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A weak functional framework for applications in statistics

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Abstract: For the problem of estimating a general loss of the form $c(\|x-\theta\|^2)$, Stein's identity is particularly relevant in deriving unbiased estimators of loss when x is used as an estimate of θ and is distributed as $\mathcal{N}_p(\theta, I)$, and when c is the identity function. In [3], Fourdrinier and Lepelletier show that extensions to other distributions (actually, to spherically symmetric distributions) and to general functions c are conceivable, but through another approach involving a Green's formula. Somewhat surprisingly, the statistical context induces an unusual weak functional framework. The main goal of this paper is to present such an analytic context.

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1. Introduction: a statistical motivation

Let X be a random vector in $\mathbb{R}^p$ from a distribution P_θ where the unknown parameter θ is a fixed vector in $\mathbb{R}^p$. A basic statistical problem is to estimate θ using an estimator, which is a function of X, say $\varphi(X)$, under a loss function $L(\theta, \varphi(X))$. Actually, the evaluation of the estimator φ is made through the risk function $R(\theta, \varphi) = E_\theta[L(\theta, \varphi(X))]$ (where E_θ denotes the expectation with respect to P_θ).

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However it is often of interest to assess the loss $L(\theta, \varphi(X))$ itself and a wide literature is devoted to this subject. See Johnstone [7] for a rationale, Lu and Berger [9], and Fourdrinier and Wells [6] for more details on this approach and Fourdrinier and Strawderman [4] for a Bayesian perspective. Assessment of the loss $L(\theta, \varphi(X))$ is usually made through the use of a function of X, say $\gamma(X)$, inducing a new type of estimator, γ, called a loss estimator. For evaluating the precision of γ, another loss is required and it has become standard to use the squared error

$$L^*\Big(\theta, \varphi(X), \gamma(X)\Big) = \Big(\gamma(X) - L\left(\theta, \varphi(X)\right)\Big)^2 .$$

More precisely this evaluation is done through the new quadratic risk function

$$\mathcal{R}(\theta, \varphi, \gamma) = E_\theta\Big[L^*(\theta, \varphi(X), \gamma(X))\Big] = E_\theta\Big[\Big(\gamma(X) - L(\theta, \varphi(X))\Big)^2\Big].$$

Here we consider the general assumption that P_θ is a spherically symmetric distribution around θ and, more specifically, that P_θ has a density with respect to the Lebesgue measure on $\mathbb{R}^p$ of the form $x \mapsto f(\|x-\theta\|^2)$ for some function f. Furthermore, as an estimator φ of θ, we consider the usual estimator $\varphi(X) = X$ under a loss of the form

$$L\left(\theta, \varphi(X)\right) = L(\theta, X) = c(\|X-\theta\|^2) \tag{1.1}$$

for a certain function c.

As a first estimate of $c(\|X-\theta\|^2)$, a simple estimator is the constant (and hence unbiased) estimator $\gamma_0 = E_0\left[c(\|X\|^2)\right]$. In the Gaussian case (that is, $P_\theta = \mathcal{N}_p(\theta, I)$ where I is the $p\times p$ identity matrix) and when c is the identity function, we clearly have $\gamma_0 = p$. In that setting, Johnstone [7] considers alternative estimators γ of the form $\gamma(X) = p + s(X)$ and yields an unbiased estimator δ of the loss of γ (in the sense that $E_\theta[\delta(X)] = \mathcal{R}(\theta, X, \gamma)$) through a repeated use of Stein's identity [10].

Indeed, denoting by $\cdot$ the usual inner product in $\mathbb{R}^p$, Stein's lemma states that

$$E_\theta\left[(X-\theta)\cdot g(X)\right] = E_\theta\left[\operatorname{div} g(X)\right] \tag{1.2}$$

for any suitable function g from $\mathbb{R}^p$ into $\mathbb{R}^p$ (for which, in an appropriate sense, the divergence $\operatorname{div} g(x) = \sum_{i=1}^p \partial_i g_i(x)$ exists) provided the above expectations exist (see further for more details on the conditions on g). As a consequence, for any suitable function s from $\mathbb{R}^p$ into $\mathbb{R}$, we have

$$E_\theta\left[\|X-\theta\|^2 s(X)\right] = E_\theta\left[p\, s(X) + \Delta s(X)\right], \tag{1.3}$$

using (1.2) with $g(X) = (X-\theta)s(X)$ (here $\Delta s(x) = \sum_{i=1}^p \partial_{ii} s(x)$ is the Laplacian of s). Hence, expanding the terms in the expression of the risk of $\gamma = p + s$, we have

$$\begin{aligned} \mathcal{R}(\theta, X, \gamma) &= E_\theta\left[(p + s(X) - \|X-\theta\|)^2\right] \\ &= E_\theta\left[(p - \|X-\theta\|^2) + 2\left(p - \|X-\theta\|^2\right)s(X) + s^2(X)\right] \end{aligned} \tag{1.4}$$

and it follows that

$$\mathcal{R}(\theta, X, \gamma) = E_\theta\left[2p - 2\Delta s(X) + s^2(X)\right]. \tag{1.5}$$

Equality (1.5) means that $\delta(X) = 2p - 2\Delta s(X) + s^2(X)$ is an unbiased estimator of the loss and leads to a simple sufficient condition of domination of γ over $\gamma_0 = p$, that is,

$$-2\Delta s(x) + s^2(x) \leq 0 \tag{1.6}$$

for any $x \in \mathbb{R}^p$, with a strict inequality on a set of positive Lebesgue measure. Actually, the above condition implies that $R(\theta, X, \gamma) < R(\theta, X, p)$, for every $\theta \in \mathbb{R}^p$.

Such a situation does not recur when we depart from the normal case and, all the more, when c is not necessarily the identity function. Although Fourdrinier and Wells [6] could construct improved estimators $\gamma = \gamma_0 + s$, in the framework of spherically symmetric distributions, using an *ad hoc* version of Stein's identity, they had to use more involved arguments to exhibit a sufficient domination condition similar to (1.6).

Note that the case where c is not the identity function is more complicated. Recently, Fourdrinier and Lepelletier [3] gave formal improvements of $\gamma = \gamma_0 + s$ over γ_0 (that is, with no recourse to simulations) in the case where c is an indicator function, say $\mathbb{1}[0, c_\alpha]$ (for some constant c_α associated to a fixed number $\alpha \in [0, 1]$). This corresponds to a confidence level estimation of a confidence region with nominal confidence coefficient $1 - \alpha$, and should be interpreted as a loss estimation problem in a wide sense.

For the general estimation problem of $c(\|X - \theta\|^2)$ under a spherical density $x \mapsto f(\|x - \theta\|^2)$, a calculus analogue to (1.4) shows that the risk expression of $\gamma = \gamma_0 + s$ is

$$\mathcal{R}(\theta, \gamma, X) = V_0 + \delta_\theta \tag{1.7}$$

where V_0 is the variance of $c(\|X - \theta\|^2)$ when $\theta = 0$ and δ_θ is the risk difference

$$\delta_\theta = E_\theta\Big[2\left(\gamma_0 - c(\|X - \theta\|^2\right) s(X) + s^2(X)\Big]. \tag{1.8}$$

The approach used in [3] consists in introducing the Laplacian of s in δ_θ and in using a Green's formula of the form

$$\int_{\mathbb{R}^p} u(x)\, \Delta v(x)\, dx = \int_{\mathbb{R}^p} v(x)\, \Delta u(x)\, dx \tag{1.9}$$

for functions u and v satisfying suitable weak conditions. Note that these conditions are imposed by the statistical problem.

The paper is organized as follows. In Section 2, we specify the spaces where the functions u and v live and give accurate conditions for the Green formula (1.9) to be valid. In Section 3, we make a link with Stoke's theorem. In particular, we show that the spaces used in Section 2 are the appropriate spaces for deriving Stein type identities. We specify, in Section 4, how our result applies in the statistical context described in Section 1. In Section 5, we give some conclusions and, finally, Section 6 is an appendix gathering analytic material (in particular, on Sobolev spaces) which underlies to a Green's formula in our context.

2. A Green's formula in a weak functional framework

Our purpose, in this section, is to yield some weak frameworks for functions with little regularity and in which a Green's formula makes sense. Our main result, consisting in a Green's formula of the type (1.9) under weak analytic conditions, is given in Theorem 2.5. For the material on functional spaces used in this section, we refer to the appendix.

Let us first recall the well known Green formula (the second Green formula) when the functions under consideration are regular enough and when the integration domain is a Lipschitz domain (see [2] *e.g.*).

Theorem 2.1. *Let Ω be a Lipschitz domain with outward unit normal ν. Let u and v two vectors fields whose components are in $\mathcal{C}^2(\Omega)$ and such that their normal derivatives $\dfrac{\partial u}{\partial \nu} = \nabla u \cdot \nu$ and $\dfrac{\partial v}{\partial \nu} = \nabla v \cdot \nu$ are defined on the boundary $\partial\Omega$ of Ω. Then*

(2.10)
$$\int_\Omega u(x)\,\Delta v(x)\,dx - \int_\Omega v(x)\,\Delta u(x)\,dx = \int_{\partial\Omega} u(x)\,\frac{\partial v}{\partial \nu}(x)\,d\sigma(x) - \int_{\partial\Omega} v(x)\,\frac{\partial u}{\partial \nu}(x)\,d\sigma(x),$$

where σ is the superficial measure on $\partial\Omega$.

The regularity of u and v in Theorem 2.1 is sufficient to give sense to the integrals and ensures equality of the right and left hand sides of (2.10). When $\Omega = \mathbb{R}^p$, the particular case of formula (1.9) (that is, when the right hand side of (2.10) vanishes) occurs under the simple regularity assumption that u and v are twice continuously differentiable on Ω with compact support.

In the following lemmas, we consider first the weakening of regularity conditions on u and v which preserve (1.9). The case where u and v both belong to the Sobolev space $H^2(\mathbb{R}^p)$ is immediate.

Lemma 2.1. *For $p \geq 1$, if u and v are two functions of $H^2(\mathbb{R}^p)$, then (1.9) is satisfied.*

Proof. Since the function u is in $H^2(\mathbb{R}^p)$, its Laplacian Δu is in $L^2(\mathbb{R}^p)$ and its Fourier transform $\widehat{u}$ verifies $\widehat{\Delta u}(\xi) = \|\xi\|^2\ \widehat{u}(\xi)$ and is also in $L^2(\mathbb{R}^p)$. Then, by virtue of Lemma 6.10, the result follows from

$$\begin{aligned}
(2\pi)^p \int_{\mathbb{R}^p} u(x)\,\Delta v(x)\,dx &= \int_{\mathbb{R}^p} \widehat{u}(x)\,\widehat{\Delta v(x)}\,dx \\
&= \int_{\mathbb{R}^p} \widehat{u}(x)\,\|x\|^2\,\widehat{v(x)}\,dx \\
&= \int_{\mathbb{R}^p} \widehat{\Delta u(x)}\,\widehat{v(x)}\,dx \\
&= (2\pi)^p \int_{\mathbb{R}^p} v(x)\,\Delta u(x)\,dx.
\end{aligned}$$

□

We consider, now, the case when the function u belongs to $L^q(\mathbb{R}^p)$ and v is a $\mathcal{C}^\infty$-rapidly decreasing function of $\mathcal{S}$.

Lemma 2.2. *Let $u \in L^q(\mathbb{R}^p)$ with $q \in [1, +\infty]$ and $v \in \mathcal{S}$. Then we have the identities*

$$< \Delta u, v >_{\mathcal{S}',\mathcal{S}} = < u, \Delta v >_{\mathcal{S}',\mathcal{S}} = \int_{\mathbb{R}^p} u(x)\,\Delta v(x)\,dx\,.$$

Moreover, if $\Delta u \in L^q(\mathbb{R}^p)$, then (1.9) is satisfied.

Proof. These identities rely on the duality $<\,,\,>$ introduced in the appendix (Definition 6.2) and on the fact that all (classes of) functions u belonging to some $L^q(\mathbb{R}^p)$ are (identified to) tempered distributions through the continuity of the mapping $\varphi \mapsto \int u(x)\,\varphi(x)\,dx$ on the space $\mathcal{C}_c^\infty$ (for the topology induced by $\mathcal{S}$, see Lemma 6.9 and Remark 6.1 for details). □

Combining a rapidly decreasing function v and a slowly increasing function u at infinity gives a similar result (see Appendix, Definition 6.4).

Lemma 2.3. *Let $u \in O_M(\mathbb{R}^p)$ and $v \in \mathcal{S}$. Then we have*

$$\int_{\mathbb{R}^p} \Delta u(x)\, v(x)\, dx = < \Delta u, v >_{\mathcal{S}',\mathcal{S}} = < u, \Delta v >_{\mathcal{S}',\mathcal{S}} = \int_{\mathbb{R}^p} u(x)\, \Delta v(x)\, dx\,.$$

Above, we presented some situations where a Green formula of the type (1.9) is relatively immediate. We now give our main result which, through the spaces where the functions u and v live, is a weak version of this Green formulation.

Theorem 2.2. *Assume that $u \in W^{2,1}_{loc}(\mathbb{R}^p)$ and that $v \in W^{2,\infty}(\mathbb{R}^p)$. Assume also that there exist R_0 and $\varepsilon > 0$ such that $u \in \mathcal{C}^2_b(\mathbb{R}^p \setminus B^p_r)$ and $v \in \mathcal{S}^{2,p+\varepsilon}(\mathbb{R}^p \setminus B^p_r)$. Then the functions $u\,\Delta v$ and $v\,\Delta u$ are in $L^1(\mathbb{R}^p)$ and, furthermore, we have the equality*

$$\int_{\mathbb{R}^p} u(x)\, \Delta v(x)\, dx = \int_{\mathbb{R}^p} v(x)\, \Delta u(x)\, dx\,.$$

The proof makes an essential use of a series of known results based on mollification, a basic technique in analysis which is classically performed by convolution with a compactly supported mollifier (see Appendix, Notations 6.1). We recall these results in the following lemmas.

Lemma 2.4. *Let $f \in \mathcal{C}^\infty(\mathbb{R}^p)$ and $g \in L^1_{loc}(\mathbb{R}^p)$. Then $f * g \in \mathcal{C}^\infty(\mathbb{R}^p)$ and $\partial^\alpha(f * g) = \partial^\alpha f * g$, for any multi-index α.*

Lemma 2.5. *Let $f \in L^1_{loc}(\mathbb{R}^p)$ and let $(\rho_n)_n$ be a standard mollifier. Then, for all compact $K \subset \mathbb{R}^p$, the sequence $(\mathbb{1}_K f_n)_n = (\mathbb{1}_K(f * \rho_n))_n$ approximates $\mathbb{1}_K f$ in the sense that $\mathbb{1}_K f_n \to \mathbb{1}_K f$ strongly in $L^1(\mathbb{R}^p)$.*

Lemma 2.6. *Let $v \in W^{2,\infty}(\mathbb{R}^p)$. There exists a sequence $(v_n)_n$ in $\mathcal{C}^\infty(\mathbb{R}^p)$ such that $v_n \rightharpoonup v$ weakly-$*$ in $W^{2,\infty}(\mathbb{R}^p)$.*

Proof. We know that the sequence $v_n = v * \rho_n \rightharpoonup v$ weakly-$*$ in $L^\infty(\mathbb{R}^p)$. Moreover, since the sequence v_n is bounded a.e., there exists a subsequence (still denoted by) v_n such that $v_n \rightharpoonup w$ weakly-$*$ in $W^{2,\infty}(\mathbb{R}^p)$. Then $v_n \rightharpoonup w$ weakly-$*$ in $L^\infty(\mathbb{R}^p)$ and $v = w$. $\square$

Lemma 2.7. *The space $W^{1,\infty}(\mathbb{R}^p)$ is a multiplicator of $W^{1,1}_{loc}(\mathbb{R}^p)$, i.e., if $f \in W^{1,1}_{loc}(\mathbb{R}^p)$ and $g \in W^{1,\infty}(\mathbb{R}^p)$, then $fg \in W^{1,1}_{loc}(\mathbb{R}^p)$.*

Lemma 2.7 is crucial in the sense that it allows us to derive a Leibniz formula

$$\partial_i(f\,g) = g\,\partial_i f + f\,\partial_i g,\ \forall i \in \{1, \ldots, p\}\,. \tag{2.11}$$

Indeed, for a given non negative constant $R \in \mathbb{R}^*_+$ and a sequence $(f_n)_n$ of $\mathcal{C}^\infty_c(B_R)$ such that $f_n \to f$ in $\mathcal{C}^\infty_c(B_R)$ strongly, we have, for any $i \in \{1, \ldots, p\}$ and any $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^p)$,

$$\begin{aligned} < f_n g, \partial_i \phi > &= < g, f_n \partial_i \phi + \phi \partial_i f_n > - < g, \phi \partial_i f_n > \\ &= - < f_n \partial_i g + \partial_i f_n g, \phi >, \end{aligned}$$

using the membership of f_n and ϕ to $\mathcal{C}^\infty_c(\mathbb{R}^p)$ and Lemma 6.8. Now, since the sequence f_n converges strongly to f, we obtain, when n goes to infinity,

$$< f_n g, \partial_i \phi > \to < f g, \partial_i \phi >$$

and hence

$$< f_n \partial_i g + \partial_i f_n g, \phi > \to < f \partial_i g + \partial_i f g, \phi > .$$

Thus the formula (2.11) is satisfied.

Note, however, that the space $W^{1,1}_{loc}(\mathbb{R}^p)$ is not an algebra. Indeed, the function f defined on $(\mathbb{R}^p)^*$ by

$$f(x) = \|x\|^{-3}$$

belongs to $W^{1,1}_{loc}(\mathbb{R}^p)$ when $p = 6$ while f^2 does not.

Proof of Theorem 2.2. Thanks to Lemma 2.4, the sequences $(u_n)_n = (u * \rho_n)_n$ and $(v_n)_n = (v * \rho_n)_n$ are in $\mathcal{C}^\infty(\mathbb{R}^p)$. Since $L^\infty(\mathbb{R}^p) \subset L^1_{loc}(\mathbb{R}^p)$, for $R > 0$, the classical Green formula on the ball B_R states that

(2.12)

$$\int_{B_R} [u_n(x)\, \Delta v_n(x) - v_n(x)\, \Delta u_n(x)]\, dx = \int_{S_R} \left[u_n(x)\, \frac{\partial v_n}{\partial \nu}(x) - v_n(x)\, \frac{\partial u_n}{\partial \nu}(x) \right] d\sigma_R(x)$$

where $S_R = \partial B_R$ is the sphere of radius R and σ_R is the superficial measure on S_R. Now, by Lemmas 2.6 and 2.5 respectively, we have $v_n \rightharpoonup v$ weakly-$*$ in $W^{2,\infty}(\mathbb{R}^p)$ and, for a given compact K of $\mathbb{R}^p$, we also have the strong convergence $\mathbb{1}_K u_n \to \mathbb{1}_K u$ in L^1. Since $(L^1(\mathbb{R}^p))' = L^\infty(\mathbb{R}^p)$, for $K = B_R$, it follows that

$$\int_{B_R} u_n(x)\, \Delta v_n(x)\, dx \longrightarrow \int_{B_R} u(x)\, \Delta v(x)\, dx$$

and

$$\int_{B_R} v_n(x)\, \Delta u_n(x)\, dx \longrightarrow \int_{B_R} v(x)\, \Delta u(x)\, dx\,.$$

Furthermore, as the functions u and v are in $C^2(\mathbb{R}^p \backslash B_r)$, for $R > r$, we also have the following convergences

$$\int_{S_R} v_n(x)\, \frac{\partial u_n}{\partial \nu}(x)\, d\sigma_R(x) \longrightarrow \int_{S_R} v(x)\, \frac{\partial u}{\partial \nu}(x)\, d\sigma_R(x)$$

and

$$\int_{S_R} u_n(x)\, \frac{\partial v_n}{\partial \nu}(x)\, d\sigma_R(x) \longrightarrow \int_{S_R} u(x)\, \frac{\partial v}{\partial \nu}(x)\, d\sigma_R(x)\,.$$

Thus, with n going to infinity in (2.12), we obtain as a preliminary result

(2.13)

$$\int_{B_R} [u(x)\, \Delta v(x) - v(x)\, \Delta u(x)]\, dx = \int_{S_R} \left[u(x)\, \frac{\partial v}{\partial \nu}(x) - v(x)\, \frac{\partial u}{\partial \nu}(x) \right] d\sigma_R(x)\,.$$

In a second step, we prove that the right hand side of (2.13) vanishes when R goes to infinity. First, estimate it as

$$\begin{aligned}
A_R &= \left| \int_{S_R} v(x) \frac{\partial u}{\partial \nu}(x)\, d\sigma_R(x) + \int_{S_R} u(x) \frac{\partial v}{\partial \nu}(x)\, d\sigma_R(x) \right| \\
&\leq \int_{S_R} |v(x)||\nabla u(x) \cdot \nu(x)|\, d\sigma_R(x) + \int_{S_R} |\nabla v(x) \cdot \nu(x)||u(x)|\, d\sigma_R(x) \\
&\leq \sum_{i=1}^{p} \|\partial_i u\|_{1,\infty,\mathbb{R}^p} \int_{S_R} |v(x)|\, d\sigma_R(x) + \|u\|_{2,\infty,\mathbb{R}^p} \int_{S_R} \sum_{i=1}^{p} |\partial_i v(x)|\, d\sigma_R(x)
\end{aligned}$$

using (6.22) (see Appendix, Definition 6.6) since $u \in \mathcal{C}_b^2(\mathbb{R}^p \setminus B_r^p)$ and $R > r$. Multiplying through by $\|x\|^p$ and dividing through by R^p inside the two last integrals give

$$A_R \leq (p+1)\|u\|_{2,\infty,\mathbb{R}^p}\|v\|^{\mathcal{S}}_{2,p+\varepsilon,\mathbb{R}^p\setminus S_r}\frac{\sigma_R(S_R)}{R^p}\,.$$

Thus, as $\frac{\sigma_R(S_R)}{R^p} = \frac{2\pi^{p/2}}{\Gamma(p/2)}\frac{1}{R}$, we have $\lim_{R\to\infty} A_R = 0$, which is the desired result.

It remains to show that the left hand side of (2.13) converges, when R goes to infinity, towards the corresponding integral on $\mathbb{R}^p$. First, let us prove that $u\,\Delta v$ and $v\,\Delta u$ are in $L^1(\mathbb{R}^p)$. Note that, on the ball B_R, we have

$$\|u\,\partial_{ii}v\|_{1,B_r} \leq \|u\|_{1,B_r}\|\partial_{ii}v\|_{\infty,B_r} < +\infty$$

since $u \in L^1_{loc}(\mathbb{R}^p)$ and $\partial_{ii}v \in L^\infty(\mathbb{R}^p)$. As, on $\mathbb{R}^p\setminus B_R$, the function u is bounded by a positive constant M, it follows that

$$\|u\,\partial_{ii}v\|_{1,\mathbb{R}^p\setminus B_R} \leq M\int_{\mathbb{R}^p\setminus B_R}|\partial_{ii}v|\,dx \leq M\,\|v\|^{\mathcal{S}}_{2,p+1,\mathbb{R}^p\setminus B_R}\left\|\frac{1}{\|x\|^{p+1}}\right\|_{1,\mathbb{R}^p\setminus B_R} < \infty$$

since $v \in \mathcal{S}^2(\mathbb{R}^p\setminus B_R)$. Thus $u\,\partial_{ii}v$ and $v\,\partial_{ii}u$ are in $L^1(\mathbb{R}^p)$ and hence $u\,\Delta v$ and $v\,\Delta u$ are in $L^1(\mathbb{R}^p)$ as well. Finally, the desired convergence follows from an application of the Lebesgue dominated convergence theorem. □

It is worth noting that, for a function $v \in W^{2,\infty}(\mathbb{R}^p) \cap \mathcal{S}^2(\mathbb{R}^p\setminus B_r^p)$, with $p > 2$, the Green formula in Theorem 2.2 does not remain valid if, for a function u, we choose the fundamental harmonic function. This reflects the fact that this function does not belong to the space $W^{2,1}_{loc}(\mathbb{R}^p)$. However note that the choice of a constant function u leads to

$$\int_{\mathbb{R}^p}\Delta v(x)\,dx = 0\,.$$

As a last remark, the fact that the function v in Theorem 2.2 has derivatives of order 2 which, multiplied by $\|x\|^{p+\varepsilon}$, converges towards 0 for $\|x\| \to \infty$ is a minimal assumption for the theorem to work. A simpler assumption is naturally that $v \in \mathcal{S}^2(\mathbb{R}^p \setminus B_r^p)$ or, more specifically, that $v \in \mathcal{S}(\mathbb{R}^p \setminus B_r^p)$.

3. A link with Stoke's theorem

A proof of the Green formula given in Section 2 can be established through the Stokes theorem. Recall that, with the notations of Section 2, this theorem states that

$$\int_\Omega \operatorname{div} g(x)\,dx = \int_{\partial\Omega} g(x)\cdot\nu(x)\,d\sigma(x) \tag{3.14}$$

for a suitable open set Ω in $\mathbb{R}^p$ and for a sufficiently regular function g from $\mathbb{R}^p$ into $\mathbb{R}^p$. For what we need, Ω will be a ball in $\mathbb{R}^p$ while, classically in the literature, the function g is continuously differentiable with compact support.

In [8], Lepelletier gives an extension of (3.14) to the case where $g \in W^{1,1}_{loc}(\mathbb{R}^p)$ which corresponds to the functional framework of Section 2. Thus, in this context, coming back to the functions u and v in Theorem 2.2, note that, for $g = u\nabla v - v\nabla u$, formula (3.14) yields formally

$$\int_\Omega (u(x)\,\Delta v(x) - v(x)\,\Delta u(x))\,dx = \int_{\partial\Omega}(u(x)\,\nabla v(x) - v(x)\,\nabla u(x))\cdot\nu(x)\,d\sigma(x)\,. \tag{3.15}$$

Choosing $\Omega = B_R$ (the ball of radius R centered at the origin) Lepelletier [8] proves that, under the conditions of Theorem 2.2, the right hand side of (3.15) goes to 0 when R goes to infinity, which is exactly the result of Theorem 2.2. This proof is less direct than the one given in Section 2 and we refer to [8] for more details.

The fact that Stoke's Theorem is underlying to our context is interesting in the sense that this theorem also intervenes naturally in Stein's identity (1.2) and in its extensions to spherically symmetric distributions. This is the purpose of the following proposition for which we follow the line of Fourdrinier and Strawderman [5].

Proposition 3.1. *Let X be a random vector in $\mathbb{R}^p$ with density of the form $x \mapsto f(\|x-\theta\|^2)$ for some function f. For $g \in W^{1,1}_{loc}(\mathbb{R}^p)$ such that*

$$E_\theta\Big[|(X-\theta)\cdot g(X)|\Big] < \infty \tag{3.16}$$

we have

$$E_\theta\left[(X-\theta)\cdot g(X)\right] = E_\theta\left[Q(\|X-\theta\|^2)\,\mathrm{div}\,g(X)\right] \tag{3.17}$$

where, for any $t \geq 0$,

$$Q(t) = \frac{1}{2f(t)}\int_t^\infty f(u)\,du.$$

Proof. By definition, we have

$$\begin{aligned} E_\theta\left[(X-\theta)\cdot g(X)\right] &= \int_{\mathbb{R}^p}(x-\theta)\cdot g(x)\,f(\|x-\theta\|^2)\,dx \\ &= \int_0^\infty R\int_{S_{R,\theta}}\frac{x-\theta}{\|x-\theta\|}\cdot g(x)\,d\sigma_{R,\theta}(x)f(R^2)\,dR \end{aligned}$$

where $S_{R,\theta}$ is the sphere of radius R centered at θ and $\sigma_{R,\theta}$ is the superficial measure on $S_{R,\theta}$. Then, using the Stokes theorem, we obtain

$$\begin{aligned} E_\theta\left[(X-\theta)\cdot g(X)\right] &= \int_0^\infty\int_{B_{R,\theta}}\mathrm{div} g(x)\,dx\,R\,f(R^2)\,dR \\ &= \int_{\mathbb{R}^p}\int_{\|x-\theta\|}^\infty R\,f(R^2)\,dR\,\mathrm{div} g(x)\,dx \\ &= \frac{1}{2}\int_{\mathbb{R}^p}\int_{\|x-\theta\|^2}^\infty f(u)\,du\,\mathrm{div} g(x)\,dx \end{aligned}$$

by Fubini's theorem and with the change of variable $u = R^2$. Finally, according to the definition of Q, we have

$$E_\theta\left[(X-\theta)\cdot g(X)\right] = \int_{\mathbb{R}^p} Q(\|x-\theta\|^2)\,\mathrm{div}\,g(x)\;f(\|x-\theta\|^2)\,dx$$

which is the desired result. $\square$

Note that, in the normal case $\big(f(t) = (1/(2\pi)^{p/2})\,e^{-t/2}\big)$, we have $Q \equiv 1$ and the proof of Proposition 3.1 is an alternative to the proof of Stein [10]. Note also that the assumption $g \in W^{1,1}_{loc}(\mathbb{R}^p)$ is the smoothness needed in the extension of the Stokes theorem mentioned above. This weak regularity was required by Stein [10], as noticed by Johnstone [7], and, through (3.15), the role of the Sobolev spaces membership in the Green formula of Theorem is perceptible.

As a complement of this section, let us see how Proposition 3.1 can be used to derive improved estimators $\varphi(X)$ of θ which dominate the standard estimator X under quadratic loss. Provided that $E_\theta[X] < \infty$ and that $E_\theta[\|g(X)\|^2] < \infty$, any estimator of the form $\varphi(X) = X + g(X)$ has a finite risk function $R(\theta, \varphi) = E[\|\varphi(X) - \theta\|^2]$. Then its risk difference with X equals

$$\delta_\theta = 2E_\theta\left[(X-\theta)\cdot g(X)\right] + E_\theta\left[\|g(X)\|^2\right]$$

and it is easy to show that (3.17) applies and, hence, that

$$\delta_\theta = 2E_\theta\left[Q(\|X-\theta\|^2)\,\mathrm{div} g(X)\right] + E_\theta\left[\|g(X)\|^2\right] \,.$$

To obtain that $\delta_\theta \leq 0$ (with strict inequality for some θ), note the two following ways. Either there exists a positive constant c such that $Q(t) \geq c$, for any $t \geq 0$, and it suffices that the function g satisfies, for any $x \in \mathbb{R}^p$,

$$2\,c\;\mathrm{div} g(x) + \|g(x)\|^2 \leq 0$$

(with strict inequality on a set of positive Lebesgue measure). Or the function Q is non-increasing (respectively non-decreasing) and the function g is such that div$g$ is a sub-harmonic (respectively super-harmonic) function.

4. Application in statistics

A statistical context to which the Green formula established in Section 2 applies is the loss estimation problem mentioned in Section 1 (see [3] for more details). Recall that we wish to estimate a function of $\|X-\theta\|^2$ when X has a spherically symmetric distribution around $\theta \in \mathbb{R}^p$. More precisely, we estimate $c(\|X-\theta\|^2)$ in (1.1) and X has a density of the form $x \mapsto f(\|x-\theta\|^2)$. Comparing the standard estimator $\gamma_0 = E_0[c(\|X\|^2)]$ to a competitive estimator $\gamma_s = \gamma_0 + s(X)$, for some function s, leads to the difference in risk between γ_s and γ_0

$$\delta_\theta = E_\theta\left[2\left(\gamma_0 - c(\|X-\theta\|^2)\right)s(X) + s^2(X)\right]$$

under the finiteness risk condition $E_0[c^2(\|X\|^2)] < \infty$ and $E_\theta[s^2(X)] < \infty$. Obtaining a more tractable expression of δ_θ was possible in [3] using Theorem 2.2 with $u(x) = s(x)$ and $v(x) = K(\|x-\theta\|^2)$ where

$$K(t) = \frac{1}{p-2}\int_t^\infty \left[\left(\frac{y}{t}\right)^{p/2-1} - 1\right](\gamma_0 - c(y))\,f(y)\,dy\,.$$

The statistical context does not right away impose strong regularity conditions on the functions u and v. Indeed, although we have the choice of the function u (which is the correction s brought to γ_0) and may consider a regular function of the form $u(x) = \frac{a}{\|x\|^2+b}$ where a and b are positive constants, it is usual in the literature to consider the case $b = 0$ for which the function $\frac{a}{\|x\|^2}$ blows up at 0. Note that, in that case, the corresponding function u is in $W^{2,1}_{loc}(\mathbb{R}^p) \cap C^2_b(\mathbb{R}^p \setminus B_r)$ for some $r > 0$ as required in Theorem 2.2. As for the regularity of v, it is worth noting that the fact that $v \in W^{2,\infty}(\mathbb{R}^p) \cap \mathcal{S}^{2,p+\varepsilon}(\mathbb{R}^p \setminus B_r)$ for some $\varepsilon > 0$ is obtained through regularity conditions on the functions f and c (f and fc are in $\mathcal{S}^{0,p/2+1+\varepsilon}(\mathbb{R}^*_+ \setminus T)$ where T is a finite set). We see how the statistical framework imposes the specific conditions of our Green type formula.

5. Conclusions

Statistical motivation led us to establish a Green type formula. Somewhat surprisingly, the natural statistical context imposes a functional framework that is weaker than the classical one. We also made a link with Stoke's theorem and showed how this theorem is related to the famous Stein's identity.

The main goal of this paper was to specify this context and to provide a full proof of the Green formula at hand. It seems to us that such an approach is also potentially important for further applications in Statistics.

6. Appendix

We aim to provide, here, a selection of the most important results in the theories of Sobolev and distributions spaces used throughout the paper. For more details we refer to [1] and [11], for example.

Notations 6.1. *Throughout we reserve the symbol* Ω *for a non-empty open subset of* $\mathbb{R}^p$, $p \geq 1$, *with closure* $\bar{\Omega}$ *and boundary* $\partial\Omega$ *which is assumed to be regular enough. We denote by* $\mathcal{D}(\Omega)$ *the space of functions of class* $\mathcal{C}^\infty$ *with compact support. The function* ρ *defined, for any* $x \in \mathbb{R}^p$, *by*

$$\rho(x) = \begin{cases} \exp\left(-\frac{1}{1-\|x\|^2}\right) & \text{if } \|x\| < 1, \\ 0 & \text{else} \end{cases}$$

where $\|x\|$ *denotes the euclidean norm of* x *defined by* $\|x\|^2 = \sum_{i=1}^p x_i^2$, *is a well known example in* $\mathcal{D}(\mathbb{R}^p)$. *For any* $\epsilon > 0$ *and for any* $x_0 \in \mathbb{R}^p$, *it allows to derive the standard mollifier* ρ_ε *defined, for any* $x \in \mathbb{R}^p$, *by*

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^p}\,\rho\Big(\frac{x - x_0}{\varepsilon}\Big)\,.$$

This is a positive C^∞ *function on* $\mathbb{R}^p$ *such that* $\int_{\mathbb{R}^p} \rho_\varepsilon(x)\,dx = 1$ *with support the ball* $B^p_{\varepsilon,x_0} = \{x \in \mathbb{R}^p \,|\, \|x_0 - x\| \leq \varepsilon\}$ *of center* x_0 *and radius* ε.

For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, *we set*

$$|\alpha| = \sum_{j=1}^n \alpha_j, \quad \alpha! = \prod_{j=1}^n \alpha_j!, \quad x^\alpha = \prod_{j=1}^n x_j^{\alpha_j}, \quad D^\alpha = \prod_{j=1}^n D_j^{\alpha_j},$$

where $D_j^{\alpha_j} = \partial^{\alpha_j}/\partial x_j^{\alpha_j}$ *is the i-th partial derivative operator of order* α_j.

There is a locally convex topology on the space $\mathcal{D}(\Omega)$. *That topology is defined sequentially as follows and leads to the notion of distribution.*

Definition 6.1. *Let* (φ_i) *be a sequence of functions of* $\mathcal{D}(\Omega)$. *We say that*

$$\varphi_i \longrightarrow \varphi \quad \text{in } \mathcal{D}(\Omega)$$

if, for all $i \geq 0$, φ_i *has its support included in a compact subset* K *of* Ω *and if*

$$D^\alpha\varphi_i \longrightarrow D^\alpha\varphi \quad \text{in } \mathcal{D}(\Omega) \quad \text{uniformly on } K, \ \forall\alpha \in \mathbb{N}^n.$$

A distribution T *is a linear form on* $\mathcal{D}(\Omega)$ *such that*

$$\lim_i T(\varphi_i) = T(\varphi)\,,$$

for any sequence φ_i *converging towards* φ *in* $\mathcal{D}(\Omega)$.

Thus a distribution is an element of the topological space of $\mathcal{D}(\Omega)$. We denote such a space $\mathcal{D}'(\Omega)$ and also adopt the common notation

$$T(\varphi) = \langle T, \varphi \rangle_{\mathcal{D}',\mathcal{D}}$$

or

$$T(\varphi) = \langle T, \varphi \rangle$$

for sake of simplicity.

If f is locally integrable on Ω, that is, if f belongs to the space

$$L^1_{loc}(\Omega) = \{v : \Omega \to \mathbb{R} \,|\, v \in L^1(K), \forall\, K \text{ compact of } \Omega\}\,,$$

then

$$\langle T_f, \varphi \rangle = \int_\Omega f(x)\,\varphi(x)\,dx$$

defines a distribution, T_f, on Ω. Indeed the continuity of T_f is an easy consequence of

$$|\langle T_f, \varphi_i - \varphi \rangle| = \Big| \int_\Omega f(x)(\varphi_i(x) - \varphi(x))\,dx \Big| \leq \max_K |\varphi_i - \varphi| \int_\Omega |f(x)|\,dx\,,$$

where K is the compact subset of Ω containing the supports of the φ_i's.

One key feature of distributions is the fact that one can differentiate them, in some sense, indefinitely.

Lemma 6.8. *Let $T \in \mathcal{D}'(\Omega)$. Then we have*

$$\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega).$$

We now give some definitions and results about the so-called Schwartz space.

Definition 6.2. *A function φ in $\mathcal{C}^\infty(\mathbb{R}^p)$ is called rapidly decreasing if any derivative of φ, multiplied by any power of $\|x\|$, converges towards 0 when $\|x\| \to \infty$. In other words, for all $k \in \mathbb{N}$ and for any couple of multi-indices $(\alpha,\, \beta)$, we have*

$$\sup_{x \in \mathbb{R}^p, |\alpha|, |\beta| \leq k} |x^\beta D^\alpha \varphi(x)| \longrightarrow 0 \quad \textit{as} \quad \|x\| \to \infty\,.$$

The Schwartz space $\mathcal{S}(\mathbb{R}^p)$ is the space of all infinitely differentiable rapidly decreasing functions φ on $\mathbb{R}^p$.

The Schwartz space is a complete topological vector space with a suitably defined family of semi-norms, that is,

$$\|\varphi\|_{k,\mathcal{S}} = \sup_{x \in \mathbb{R}^p, |\alpha|, |\beta| \leq k} |x^\beta D^\alpha \varphi(x)|\,.$$

It is clear that the functions of $\mathcal{D}(\mathbb{R}^p)$ are elements of $\mathcal{S}(\mathbb{R}^p)$. Another well known subclass of $\mathcal{S}(\mathbb{R}^p)$ is the space of Gaussian functions, that is, the functions of the type $e^{-a|x|^2}$ where a is a non-negative number.

The main feature of the functions in $\mathcal{S}(\mathbb{R}^p)$ is that they decrease towards 0 as fast as any polynomial. Requirement about differentiability can be weakened in some cases and the following space of functions on Ω may be sufficient for applications:

$$\mathcal{S}^{l,r}(\Omega) = \left\{ \varphi \in \mathcal{C}^l(\Omega) \,|\, \sup_{x \in \Omega, |\alpha| \leq l; \beta \leq r} \|x\|^\beta \, |D^\alpha \varphi(x)| < \infty \right\}$$

for fixed $(l, r) \in \mathbb{N}^2$.

The Schwartz space gives rise to a subspace of $\mathcal{D}'(\mathbb{R}^p)$ through the following definition.

Definition 6.3. *A tempered distribution T is a linear form on the space $\mathcal{S}(\mathbb{R}^p)$ which is continuous is the following sense: there exist an integer k and a constant C such that*

$$|< T, \varphi >| \leq C||\varphi||_{k,S},$$

for all φ in $\mathcal{S}(\mathbb{R}^p)$. We denote by $\mathcal{S}'(\mathbb{R}^p)$ the space of tempered distribution. Finally, we say that T_i converges to T in $\mathcal{S}'(\mathbb{R}^p)$ if $< T_i - T, \varphi > \to 0$, for any function φ in $\mathcal{S}(\mathbb{R}^p)$.

For $q \in [1, +\infty]$, we denote by $L^q_M(\mathbb{R}^p)$ the set of locally integrable functions f such that there exists an integer d for which the function $x \to (1 + \|x\|)^{-d} f(x)$ belongs to $L^q(\mathbb{R}^p)$.

Lemma 6.9. *Let $f \in L^q_M(\mathbb{R}^p)$. The linear form S_f defined, for any $\varphi \in \mathcal{S}(\mathbb{R}^p)$, by*

$$|< S_f, \varphi >| = \int_{\mathbb{R}^p} f(x)\, \varphi(x) dx$$

is a tempered distribution.

Proof. Let $d \in \mathbb{N}$ such that $(1 + \|x\|^{-d}) f(x) \in L^1(\mathbb{R}^d)$. The continuity of S_f follows immediately from

$$|< S_f, \phi >| \leq \|(1 + \|.\|)^{-d}\|_{L^1} \, \|\phi\|_{d,S} \,.$$

□

Remark 6.1. *For all $q \geq 1$, the $L^q(\mathbb{R}^p)$-functions are in $L^1_M(\mathbb{R}^p)$. Then the $L^q(\mathbb{R}^p)$-functions are identified to the distributions in $\mathcal{S}(\mathbb{R}^p)$.*

In contrast to the rapidly decreasing functions are the slowly increasing functions.

Definition 6.4. *A function f in $\mathcal{C}^\infty(\mathbb{R}^p)$ is said to be slowly increasing at infinity if*

$$\forall k \in \mathbb{N} \quad \exists N \in \mathbb{N} \quad \exists C > 0 \qquad \sup_{|\alpha|=k} |D^\alpha f(x)| \leq C(1 + |x|)^N. \tag{6.18}$$

We denote by $\mathcal{O}_M$ the space of $\mathcal{C}^\infty(\mathbb{R}^p)$-functions which are slowly increasing at infinity.

For $n \in \mathbb{N}$ fixed, an example of an element of $\mathcal{O}_M$ is the function f_n defined, for any $x \in \mathbb{R}$, by

$$f_n(x) = \frac{1}{||x||^2 + 1/n} \,.$$

The space $\mathcal{O}_M$ induces multiplicative operators in $\mathcal{S} = \mathcal{S}(\mathbb{R})$ and $\mathcal{S}' = \mathcal{S}'(\mathbb{R})$. Indeed, for fixed f in $\mathcal{O}_M$, the multiplication $\varphi \in \mathcal{S} \mapsto f\varphi$ (respectively $S \in \mathcal{S}' \mapsto fS$) is a continuous linear mapping from $\mathcal{S}$ into $\mathcal{S}$ (respectively from $\mathcal{S}'$ into $\mathcal{S}'$).

Definition 6.5. *(Algebraic operations on $\mathcal{S}'$) For $S \in \mathcal{S}'$, we define*

i) the derivation by

$$< D^\alpha S, \phi >= (-1)^{|\alpha|} < S, D^\alpha \phi > ;$$

ii) the convolution with a function f in L^1_M by

$$< f \star S, \phi > = < S, f \star \phi > ;$$

iii) the Fourier transform by

$$< \widehat{S}, \phi > = < S, \widehat{\phi} > \ .$$

Lemma 6.10. *For any function f and g in L^2, we have*

$$\int_{\mathbb{R}^p} f(x)\, g(x)\, dx = (2\pi)^{-p} \int_{\mathbb{R}^p} \widehat{f}(x)\, \widehat{g}(x)\, dx, \tag{6.19}$$

and, in particular,

$$||f||_{L^2(\mathbb{R}^p)} = (2\pi)^{-p/2} ||\widehat{f}||_{L^2(\mathbb{R}^p)}\,. \tag{6.20}$$

Proof. Recall first that, for any function ϕ in $\mathcal{S}$, Plancherel's formula states

$$||\phi||_{L^2(\mathbb{R}^p)} = (2\pi)^{-p/2} ||\widehat{\phi}||_{L^2(\mathbb{R}^p)}.$$

Now let $(f_n)_n$ be a sequence of functions in $\mathcal{S}$ which converges strongly to f in L^2. Since the norm $||.||_{L^2}$ and the Fourier transform are continuous, we have

$$||\widehat{f}||_{L^2(\mathbb{R}^p)} = \lim_{n\to\infty} ||\widehat{f}_n||_{L^2(\mathbb{R}^p)} = \lim_{n\to\infty} (2\pi)^{p/2} ||f_n||_{L^2(\mathbb{R}^p)} = (2\pi)^{p/2} ||f||_{L^2(\mathbb{R}^p)}.$$

So that we first obtain (6.20). Then formula (6.19) follows from the classical identity

$$\int_{\mathbb{R}^p} f(x)\, g(x)\, dx = \frac{1}{4}\Big(||f+g||^2_{L^2(\mathbb{R}^p)} - ||f-g||^2_{L^2(\mathbb{R}^p)} + i||f+ig||^2_{L^2(\mathbb{R}^p)} - i||f-ig||^2_{L^2(\mathbb{R}^p)} \Big)\,.$$

□

The Sobolev spaces are a very useful tool to solve partial differential equation. The definition of such spaces makes an essential use of the derivation in the sense of $\mathcal{D}'$ or $\mathcal{S}'$ seen above.

Definition 6.6. *For $m \in \mathbb{N}$ and $1 \le n \le +\infty$, the Sobolev space of integrable functions of order m and n on Ω is defined by*

$$W^{m,n}(\Omega) = \{u \in L^n(\Omega) \,|\, D^\alpha u \in L^n(\Omega), \forall \alpha, |\alpha| \le m\}$$

so that, for $u \in W^{m,n}(\Omega)$, we have

$$\int_\Omega u(x)\, D^\alpha \phi(x)\, dx = (-1)^{|\alpha|} \int_\Omega D^\alpha u(x)\, \phi(x)\, dx\,.$$

Similarly, the Sobolev space of locally integrable functions of order m and n on Ω is defined by

$$W^{m,n}_{loc}(\Omega) = \{u \in L^n_{loc}(\Omega) \,|\, D^\alpha u \in L^n_{loc}(\Omega), \forall \alpha, |\alpha| \le m\}\,.$$

These spaces are equipped with the following norms

$$\|u\|_{m,n,\Omega} = \left(\sum_{0\le|\alpha|\le m} \|D^\alpha u\|^n_{n,\Omega} \right)^{1/n} \qquad \textit{if } 1 \le n < \infty \tag{6.21}$$

and

$$\|u\|_{m,\infty,\Omega} = \max_{0\leq|\alpha|\leq m} \|D^\alpha u\|_{\infty,\Omega} . \tag{6.22}$$

As an important case for applications, the space $W^{m,2}(\Omega)$ is an Hilbert space whose the norm (6.21) derives from the scalar product

$$(u, v)_{m,2} = \sum_{|\alpha|\leq m} \int_\Omega D^\alpha u(x) D^\alpha v(x)\, dx .$$

This space is usually denoted by $H^m(\Omega)$.

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