are locally compact and σ-compact Hausdorff spaces and assume that G is a locally compact topological group that acts on $\mathcal{X}$ in such a way that $\mathcal{X}$ is a homogeneous space. It is also assumed that μ_1 is a G-invariant Radon measure on $\mathcal{X}$ so the integral

$$J_1(f_1) \equiv \int f_1(x) \mu_1(dx), \qquad f_1 \in \mathcal{K}(\mathcal{X})$$

is G-invariant, and is unique up to a positive constant.

Proposition 6.10. Assume the conditions above on $\mathcal{X}$, $\mathcal{Y}$, G, and J_1. Define G acting on the locally compact and σ-compact space $\mathcal{X} \times \mathcal{Y}$ by $g(x, y) = (gx, y)$. If m is a G-invariant Radon measure on $\mathcal{X} \times \mathcal{Y}$, then $m = \mu_1 \times \nu$ for some Radon measure ν on $\mathcal{Y}$.

Proof. By assumption, the integral

$$J(f) \equiv \iint_{\mathcal{Y}\mathcal{X}} f(x, y) m(dx, dy), \qquad f \in \mathcal{K}(\mathcal{X} \times \mathcal{Y})$$

satisfies

$$J(gf) = \iint_{\mathcal{Y}\mathcal{X}} f(g^{-1}x, y)m(dx, dy) = J(f).$$

For $f_2 \in \mathcal{K}(\mathcal{Y})$ and $f_1 \in \mathcal{K}(\mathcal{X})$, the product f_1f_2, defined by $(f_1f_2)(x, y) = f_1(x)f_2(y)$, is in $\mathcal{K}(\mathcal{X} \times \mathcal{Y})$ and

$$J(f_1f_2) = \iint_{\mathcal{Y}\mathcal{X}} f_1(x)f_2(y)m(dx, dy).$$

Fix $f_2 \in \mathcal{K}(\mathcal{Y})$ such that $f_2 \geqslant 0$ and let

$$H(f_1) \equiv \iint_{\mathcal{Y}\mathcal{X}} f_1(x)f_2(y)m(dx, dy), \qquad f_1 \in \mathcal{K}(\mathcal{X}).$$

Since $J(gf) = J(f)$, it follows that

$$H(gf_1) = H(f_1) \quad \text{for } g \in G \text{ and } f_1 \in \mathcal{K}(\mathcal{X}).$$

Therefore H is a G-invariant integral on $\mathcal{K}(\mathcal{X})$. Hence there exists a non-negative constant $c(f_2)$ depending on f_2 such that

$$H(f_1) = c(f_2)J_1(f_1)$$

and $c(f_2) = 0$ iff $H(f_1) = 0$ for all $f_1 \in \mathcal{K}(\mathcal{X})$. For an arbitrary $f_2 \in \mathcal{K}(\mathcal{Y})$, write $f_2 = f_2^+ - f_2^-$ where $f_2^+ = \max(f_2, 0)$ and $f_2^- = \max(-f_2, 0)$ are in $\mathcal{K}(\mathcal{Y})$. For such an f_2, it is easy to show

$$J(f_1f_2) = c(f_2^+)J_1(f_1) - c(f_2^-)J_1(f_1) = \big(c(f_2^+) - c(f_2^-)\big)J_1(f_1).$$

Thus defining c on $\mathcal{K}(\mathcal{Y})$ by $c(f_2) = c(f_2^+) - c(f_2^-)$, it is easy to show that c is an integral on $\mathcal{K}(\mathcal{Y})$. Hence

$$c(f_2) = \int_{\mathcal{Y}} f_2(y)\nu(dy)$$

for some Radon measure ν. Therefore,

$$\iint_{\mathcal{Y}\mathcal{X}} f_1(x)f_2(y)m(dx, dy) = \iint_{\mathcal{Y}\mathcal{X}} f_1(x)f_2(y)\mu_1(dx)\nu(dy).$$

A standard approximation argument now implies that m is the product measure $\mu_1 \times \nu$. □

Proposition 6.10 provides one technique for establishing the stochastic independence of two random vectors. This technique is used in the next chapter. The one application of Proposition 6.10 given here concerns the space of positive definite matrices.

◆ **Example 6.21.** Let $\mathcal{Z}$ be the set of all $p \times p$ positive definite matrices that have distinct eigenvalues. That $\mathcal{Z}$ is an open subset of $\mathcal{S}_p^+$ follows from the fact that the eigenvalues of $S \in \mathcal{S}_p^+$ are continuous functions of the elements of the matrix $\mathcal{S}$. Thus $\mathcal{Z}$ has nonzero Lebesgue measure in $\mathcal{S}_p^+$. Also, let $\mathcal{Y}$ be the set of $p \times p$ diagonal matrices Y with diagonal elements $y_1, \dots, y_p$ that satisfy $y_1 > y_2 > \cdots > y_p$. Further, let $\mathcal{X}$ be the quotient space $\mathcal{O}_p/\mathcal{D}_p$ where $\mathcal{D}_p$ is the group of sign changes introduced in Example 6.6. We now construct a natural one-to-one onto map from $\mathcal{X} \times \mathcal{Y}$ to $\mathcal{Z}$. For $X \in \mathcal{X}$, $X = \Gamma\mathcal{D}_p$ for some $\Gamma \in \mathcal{O}_p$. Define ϕ by

$$\phi(X, Y) = \Gamma Y \Gamma', \qquad X = \Gamma\mathcal{D}_p,\ Y \in \mathcal{Y}.$$

To verify that ϕ is well defined, suppose that $X = \Gamma_1\mathcal{D}_p = \Gamma_2\mathcal{D}_p$. Then

$$\phi(X, Y) = \Gamma_1 Y \Gamma_1' = \Gamma_2\Gamma_2'\Gamma_1 Y \Gamma_1'\Gamma_2\Gamma_2' = \Gamma_2 Y \Gamma_2'$$

since $\Gamma_2'\Gamma_1 \in \mathcal{D}_p$ and every element $D \in \mathcal{D}_p$ satisfies $DYD = Y$ for all $Y \in \mathcal{Y}$. It is clear that $\phi(X, Y)$ has ordered eigenvalues $y_1 > y_2 > \cdots > y_p > 0$, the diagonal elements of Y. Clearly, the function ϕ is onto and continuous. To show ϕ is one-to-one, first note that, if Y is any element of $\mathcal{Y}$, then the equation

$$\Gamma Y \Gamma' = Y, \qquad \Gamma \in \mathcal{O}_p$$

implies that $\Gamma \in \mathcal{D}_p$ ($\Gamma Y \Gamma' = Y$ implies that $\Gamma Y = Y\Gamma$ and equating the elements of these two matrices shows that Γ must be diagonal so $\Gamma \in \mathcal{D}_p$). If

$$\phi(X_1, Y_1) = \phi(X_2, Y_2),$$

then $Y_1 = Y_2$ by the uniqueness of eigenvalues and the ordering of the diagonal elements of $Y \in \mathcal{Y}$. Thus

$$\Gamma_1 Y_1 \Gamma_1' = \Gamma_2 Y_1 \Gamma_2'$$

when

$$\phi(X_1, Y_1) = \phi(X_2, Y_1).$$

Therefore,

$$\Gamma_2'\Gamma_1 Y_1 \Gamma_1' \Gamma_2 = Y_1,$$

which implies that $\Gamma_2'\Gamma_1 \in \mathfrak{D}_p$. Since $X_i = \Gamma_i \mathfrak{D}_p$ for $i = 1, 2$, this shows that $X_1 = X_2$ and that ϕ is one-to-one. Therefore, ϕ has an inverse and the spectral theorem for matrices specifies just what ϕ^{-1} is. Namely, for $Z \in \mathfrak{Z}$, let $y_1 > \cdots > y_p > 0$ be the ordered eigenvalues of Z and write Z as

$$Z = \Gamma Y \Gamma', \qquad \Gamma \in \mathcal{O}_p$$

where $Y \in \mathcal{Y}$ has diagonal elements $y_1 > \cdots > y_p > 0$. The problem is that $\Gamma \in \mathcal{O}_p$ is not unique since

$$\Gamma Y \Gamma' = \Gamma D Y D \Gamma' \quad \text{for } D \in \mathfrak{D}_p.$$

To obtain uniqueness, we simply have "quotiented out" the subgroup $\mathfrak{D}_p$ in order that ϕ^{-1} be well defined. Now, let

$$\mu(dZ) = dZ$$

be Lebesgue measure on $\mathfrak{Z}$ and consider $\nu = \mu \circ \phi$—the induced measure on $\mathcal{X} \times \mathcal{Y}$. The problem is to obtain some information about the measure ν. Since ϕ is continuous, ν is a Radon measure on $\mathcal{X} \times \mathcal{Y}$, and ν satisfies

$$\iint f(X, Y)\nu(dX, dY) = \int f(\phi^{-1}(Z))\, dZ$$

for $f \in \mathcal{K}(\mathcal{X} \times \mathcal{Y})$. The claim is that the measure ν is invariant under the action of $\mathcal{O}_p$ on $\mathcal{X} \times \mathcal{Y}$ defined by

$$\Gamma(X, Y) = (\Gamma X, Y).$$

To see this, we have

$$\iint f(\Gamma'(X, Y))\nu(dX, dY) = \int f(\Gamma'\phi^{-1}(Z))\, dZ.$$

But a bit of reflection shows that $\Gamma'\phi^{-1}(Z) = \phi^{-1}(\Gamma' Z \Gamma)$. Since the Jacobian of the transformation $\Gamma' Z \Gamma$ is equal to one, it follows that ν is $\mathcal{O}_p$-invariant. By Proposition 6.10, the measure ν is a product measure $\nu_1 \times \nu_2$ where ν_1 is an $\mathcal{O}_p$-invariant measure on $\mathfrak{X}$. Since $\mathcal{O}_p$ is compact and $\mathfrak{X}$ is compact, the measure ν_1 is finite and we take $\nu_1(\mathfrak{X}) = 1$ as a normalization. Therefore,

$$\int f(\phi^{-1}(Z))\, dZ = \int\int f(X, Y)\nu_1(dX)\nu_2(dY)$$

for all $f \in \mathcal{K}(\mathfrak{X} \times \mathfrak{Y})$. Setting $h = f\phi^{-1}$ yields

$$\int h(Z)\, dZ = \int\int h(\phi(X, Y))\nu_1(dX)\nu_2(dY)$$

for $h \in \mathcal{K}(\mathfrak{Z})$. In particular, if $h \in \mathcal{K}(\mathfrak{Z})$ satisfies $h(Z) = h(\Gamma Z \Gamma')$ for all $\Gamma \in \mathcal{O}_p$ and $Z \in \mathfrak{Z}$, then $h(\phi(X, Y)) = h(Y)$ and we have the identity

$$\int h(Z)\, dZ = \int h(Y)\nu_2(dY).$$

It is quite difficult to give a rigorous derivation of the measure ν_2 without the theory of differential forms. In fact, it is not obvious that ν_2 is absolutely continuous with respect to Lebesgue measure on $\mathfrak{Y}$. The subject of this example is considered again in later chapters. ◆

PROBLEMS

1. Let M be a proper subspace for V and set

$$G(M) = \{g | g \in Gl(V), g(M) = M\}$$

where $g(M) = \{x | x = gv \text{ for some } v \in M\}$.

(i) Show that $g(M) = M$ iff $g(M) \subseteq M$ for $g \in Gl(V)$ and show that $G(M)$ is a group.

Now, assume $V = R^p$ and, for $x \in R^p$, write $x = \binom{y}{z}$ with $y \in R^q$ and $z \in R^r$, $q + r = p$. Let $M = \{x | x = \binom{y}{0}, y \in R^q\}$.

(ii) For $g \in Gl_p$, partition g as

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \qquad g_{11} \text{ is } q \times q.$$

Show that $g \in G(M)$ iff $g_{11} \in Gl_q$, $g_{22} \in Gl_r$, and $g_{21} = 0$. For such g show that

$$g^{-1} = \begin{pmatrix} g_{11}^{-1} & -g_{11}^{-1} g_{12} g_{22}^{-1} \\ 0 & g_{22}^{-1} \end{pmatrix}.$$

(iii) Verify that $G_1 = \{g \in G(M) | g_{11} = I_q, g_{12} = 0\}$ and $G_2 = \{g \in G(M) | g_{22} = I_r\}$ are subgroups of $G(M)$ and G_2 is a normal subgroup of $G(M)$.

(iv) Show that $G_1 \cap G_2 = \{I\}$ and show that each g can be written uniquely as $g = hk$ with $h \in G_1$ and $k \in G_2$. Conclude that, if $g_i = h_i k_i$, $i = 1, 2$, then $g_1 g_2 = h_3 k_3$, where $h_3 = h_1 h_2$ and $k_3 = h_2^{-1} k_1 h_2 k_2$, is the unique representation of $g_1 g_2$ with $h_3 \in G_1$ and $k_3 \in G_2$.

2. Let $G(M)$ be as in Problem 1. Does $G(M)$ act transitively on $V - \{0\}$? Does $G(M)$ act transitively on $V \cap M^c$ where M^c is the complement of the set M in V?

3. Show that $\mathcal{O}_n$ is a compact subset of R^m with $m = n^2$. Show that $\mathcal{O}_n$ is a topological group when $\mathcal{O}_n$ has the topology inherited from R^m. If χ is a continuous homomorphism from $\mathcal{O}_n$ to the multiplicative group $(0, \infty)$, show that $\chi(\Gamma) = 1$ for all $\Gamma \in \mathcal{O}_n$.

4. Suppose χ is a continuous homomorphism on $(0, \infty)$ to $(0, \infty)$. Show that $\chi(x) = x^\alpha$ for some real number α.

5. Show that $\mathcal{O}_n$ is a compact subgroup of Gl_n and show that G_U^+ (of dimension $n \times n$) is a closed subgroup of Gl_n. Show that the uniqueness of the representation $A = \Gamma U$ ($A \in Gl_n$, $\Gamma \in \mathcal{O}_n$, $U \in G_U^+$) is equivalent to $\mathcal{O}_n \cap G_U^+ = \{I_n\}$. Show that neither $\mathcal{O}_n$ nor G_U^+ is a normal subgroup of Gl_n.

6. Let $(V, (\cdot, \cdot))$ be an inner product space.
   (i) For fixed $v \in V$, show that χ defined by $\chi(x) = \exp[(v, x)]$ is a continuous homomorphism on V to $(0, \infty)$. Here V is a group under addition.

(ii) If χ is a continuous homomorphism on V, show that $\chi(x) = \log \chi(x)$ is a linear function on V. Conclude that $\chi(x) = \exp[(v, x)]$ for some $v \in V$.

7. Suppose χ is a continuous homomorphism defined on Gl_n to $(0, \infty)$. Using the steps outlined below, show that $\chi(A) = |\det A|^{\alpha}$ for some real α.
   (i) First show that $\chi(\Gamma) = 1$ for $\Gamma \in \mathcal{O}_n$.
   (ii) Write $A = \Gamma D \Delta$ with $\Gamma, \Delta \in \mathcal{O}_n$ and D diagonal with positive diagonals $\lambda_1, \ldots, \lambda_n$. Show that $\chi(A) = \chi(D)$.
   (iii) Next, write $D = \prod D_i(\lambda_i)$ where $D_i(c)$ is diagonal with all diagonal elements equal to one except the ith diagonal element, which is c. Conclude that $\chi(D) = \prod \chi(D_i(\lambda_i))$.
   (iv) Show that $D_i(c) = PD_1(c)P'$ for some permutation matrix $P \in \mathcal{O}_n$. Using this, show that $\chi(D) = \chi(D_1(\lambda))$ where $\lambda = \prod \lambda_i$.
   (v) For $\lambda \in (0, \infty)$, set $\xi(\lambda) = \chi(D_1(\lambda))$ and show that ξ is a continuous homomorphism on $(0, \infty)$ to $(0, \infty)$ so $\xi(\lambda) = \lambda^{\beta}$ for some real β. Now, complete the proof of $\chi(A) = |\det A|^{\alpha}$.

8. Let $\mathfrak{X}$ be the set of all rank r orthogonal projections on R^n to R^n $(1 \leqslant r \leqslant n - 1)$.
   (i) Show that $\mathcal{O}_n$ acts transitively on $\mathfrak{X}$ via the action $x \to \Gamma x \Gamma'$, $\Gamma \in \mathcal{O}_n$. For

$$x_0 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{X}.$$

   what is the isotropy subgroup? Show that the representation of x in this case is $x = \psi\psi'$ where $\psi : n \times r$ consists of the first r columns of $\Gamma \in \mathcal{O}_n$.
   (ii) The group $\mathcal{O}_r$ acts on $\mathfrak{F}_{r,n}$ by $\psi \to \psi\Delta'$, $\Delta \in \mathcal{O}_r$. This induces an equivalence relation on $\mathfrak{F}_{r,n}$ ($\psi_1 \cong \psi_2$ iff $\psi_1 = \psi_2\Delta'$ for some $\Delta \in \mathcal{O}_r$), and hence defines a quotient space. Show that the map $[\psi] \to \psi\psi'$ defines a one-to-one onto map from this quotient space to $\mathfrak{X}$. Here $[\psi]$ is the equivalence class of ψ.

9. Following the steps outlined below, show that every continuous homomorphism on G_T^+ to $(0, \infty)$ has the form $\chi(T) = \prod_1^p (t_{ii})^{c_i}$ where $T : p \times p$ has diagonal elements $t_{11}, \ldots, t_{pp}$ and $c_1, \ldots, c_p$ are real numbers.

(i) Let

$$G_1 = \left\{ T | T = \begin{pmatrix} T_{11} & 0 \\ 0 & 1 \end{pmatrix}, T_{11} : (p-1) \times (p-1) \right\}$$

and

$$G_2 = \left\{ T | T = \begin{pmatrix} I_{p-1} & 0 \\ T_{21} & t_{pp} \end{pmatrix} \right\}.$$

Show that G_1 and G_2 are subgroups of G_T^+ and G_2 is normal. Show that every T has a unique representation as $T = hk$ with $h \in G_1$, $k \in G_2$.

(ii) An induction assumption yields $\chi(h) = \prod_1^{p-1}(t_{ii})^{c_i}$. Also for $T = hk$, $\chi(T) = \chi(h)\chi(k)$.

(iii) Show that $\chi(k) = (t_{pp})^{c_p}$ for some real c_p.

10. Evaluate the integral $I_\gamma = \int |X'X|^\gamma \exp[-\frac{1}{2} \operatorname{tr} X'X]\, dX$ where X ranges over all $n \times p$ matrices of rank p. In particular, for what values of γ is this integral finite?

11. In the notation of Problems 1 and 2, find all of the relatively invariant integrals on $R^p \cap M^c$ under the action of $G(M)$.

12. In R^n, let $\mathcal{X} = \{x | x \in R^n, x \notin \operatorname{span}\{e\}\}$. Also, let $S_{n-1}(e) = \{x | \|x\| = 1, x \in R^n, x'e = 0\}$ and let $\mathcal{Y} = R^1 \times (0, \infty) \times S_{n-1}(e)$. For $x \in \mathcal{X}$, set $\bar{x} = n^{-1}e'x$ and set $s^2(x) = \Sigma(x_i - \bar{x})^2$. Define a mapping τ on $\mathcal{X}$ to $\mathcal{Y}$ by $\tau(x) = \{\bar{x}, s, (x - \bar{x}e)/s\}$.

(i) Show that τ is one-to-one, onto and find τ^{-1}. Let $\mathcal{O}_n(e) = \{\Gamma | \Gamma \in \mathcal{O}_n, \Gamma e = e\}$ and consider a group G defined by $G = \{(a, b, \Gamma) | a \in (0, \infty),\ b \in R^1,\ \Gamma \in \mathcal{O}_n(e)\}$ with group composition given by $(a_1, b_1, \Gamma_1)(a_2, b_2, \Gamma_2) = (a_1a_2, a_1b_2 + b_1, \Gamma_1\Gamma_2)$. Define G acting on $\mathcal{X}$ and $\mathcal{Y}$ by $(a, b, \Gamma)x = a\Gamma x + be$, $x \in \mathcal{X}$, $(a, b, \Gamma)(u, v, w) = (au + b, av, \Gamma w)$ for $(u, v, w) \in \mathcal{Y}$.

(ii) Show that $\tau(gx) = g\tau(x)$, $g \in G$.

(iii) Show that the measure $\mu(dx) = dx/s^n$ is an invariant measure on $\mathcal{X}$.

(iv) Let $\gamma(dw)$ be the unique $\mathcal{O}_n(e)$ invariant probability measure on $S_{n-1}(e)$. Show that the measure

$$\nu(d(u, v, w)) = du \frac{dv}{v^2} \gamma(dw)$$

is an invariant measure on $\mathcal{Y}$.

(v) Prove that $\int_{\mathcal{X}} f(x)\mu(dx) = k\int_{\mathcal{Y}} f(\tau^{-1}(y))\nu(dy)$ for all integrable f where k is a fixed constant. Find k.

(vi) Suppose a random vector $X \in \mathcal{X}$ has a density (with respect to dx) given by

$$f(x) = \frac{1}{\sigma^n} h\left(\frac{\|x - \delta e\|^2}{\sigma^2}\right), \qquad x \in \mathcal{X}$$

where $\delta \in R^1$ and $\sigma > 0$ are parameters. Find the joint density of $\bar{X}$ and s.

13. Let $\mathcal{X} = R^n - \{0\}$ and consider $X \in \mathcal{X}$ with an $\mathcal{O}_n$-invariant distribution. Define ϕ on $\mathcal{X}$ to $(0, \infty) \times \mathcal{F}_{1,n}$ by $\phi(x) = (\|x\|, x/\|x\|)$. The group $\mathcal{O}_n$ acts on $(0, \infty) \times \mathcal{F}_{1,n}$ by $\Gamma(u, v) = (u, \Gamma v)$. Show that $\phi(\Gamma x) = \Gamma\phi(x)$ and use this to prove that:

(i) $\|X\|$ and $X/\|X\|$ are independent.

(ii) $X/\|X\|$ has a uniform distribution on $\mathcal{F}_{1,n}$.

14. Let $\mathcal{X} = \{x \in R^n | x_i \neq x_j \text{ for all } i \neq j\}$ and let $\mathcal{Y} = \{y \in R^n | y_1 < y_2 < \cdots < y_n\}$. Also, let $\mathcal{P}_n$ be the group of $n \times n$ permutation matrices so $\mathcal{P}_n \subseteq \mathcal{O}_n$ and $\mathcal{P}_n$ acts on $\mathcal{X}$ by $x \to gx$.

(i) Show that the map $\phi(g, y) = gy$ is one-to-one and onto from $\mathcal{P}_n \times \mathcal{Y}$ to $\mathcal{X}$. Describe ϕ^{-1}.

(ii) Let $X \in \mathcal{X}$ be a random vector such that $\mathcal{L}(X) = \mathcal{L}(gX)$ for $g \in \mathcal{P}_n$. Write $\phi^{-1}(X) = (P(X), Y(X))$ where $P(X) \in \mathcal{P}_n$ and $Y(X) \in \mathcal{Y}$. Show that $P(X)$ and $Y(X)$ are independent and that $P(X)$ has a uniform distribution on $\mathcal{P}_n$.

NOTES AND REFERENCES

1. For an alternative to Nachbin's treatment of invariant integrals, see Segal and Kunze (1978).

2. Proposition 6.10 is the Radon measure version of a result due to Farrell (see Farrell, 1976). The extension of Proposition 6.10 to relatively invariant integrals that are unique up to constant is immediate—the proof of Proposition 6.10 is valid.

3. For the form of the measure ν_2 in Example 6.21, see Deemer and Olkin (1951), Farrell (1976), or Muirhead (1982).