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Optimal Asset Allocation under Forward Exponential Performance Criteria

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Abstract: This work presents a novel concept in stochastic optimization, namely, the notion of forward performance. As an application, we analyze a portfolio management problem with exponential criteria. Under minimal model assumptions we explicitly construct the forward performance process and the associated optimal wealth and asset allocations. For various model parameters, we recover a range of investment policies that correspond to distinct financial applications.

1. Introduction

Optimal asset allocation problems can be formulated as classical stochastic optimization problems. They typically consist of a time horizon, a controlled process (investor's wealth) and an optimization criterion represented as the conditional expectation of a wealth functional, given a relevant filtration. Maximizing this expectation, over a given set of admissible policies, yields the so-called value function.

To facilitate the exposition, we denote the state controlled process by X, the set of admissible controls by $\mathcal{A}$ and the relevant filtration by $\mathcal{F}_t$, $0 \leq t \leq T$. The criterion to be optimized is of the form $J = E_{\mathbb{P}}^{(x,t)}\left(U\left(X_T\right)\right)$ with U being a concave and increasing function, often referred to as the investor's bequest or utility. The value function V is, in turn, defined as

$$V\left(x,t;T\right) = \sup_{\mathcal{A}} E_{\mathbb{P}}^{(x,t)}\left(U\left(X_T\right)\right). \tag{1.1}$$

At $t = T$, it coincides with the utility datum and for previous times, it satisfies—under weak model assumptions—the Dynamic Programming Principle. Namely,

$$V\left(X_s^*, s; T\right) = \begin{cases} E_{\mathbb{P}}\left(V\left(X_{s'}^*, s'; T\right)\Big|\, \mathcal{F}_s\right) & \text{for} \quad t \leq s \leq s' < T \\ U\left(X_T^*\right) \quad \text{for} \quad s = T, \end{cases} \tag{1.2}$$

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where X^* stands for the optimized state process, with $X_t^* = x$ (see, for example, [1] and [14]).

What the above tells us is that the value function is prespecified at the end of the horizon and, for earlier times, is generated backwards in time. It is a *martingale* at the optimum, and a *supermartingale* otherwise. In essence, all that is needed in order to specify the value function, is to find a martingale that coincides with the utility at maturity. [1]

Assigning a datum at a future time is in accordance with classical control criteria, as for example, in applications in manufacturing, supply chain management, production planning and inventory control (see [14] for a concise collection of applications). In other settings, however, it might not be a very realistic modeling assumption. This was, for example, observed by the authors in utility models used in the so-called indifference valuation of claims in incomplete markets. Therein, the following issues were observed.

Firstly, fixing the trading horizon makes the valuation of claims of arbitrary maturities impossible. One might try to remedy this by allowing for infinite horizon and incorporating either a running discounted payoff in $[t, \infty]$ or asymptotic growth criteria. However, infinite horizon problems, albeit more tractable than the time dependent ones, are, often, not suitable for modeling realistic situations generated, for example, by sudden changes of the investment opportunity set, defaults, etc..

Secondly, the fact that, from one hand, the utility is exogenously chosen far ahead in the future, and on the other, it is used to make investment decisions for today, does not appear very natural. Besides, the optimal expected utility is generated backwards in time while the market moves in the opposite direction (forward), an apparently not very intuitive situation.

Motivated by the above considerations, the authors proposed an alternative approach to stochastic optimization, introduced in [4] (see, also [5] and [8]). Firstly, the horizon dependence was relaxed by removing the assumption of preassigned future data. The only standing requirement for the solution is that it is an adapted process and a supermartingale for arbitrary controls, and becomes a martingale at an optimum. Such a process is called a *dynamic performance.* For its full specification, a datum needs to be introduced. In contrast to the traditional (backwards) framework, the authors proposed to have a condition assigned at *initial* time. The solution is then called a *forward performance.* We refer the reader to [4] for a detailed exposition of the new approach and its applicability to valuation and hedging in the presence of unhedgeable risks.

The martingality property stems from the natural requirement that, if the system is currently at an optimal state, one needs to seek for controls so that the same level of average performance is preserved at all future times. On the other hand, supermartingality is associated with declining upcoming average performance and, thus, suboptimal system behavior. We comment that the latter requirement is not crucial for the construction of the (optimal) martingale process. For the applications we are interested in, it is, however, a natural consequence of the inherent concavity properties the solution process has.

Herein, we extend the results obtained in [4], for incomplete binomial models, to the case in which asset prices are modeled as Ito processes. The model is rather general and allows for market incompleteness as well as for investment in many assets. No Markovian assumptions are introduced. The solution approach is based

[1]The above conditions are easily modified when a running criterion is also incorporated. For simplicity and in order to expose the new concepts, we choose not to consider this case.

entirely on stochastic calculus and yields explicit expressions for the forward performance process and the optimal policies. The forward performance is constructed by combining differential and stochastic input, namely, a deterministic function of wealth and time, and two auxiliary processes (see, respectively, (4.2), and (4.3), (4.4)). The first auxiliary process may be interpreted as a benchmark. The other is associated with a change of measure and may be used to represent the investor's views for the market's state away from equilibrium, or to model trading constraints.

We work with exponential criteria (see (4.1)). We choose to do so for two reasons. Firstly, exponential preferences are most frequently used for pricing in incomplete markets, currently a very active area of research and applications. Their popularity is coming from the explicit solutions they generate as well as their direct connection to entropic dynamic risk measures. Secondly, the aim herein is to expose the advantages of working with forward exponential criteria instead of the backward ones. We will see that the proposed model is not only general and tractable, but it also yields a rich class of policies that capture distinct realistic situations. Indeed, we show that judicious choices of the coefficients of the market input processes generate a range of interesting strategies, including, among others, two extreme situations, namely, strategies that allocate zero or the entire wealth in the riskless asset. Our findings suggest that, if put in the right modeling perspective, exponential criteria do not produce naive, wealth-independent strategies, as it is the case in the traditional framework. Rather, they generate policies that seem suitable for a variety of applications in portfolio choice, and derivative pricing and hedging.

The paper is organized as follows. In section 2 we introduce the model and the notion of forward performance. In section 3 we present two motivational examples. In section 4 we analyze the general exponential case. We construct the solution and the optimal strategies and wealth. In section 5, we analyze the optimal investments, wealth and performance for various choices of market parameters and coefficients of the auxiliary processes. We conclude in section 6.

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2. The model and its forward performance

The market environment consists of one riskless and k risky securities. The risky securities are stocks and their prices are modeled as Ito processes. Namely, for $i = 1, ..., k$, the price S^i of the i^{th} risky asset solves

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sum_{j=1}^{d} \sigma_t^{ji} dW_t^j \right) \tag{2.1}$$

with $S_0^i > 0$. The process $W = \left(W^1, ..., W^d\right)$ is a standard $d-$dimensional Brownian motion, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For simplicity, it is assumed that the underlying filtration, $\mathcal{F}_t$, coincides with the one generated by the Brownian motion, that is $\mathcal{F}_t = \sigma\left(W_s : \ 0 \leq s \leq t\right)$.

The coefficients μ^i and σ^i, $i = 1, ..., k$, follow bounded $\mathcal{F}_t$-adapted processes with values in $\mathcal{R}$ and $\mathcal{R}^d$, respectively. For brevity, we write $\sigma = \sigma_t$ to denote the volatility matrix, i.e. the $d \times k$ stochastic matrix $\left(\sigma_t^{ji}\right)$, whose i^{th} column represents the volatility σ_t^i of the i^{th} risky asset. We may, then, alternatively write (2.1) as

$$dS_t^i = S_t^i \left(\mu_t^i dt + \sigma_t^i \cdot dW_t\right).$$

The riskless asset, the savings account, has the price process B satisfying

$$dB_t = r_t B_t dt$$

with $B_0 = 1$, and for a bounded, nonnegative, $\mathcal{F}_t$-adapted interest rate process r.

A fundamental assumption in the financial applications that motivated this study is the so-called absence of arbitrage. Consequently, it is postulated that there exists an $\mathcal{F}_t$-adapted process λ, taking values in $\mathcal{R}^d$, such that the equality

$$\mu_t^i - r_t = \sum_{j=1}^{d} \sigma_t^{ji} \lambda_t^j = \sigma_t^i \cdot \lambda_t$$

is satisfied for $t \geq 0$, for all $i = 1, ..., k$. Using vector and matrix notation, the above becomes

$$\mu_t - r_t \mathbf{1} = \sigma_t^T \lambda_t, \tag{2.2}$$

where σ^T stands for the matrix transpose of σ, and $\mathbf{1}$ denotes the $d-$dimensional vector with every component equal to one. The process λ is often referred to as a *market price of risk*. Note that, in general, it is not uniquely determined.

Starting at $t = 0$ with an initial endowment $x \in \mathcal{R}$, at future times the investor invests the amounts π_t^0 and π_t^i , $i = 1, ..., k$, respectively, in the riskless and the i^{th} risky asset. The present value of his/her investment is then given by

$$X_t = \frac{\sum_{i=0}^{k} \pi_t^i}{B_t}.$$

We will refer to X as the discounted wealth process. The investment strategies will play the role of control processes and are taken to satisfy the standard assumption of being *self-financing*, i.e. for $s \geq t$,

$$X_s = x + \sum_{i=1}^{k} \int_0^s \frac{\pi_u^i}{B_u} \left(\mu_u^i - r_u\right) du + \sum_{i=1}^{k} \int_0^s \frac{\pi_u^i}{B_u} \sigma_u^i \cdot dW_u.$$

Writing the above in differential form, yields the evolution of the discounted wealth,

$$dX_t = \sum_{i=1}^{k} \frac{\pi_t^i}{B_t} \sigma_t^i \cdot \left(\lambda_t dt + dW_t\right) = \beta_t \cdot \left(\lambda_t dt + dW_t\right). \tag{2.3}$$

Herein,

$$\beta_t = \sum_{i=1}^{k} \frac{\pi_t^i}{B_t} \sigma_t^i \tag{2.4}$$

or, equivalently,

$$B_t^{-1} \sigma_t \pi_t = \beta_t, \tag{2.5}$$

where the (column) vector, $\pi_t = \left(\pi_t^i;\ i = 1, ..., k\right)$. The set of admissible strategies, $\mathcal{A}$, consists of all self-financing $\mathcal{F}_t$-adapted processes, π, for which

$$E_{\mathbb{P}} \int_0^t \left| \sum_{i=1}^k \frac{\pi_s^i}{B_s} \sigma_s^i \right|^2 ds < \infty, \quad t > 0.$$

Whenever needed, we will be using the notation X^π to denote the solution of (2.3) when the control π is used.

We next introduce the notion of dynamic performance.

Definition 2.1. *An $\mathcal{F}_t$-adapted process $U_t(x)$ is a dynamic performance process if:*

i) the mapping $x \to U_t(x)$ is increasing and concave, for each $t \geq 0$,

ii) for each self-financing strategy, π, and $s \geq t$,

$$E_{\mathbb{P}}\left(U_s\left(X_s^\pi\right) | \mathcal{F}_t\right) \leq U_t\left(X_t^\pi\right) \tag{2.6}$$

and

iii) there exists a self-financing strategy, π^*, for which

$$E_{\mathbb{P}}\left(U_s\left(X_s^{\pi^*}\right) | \mathcal{F}_t\right) = U_t\left(X_t^{\pi^*}\right), \quad s \geq t. \tag{2.7}$$

Remark: We, easily, see that the traditional value function V, (cf. (1.1)), is a dynamic performance. Indeed, if we define,

$$U_t(x) = \begin{cases} V(x, t; T) & \text{for} \quad 0 \leq t \leq T \\ \\ V(x, T; T) & \text{for} \quad t \geq T, \end{cases}$$

then $U_t(x)$ satisfies the criteria in the above definition. Notice, however, the stringent requirement that the process U_t does not change for $t \geq T$.

Herein, we focus our attention on dynamic performance processes that are specified at *initial time,* to be henceforth called *forward performance processes.* We give their formal definition below.

Definition 2.2. *An $\mathcal{F}_t$-adapted process $U_t(x)$ is a forward performance process if it satisfies the assumptions of Definition 2.1 together with the initial condition*

$$U_0(x) = u_0(x) \tag{2.8}$$

where u_0 is a concave and increasing function of wealth.

We note that the forward performance process might *not* be unique. While lack of uniqueness is not important for the applications in mind, characterizing the class of all solutions is, in our opinion, an interesting and challenging question.

We conclude this section mentioning that forward formulations of optimal control problems have been proposed and analyzed in the past. For deterministic models we refer the reader, among others, to [3], [12] and [13]. In stochastic settings, forward optimality has been studied, primarily under Markovian assumptions, in [2] via the associated martingale problems and construction of the Nisio semigroup (see, also [9]). The object of study is

$$V_t(x) = \sup_{\mathcal{A}} E^{(x,t)}\left(U_0\left(X_t\right)\right), \quad t \geq 0, \tag{2.9}$$

with $X_0(x) = x$ and U_0 a given initial input. A rich theory has been developed which addresses a variety of questions related, among others, to the validity of the Dynamic Programming Principle and construction of the solution and optimal policies across all times. While the forward performance process introduced herein plays a different role than V_t, exploring how this theory can contribute to the study of forward solutions as well as to addressing some of the shortcomings of the existing terminal horizon (backward) problems is certainly worth pursing. [2]

3. Two examples

In order to provide intuition for the upcoming construction of the exponential forward performance process we present two representative examples. To facilitate the exposition, we assume that the market consists of a single stock and a bond and that the interest rate is zero. In the first example, we consider a binomial model while in the second we model the stock as in (2.1). To highlight the generality of the construction method, we take the binomial model to be incomplete. In both cases, the initial data is given by $u_0(x) = -e^{-x}$, $x \in \mathcal{R}$.

The solution of the binomial example, see (3.3), suggests that the forward process can be constructed using a deterministic function of wealth and time, with the latter argument replaced by an appropriately chosen process. While the form of the deterministic input is, to some extent, not too surprising - due to the specific assumptions on the initial data - changing time is by no means standard. Notice that this is performed via a positive and non-decreasing process (cf. (3.2)) which depends on market movements but not on the investor's preferences. In the second example, we use these insights and produce a similar representation of the solution.

Example 1: We consider a single stock whose levels are denoted by $S_t > 0$, $t = 0, 1, \ldots$ and define the variables ξ_{t+1} as $\xi_{t+1} = \frac{S_{t+1}}{S_t}$, $\xi_{t+1} = \xi_{t+1}^d, \xi_{t+1}^u$ with $0 < \xi_{t+1}^d < 1 < \xi_{t+1}^u$. A non-traded factor might be present whose values are denoted by Y_t, $(Y_t \neq 0)$, $t = 0, 1, \ldots$. We, then, view $\{(S_t, Y_t) : t = 0, 1, \ldots\}$ as a two-dimensional stochastic process defined on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with $\mathbb{P}$ being the historical measure. The filtration $\mathcal{F}_t$ is generated by the random variables S_i and Y_i, for $i = 0, 1, \ldots, t$.

We denote by X_t, $t = 0, 1, \ldots$, the investor's wealth process associated with a multi-period self-financing portfolio. We take α_t, $t = 0, 1, \ldots$, to be the number of shares of the traded asset held in this portfolio over the interval $[t-1, t)$. Then, denoting by $\triangle S_t$ the increment $\triangle S_t = S_t - S_{t-1}$, we have, for $s = t+1, t+2, \ldots$, the binomial analogue of (2.3), namely, $X_s = X_t + \sum_{i=t+1}^{s} \alpha_i \triangle S_i$ with $X_t = x \in \mathcal{R}$.

Proposition 3.1. *Consider, for $i = 1, \ldots$, the sets $B_i = \{\omega : \xi_i(\omega) = \xi_i^u\}$ and the associated nested risk neutral probabilities $q_i = \frac{1-\xi_i^d}{\xi_i^u - \xi_i^d}$. Let*

$$u(x,t) = -e^{-x+t} \tag{3.1}$$

and introduce the process

$$A_t = \sum_{i=1}^{t} h_i \tag{3.2}$$

[2]The authors thank an anonymous referee for bringing these results to their attention.

with $A_0 = 0$, *where*

$$h_i = q_i \log \frac{q_i}{\mathbb{P}\left(B_i \middle| \mathcal{F}_{i-1}\right)} + (1-q_i)\log \frac{1-q_i}{1-\mathbb{P}\left(B_i \middle| \mathcal{F}_{i-1}\right)}.$$

Then

$$U_t\left(x\right) = u\left(x, A_t\right) \qquad t = 0, 1, ... \tag{3.3}$$

is a forward performance process.

Sketch of the proof: Using that $\sup_{\alpha_s} E_{\mathbb{P}}\left(-e^{-\alpha_s \triangle S_s + h_s}\middle| \mathcal{F}_{s-1}\right) = -1$ (see, for example, [4]), we observe that for $s = t+1, t+2, ...,$

$$\sup_{\alpha_{t+1},\ldots,\alpha_s} E_{\mathbb{P}}\left(U_s\left(X_s\right)\middle| \mathcal{F}_t\right) = \sup_{\alpha_{t+1},\ldots,\alpha_{s-1}} E_{\mathbb{P}}\left(-\exp\left(-X_{s-1} + \sum_{i=1}^{s-1} h_i\right)\middle| \mathcal{F}_t\right)$$

and proceeding inductively we conclude.

Example 2: We consider a single stock whose price solves (cf. (2.1))

$$dS_t = S_t \sigma_t \left(\lambda_t dt + dW_t\right) \tag{3.4}$$

with $S_0 = S > 0$. The wealth process X satisfies (cf. (2.3))

$$dX_t = \sigma_t \pi_t \left(\lambda_t dt + dW_t\right) \tag{3.5}$$

with $X_0 = x$. We look for a forward solution in the form $U_t\left(x\right) = u\left(x, A_t\right)$ for some smooth concave and increasing function $u\left(x,t\right)$, with $u\left(x,0\right) = u_0\left(x\right)$. For reasons that will be apparent in the sequel, we choose $A_t = \int_0^t \lambda_s^2 ds$. For an arbitrary control π, we, then, have

$$\begin{aligned} dU_t\left(X_t\right) &= u_x\left(X_t, A_t\right)\sigma_t \pi_t dW_t \\ &\qquad + \left(u_t\left(X_t, A_t\right)\lambda_t^2 + u_x\left(X_t, A_t\right)\sigma_t\pi_t\lambda_t + \frac{1}{2}u_{xx}\left(X_t, A_t\right)\sigma_t^2\pi_t^2\right) dt \\ &= u_x\left(X_t, A_t\right)\sigma_t\pi_t dW_t \\ &\qquad + \lambda_t^2\left(u_t\left(X_t, A_t\right) + u_x\left(X_t, A_t\right)\alpha_t + \frac{1}{2}u_{xx}\left(X_t, A_t\right)\alpha_t^2\right) dt \end{aligned}$$

with $\alpha = \sigma\pi\lambda^{-1}$. We readily see that, due to the concavity assumption on u, it suffices to have that the above drift remains non positive. Because of its quadratic form, the appropriate drift sign is guaranteed if $u_t\left(x,t\right)u_{xx}\left(x,t\right) \geq \frac{1}{2}u_x^2\left(x,t\right)$, $\left(x,t\right) \in \mathcal{R}\times\left(0,+\infty\right)$. Let us now look for a concave and increasing function solving

$$u_t = \frac{1}{2}\frac{u_x^2}{u_{xx}} \quad \text{ and } \quad u\left(x,0\right) = -e^{-x}, \quad t \geq 0,\ x \in \mathcal{R}.$$

Notice that a solution to the above is given by

$$u\left(x,t\right) = -e^{-x+\frac{t}{2}}. \tag{3.6}$$

Next, consider the control policy

$$\pi_t^* = -\sigma_t^{-1}\lambda_t \frac{u_x\left(X_t^*, A_t\right)}{u_{xx}\left(X_t^*, A_t\right)}, \tag{3.7}$$

with X^* being the associated to π^* wealth process. Assuming that the appropriate regularity conditions that guarantee solution to (3.5), if π_t^* is used, hold, we easily deduce that the above drift term vanishes, yielding

$$dU_t\left(X_t^*\right) = u_x\left(X_t^*, A_t\right)\sigma_t \pi_t^* dW_t.$$

Using Definition 2.2 we conclude. We summarize these findings below.

Proposition 3.2. *Let the process λ be as in (2.2) and define*

$$A_t = \int_0^t \lambda_s^2 ds, \qquad t \geq 0. \tag{3.8}$$

Let, also, $u : (x,t) \in \mathcal{R} \times (0, +\infty)$ be given in (3.6). Then, the process

$$U_t\left(x\right) = u\left(x, A_t\right) \tag{3.9}$$

is a forward performance.

Observe that the associated optimal policy (3.7) is not only explicitly given but, also, constructed in a feedback form via the stochastic functional $\Pi_t^*\left(x\right) = -\sigma_t^{-1}\lambda_t r\left(x, A_t\right)$ with $r\left(x,t\right) = -\frac{u_x(x,t)}{u_{xx}(x,t)}$. This feedback format comes as a surprise given the non Markovian nature of the model.

4. Forward exponential performance and log-affine solutions

In this section, we construct a class of forward performance processes under the assumption that the initial datum is of the exponential form,

$$U_0\left(x\right) = -\exp\left(-\frac{x}{y}\right), \tag{4.1}$$

for $x \in \mathcal{R}$ and $y > 0$.

We recall that in the traditional exponential case, the coefficient y is a given positive constant, expressed in wealth units. It is the reciprocal of risk aversion and is often called the investor's *risk tolerance.* In the forward framework we propose herein, y will *not* be a constant. Rather, it will parametrize, as its initial condition, an auxiliary state process (see (4.3) below).

Following the insights gained by the two examples presented in the previous section, we seek a solution process constructed by combining a deterministic and a stochastic input. The first is given by the function $u : \mathcal{R} \times \mathcal{R}^+ \times \mathcal{R} \to \mathcal{R}^-$,

$$u\left(x,y,z\right) = -\exp\left(-\frac{x}{y} + z\right) \tag{4.2}$$

and is called *differential performance input.* It depends on individual characteristics, i.e. on the investor's wealth and initial risk preferences.

The stochastic input consists of a pair of Ito processes, (Y, Z), solving, respectively,

$$\begin{cases} dY_t = Y_t \delta_t \cdot \left(\kappa_t dt + dW_t\right) \\ \\ \qquad Y_0 = y > 0 \end{cases} \tag{4.3}$$

and

$$
\begin{cases} dZ_t = \eta_t dt + \xi_t \cdot dW_t \\ \\ Z_0 = 0. \end{cases} \tag{4.4}
$$

Their coefficients satisfy the assumptions given in Condition 4.2 below.

In the analysis that follows, we will be using the Moore-Penrose pseudo inverse matrix, denoted by σ^+, of the volatility matrix σ . This concept was developed, independently, by Moore in 1920 and by Penrose in 1955 (see [10]). The matrix σ^+ always exists even if σ fails to be invertible. This is, often, the case in incomplete markets and, thus, this (pseudo) invertibility notion seems to be very suitable for the applications we want to study.

Definition 4.1. *Let σ be an $d \times k$ matrix. Its Moore-Penrose pseudo inverse σ^+ is the unique $k \times d$ matrix satisfying*

$$
\begin{gathered} \sigma\sigma^+\sigma = \sigma \qquad \sigma^+\sigma\sigma^+ = \sigma^+ \\ \\ \left(\sigma\sigma^+\right)^T = \sigma\sigma^+ \ \left(\sigma^+\sigma\right)^T = \sigma^+\sigma. \end{gathered} \tag{4.5}
$$

Condition 4.2. *The processes $\delta, \kappa, \eta, \xi$ are taken to be bounded and $\mathcal{F}_t$-adapted. It is, also, assumed that*

$$
\sigma\sigma^+\delta = \delta, \tag{4.6}
$$

and

$$
\delta \cdot (\kappa - \lambda) = 0. \tag{4.7}
$$

Moreover, the drift η of the process Z satisfies

$$
2\eta = \left|\delta - \sigma\sigma^+ (\lambda + \xi)\right|^2 - |\xi|^2 . \tag{4.8}
$$

We are now ready to present one of the main results.

Theorem 4.3. *Let U_0 be given by (4.1), and the processes Y and Z solving (4.3) and (4.4), respectively, with the coefficients $\delta, \kappa, \eta, \xi$ satisfying Condition 4.2.*

Then, for $x \in \mathcal{R}$ and $t \geq 0$, the process

$$
U_t(x) = -\exp\left(-\frac{x}{Y_t} + Z_t\right) \tag{4.9}
$$

is a forward exponential performance.

Proof. We first observe that (4.1) is automatically satisfied by the choice of the initial conditions of Y and Z. The fact that $U_t(x)$ is $\mathcal{F}_t$-adapted is, also, immediate.

We continue with the derivation of the semimartingale representation for the process $U_t(X_t)$ where X_t satisfies (2.3), for a fixed π.

We set, for $\mathbf{x} = (x, y, z)$, $F(\mathbf{x}) = u(x, y, z)$ with u as in (4.2). Setting

$$
\mathbb{X}_t = (X_t, Y_t, Z_t) ,
$$

we have

$$
dF(\mathbb{X}_t) = DF(\mathbb{X}_t) \cdot d\mathbb{X}_t + \frac{1}{2} D^2 F(\mathbb{X}_t) \cdot d\langle \mathbb{X}\rangle_t ,
$$

where " $\cdot$ " stands for the inner product in the appropriate space. Direct calculations yield

$$DF(\mathbf{x}) = F(\mathbf{x}) \begin{pmatrix} -y^{-1} \\ xy^{-2} \\ 1 \end{pmatrix}$$

and

$$D^2 F(\mathbf{x}) = F(\mathbf{x}) \begin{pmatrix} y^{-2} & y^{-2} - xy^{-3} & -y^{-1} \\ y^{-2} - xy^{-3} & x^2 y^{-4} - 2xy^{-3} & xy^{-2} \\ -y^{-1} & xy^{-2} & 1 \end{pmatrix}.$$

Moreover, the joint quadratic variation $\langle \mathbb{X} \rangle$ satisfies

$$\frac{d\langle \mathbb{X} \rangle_t}{dt} = \begin{pmatrix} |\beta_t|^2 & Y_t \delta_t \cdot \beta_t & \xi_t \cdot \beta_t \\ Y_t \delta_t \cdot \beta_t & Y_t^2 |\delta_t|^2 & Y_t \delta_t \cdot \xi_t \\ \xi_t \cdot \beta_t & Y_t \delta_t \cdot \xi_t & |\xi_t|^2 \end{pmatrix}.$$

Therefore, for $U_t(X_t) = F(\mathbb{X}_t)$, we can write (to ease the presentation, we omit for the moment the time indices)

$$\begin{aligned} dU(X) &= U(X)\left(-Y^{-1}dX + XY^{-2}dY + dZ\right) \\ &\quad + \frac{1}{2}U(X)\Big(Y^{-2}|\beta|^2 + \left(Y^{-2} - XY^{-3}\right)Y\delta\cdot\beta - Y^{-1}\xi\cdot\beta \\ &\quad + \left(Y^{-2} - XY^{-3}\right)Y\delta\cdot\beta + \left(X^2Y^{-4} - 2XY^{-3}\right)Y^2|\delta|^2 + XY^{-2}Y\delta\cdot\xi \\ &\quad - Y^{-1}\xi\cdot\beta + XY^{-2}Y\delta\cdot\xi + |\xi|^2\Big)\,dt. \end{aligned}$$

Using the dynamics of X, Y and Z, and the definition of β (cf. (2.5)), we deduce

$$\begin{aligned} dU(X) &= U(X)\left(-Y^{-1}\beta\cdot dW + XY^{-1}\delta\cdot dW + \xi\cdot dW\right) \\ &\quad + \frac{1}{2}U(X)\Big(-2Y^{-1}\beta\cdot\lambda + 2XY^{-1}\delta\cdot\kappa + 2\eta + Y^{-2}|\beta|^2 \\ &\quad + 2\left(Y^{-1} - XY^{-2}\right)\delta\cdot\beta - 2Y^{-1}\xi\cdot\beta + 2XY^{-1}\delta\cdot\xi \\ &\quad + \left(X^2Y^{-2} - 2XY^{-1}\right)|\delta|^2 + |\xi|^2\Big)dt \\ &= U(X)\left(-Y^{-1}\beta\cdot dW + XY^{-1}\delta\cdot dW + \xi\cdot dW\right) \\ &\quad + \frac{1}{2}U(X)\Big(\left|Y^{-1}\beta - \left((\lambda+\xi) + \left(XY^{-1} - 1\right)\delta\right)\right|^2 \\ &\quad + 2XY^{-1}\delta\cdot(\kappa - \lambda) + 2\eta + |\xi|^2 - |\delta - (\lambda+\xi)|^2\Big)dt \end{aligned}$$

and, in turn,

$$\begin{aligned} (4.10)\quad dU(X) &= U(X)\left(-Y^{-1}B^{-1}\sigma\pi + XY^{-1}\delta + \xi\right)\cdot dW \\ &\quad + \frac{1}{2}U(X)\Big(\left|Y^{-1}B^{-1}\sigma\pi - \sigma\sigma^+\left((\lambda+\xi) + \left(XY^{-1} - 1\right)\delta\right)\right|^2 \\ &\quad + 2XY^{-1}\delta\cdot(\kappa-\lambda) + \left|\left(I - \sigma\sigma^+\right)(\lambda+\xi)\right|^2 \\ &\quad + 2\eta + |\xi|^2 - |\delta - (\lambda+\xi)|^2\Big)dt. \end{aligned}$$

Next, we observe that Condition 4.2, together with the orthogonality of the vectors $(I - \sigma\sigma^+)\cdot(\delta - (\lambda+\xi))$ and $\delta - \sigma\sigma^+(\lambda+\xi)$, yield

$$2XY^{-1}\delta\cdot(\kappa-\lambda) + \left|\left(I - \sigma\sigma^+\right)(\lambda+\xi)\right|^2 + 2\eta + |\xi|^2 - |\delta - (\lambda+\xi)|^2 = 0.$$

Therefore, (4.10) simplifies to

$$dU(X) = U(X)\left(-Y^{-1}B^{-1}\sigma\pi + XY^{-1}\delta + \xi\right)\cdot dW + \frac{1}{2}U(X)\left|Y^{-1}B^{-1}\sigma\pi - \sigma\sigma^{+}\left(\lambda + \left(XY^{-1}-1\right)\delta + \xi\right)\right|^{2}dt. \tag{4.11}$$

We next choose, the feedback portfolio control process

$$\pi^{*} = YB\sigma^{+}\left(\lambda + \left(X^{*}Y^{-1}-1\right)\delta + \xi\right). \tag{4.12}$$

Clearly, (2.3) has a unique solution, denoted by X^{*}, solving

$$\begin{aligned} dX^{*} &= B^{-1}\sigma\pi^{*}\cdot(\lambda dt + dW) \\ &= \left(Y\left(\sigma\sigma^{+}(\lambda+\xi) - \delta\right) + X^{*}\delta\right)\cdot(\lambda dt + dW). \end{aligned} \tag{4.13}$$

Consider now the process $U_t(X_t^{*}) = u(X_t^{*}, Y_t, Z_t)$ and recall that

$$U_t(X_t) = u(X_t, Y_t, Z_t),$$

with X solving (2.3) for a generic policy π. To complete the proof , it suffices to establish that they are, respectively, martingale and supermartingale with respect to $\mathcal{F}_t$ and under $\mathbb{P}$. The latter assertion follows directly from (4.11) and the negativity of U. For the former one, we see from (4.11) that $U_t(X_t^{*})$ satisfies

$$dU(X^{*}) = U(X^{*})\left(-Y^{-1}B^{-1}\sigma\pi^{*} + X^{*}Y^{-1}\delta + \xi\right)\cdot dW \tag{4.14}$$

and from (4.12),

$$= U(X^{*})\left(\sigma\sigma^{+}(\delta - \lambda) + \left(I - \sigma\sigma^{+}\right)\xi\right)\cdot dW.$$

The martingality property then follows from the assumptions on the coefficients and the choice of U. □

Remark 1: Note that under the assumption $\delta\cdot(\kappa - \lambda) = 0$, the dynamics of the auxiliary process Y can, also, be written as

$$dY_t = Y_t\delta_t\cdot(\lambda_t dt + dW_t) \tag{4.15}$$

with $Y_0 = y > 0$. Consequently, without loss of generality, in choosing the process Y, we assume from now on that $\kappa = \lambda$. We have

$$Y_t = y\exp\left(\int_0^t\left(\delta_s\cdot\lambda_s - \frac{1}{2}\left|\delta_s\right|^{2}\right)ds + \int_0^t\delta_s\cdot dW_s\right). \tag{4.16}$$

Remark 2: Under the choice of drift (4.8), the dynamics of the second auxiliary process Z become

$$dZ_t = \frac{1}{2}\left(\left|\delta_t - \sigma_t\sigma_t^{+}(\lambda_t + \xi_t)\right|^{2} - \left|\xi_t\right|^{2}\right)dt + \xi_t\cdot dW_t \tag{4.17}$$

with $Z_0 = 0$. Thus,

$$Z_t = \frac{1}{2}\int_0^t\left(\left|\delta_s - \sigma_s\sigma_s^{+}(\lambda_s + \xi_s)\right|^{2} - \left|\xi_s\right|^{2}\right)ds + \int_0^t\xi_s\cdot dW_s. \tag{4.18}$$

Next, we construct the optimal wealth process. For completeness, we restate some of the above findings.

The proof of (4.21) follows directly from (4.13), (4.19), and Theorem 53 in [11].

Theorem 4.4. *Let Y and Z satisfy (4.16) and (4.18). For $t \geq 0$, the associated optimal allocation process (cf. (4.12)) is given by*

$$\pi_t^* = B_t Y_t \sigma_t^+ (\lambda_t + \xi_t - \delta_t) + B_t X_t^* \sigma_t^+ \delta_t, \tag{4.19}$$

where X^* is the unique solution to the wealth equation (2.3), with π^* being used.

Thus, the optimal discounted wealth process X^* solves, for $t \geq 0$,

$$\begin{aligned} dX_t^* &= B_t^{-1} \sigma_t \pi_t^* \cdot (\lambda_t dt + dW_t) \\ &= \left(Y_t \left(\sigma_t \sigma_t^+ (\lambda_t + \xi_t) - \delta_t\right) + X_t^* \delta_t\right) \cdot (\lambda_t dt + dW_t). \end{aligned} \tag{4.20}$$

It is, in turn, given by

$$X_t^* = \mathcal{E}_t \left(x + \int_0^t \mathcal{E}_s^{-1} Y_s \left(\sigma_s \sigma_s^+ (\lambda_s + \xi_s) - \delta_s\right) \cdot ((\lambda_s - \delta_s) ds + dW_s) \right) \tag{4.21}$$

where

$$\mathcal{E}_t = \exp\left(\int_0^t \left(\delta_s \cdot \lambda_s - \frac{1}{2} |\delta_s|^2 \right) ds + \int_0^t \delta_s \cdot dW_s \right). \tag{4.22}$$

Corollary 4.5. *The optimal π^* defined in (4.19) is an affine function of the initial wealth x, namely, for $t \geq 0$,*

$$\begin{aligned} \pi_t^* = {} & x \mathcal{E}_t B_t \sigma_t^+ \delta_t + B_t Y_t \sigma_t^+ (\lambda_t + \xi_t - \delta_t) \\ & + B_t \mathcal{E}_t \Big(\int_0^t \mathcal{E}_s^{-1} Y_s \left(\sigma_s \sigma_s^+ (\lambda_s + \xi_s) - \delta_s\right) \\ & \qquad \cdot ((\lambda_s - \delta_s) ds + dW_s) \Big) \sigma_t^+ \delta_t. \end{aligned} \tag{4.23}$$

The next result yields the optimal level of the investment system's performance. It follows directly from (4.9) and (4.14).

Proposition 4.6. *At the optimum, the forward exponential performance process is given by*

$$U_t (X_t^*) = u (X_t^*, Y_t, Z_t)$$

with u as in (4.2) and X^*, Y and Z as in (4.21), (4.3) and (4.4).

It has the semimartingale representation

$$dU_t (X_t^*) = U_t (X_t^*) \left(\sigma_t \sigma_t^+ (\delta_t - \lambda_t) + \left(I - \sigma_t \sigma_t^+\right) \xi_t\right) \cdot dW_t$$

and, hence, it is given by the martingale

$$\begin{aligned} U_t (X_t^*) = - \exp \Big(& - \frac{x}{y} - \int_0^t \frac{1}{2} \left|\sigma_s \sigma_s^+ (\delta_s - \lambda_s) + \left(I - \sigma_s \sigma_s^+\right) \xi_s\right|^2 ds \\ & + \int_0^t \left(\sigma_s \sigma_s^+ (\delta_s - \lambda_s) + \left(I - \sigma_s \sigma_s^+\right) \xi_s\right) \cdot dW_s \Big). \end{aligned} \tag{4.24}$$

5. Examples

We construct the forward performance process for various choices of model coefficients. We also compute and analyze the associate optimal wealth and asset allocation, as well as the optimal performance level. For convenience, we assume that the initial datum is assigned at $t = 0$.

We recall that $\pi^* = \left(\pi^{1,*}, ..., \pi^{k,*}\right)$ is the vector of the optimal allocations in the k risky assets. It is given by (4.19) while the optimal discounted wealth, X^*, is given in (4.21). Recall that the amount $\pi^{0,*} = X^* - \mathbf{1} \cdot \frac{\pi^*}{B}$ is the optimal allocation in the riskless asset, the discounted bond.

Case 1: $\delta = \xi = 0$.

Then, $Y_t = y$, for $t \geq 0$. The forward performance process takes the form

$$U_t(x) = -\exp\left(-\frac{x}{y} + \int_0^t \frac{1}{2}\left|\sigma_s \sigma_s^+ \lambda_s\right|^2 ds\right).$$

Note that even in this simple case, the solution is equal to the classical exponential utility only at $t = 0$.

The optimal discounted wealth and optimal asset allocation are given, respectively, by

$$X_t^* = x + \int_0^t y\left(\sigma_s \sigma_s^+ \lambda_s\right) \cdot \left(\lambda_s ds + dW_s\right)$$

and

$$\pi_t^* = y B_t \sigma_t^+ \lambda_t.$$

At the optimum,

$$U_t\left(X_t^*\right) = -\exp\left(-\frac{x}{y} - \int_0^t \frac{1}{2}\left|\sigma_s \sigma_s^+ \lambda_s\right|^2 ds - \int_0^t \sigma_s \sigma_s^+ \lambda_s \cdot dW_s\right).$$

Observe that π^* is *independent* of the initial wealth x. Consequently, the total amount allocated in the risky assets is given by

$$\mathbf{1} \cdot \frac{\pi_t^*}{B_t} = \mathbf{1} \cdot y \sigma_t^+ \lambda_t$$

and, thus, the amount invested in the riskless asset is

$$\pi_t^{0,*} = X_t^* - \mathbf{1} \cdot y \sigma_t^+ \lambda_t.$$

Clearly, such an allocation is rather conservative and is often viewed as an argument against the classical exponential utility. However, as examples below demonstrate, the class of forward exponential performances is rich enough to present an interesting range of allocations.

Case 2: $\sigma\sigma^+(\delta - \lambda) + (I - \sigma\sigma^+)\xi = 0$.

We observe that this condition yields $\sigma^+(\delta - \lambda) = 0$ and $\sigma\sigma^+\xi = \xi$. It is, then, easy to see that $Z_t = \int_0^t \xi_s \cdot dW_s$ and, in turn,

$$U_t(x) = -\exp\left(-\frac{x}{Y_t} + \int_0^t \xi_s \cdot dW_s\right)$$

with Y as in (4.16).

The optimal discounted wealth is given by

$$X_t^* = \mathcal{E}_t\left(x + \int_0^t \mathcal{E}_s^{-1} Y_s \xi_s \cdot \left(\left(\lambda_s - \delta_s\right) ds + dW_s\right)\right)$$

with $\mathcal{E}$ as in (4.22). Respectively,

$$\pi_t^* = x\mathcal{E}_t B_t \sigma_t^+ \delta_t + Y_t B_t \sigma_t^+ \xi_t + B_t \mathcal{E}_t \left(\int_0^t \mathcal{E}_s^{-1} Y_s \xi_s \cdot dW_s \right) \sigma_t^+ \delta_t.$$

At the optimum,

$$U_t\left(X_t^*\right) = U_0\left(x\right) = -\exp\left(-\frac{x}{y}\right),$$

namely, the optimal level of forward performance remains constant across times.

Case 3: $\delta = 0$ and $\lambda + \xi = 0$.

In this case, $\mathcal{E}_t = 1$, $Y_t = y > 0$ and $Z_t = -\int_0^t \frac{1}{2} \left|\lambda_s\right|^2 ds - \int_0^t \lambda_s \cdot dW_s$. Then,

$$U_t\left(x\right) = -\exp\left(-\frac{x}{y} - \frac{1}{2}\int_0^t \left|\lambda_s\right|^2 ds - \int_0^t \lambda_s \cdot dW_s\right).$$

The optimal discounted wealth remains constant,

$$X_t^* = x.$$

In turn, the optimal allocations are

$$\pi_t^* = 0 \qquad \text{and} \qquad \pi_t^{0,*} = X_t^* = x. \tag{5.1}$$

Moreover,

$$U_t\left(X_t^*\right) = U_t\left(x\right).$$

It is important to notice that, for all trading times, the optimal allocation consists of putting *zero* into the risky assets and, therefore, investing the entire wealth into the riskless asset. Such a solution seems to capture quite accurately the strategy of a derivatives trader for whom the underlying objective is to hedge as opposed to the asset manager whose objective is to invest.

Case 4: $\delta = \lambda + \xi$ with $\lambda + \xi \neq 0$.

Observe that this condition implies that $\delta = \sigma\sigma^+\left(\lambda + \xi\right)$ and, in turn, that $Z_t = -\int_0^t \frac{1}{2}\left|\xi_s\right|^2 ds + \int_0^t \xi_s \cdot dW_s$. Therefore,

$$U_t\left(x\right) = -\exp\left(-\frac{x}{Y_t} - \int_0^t \frac{1}{2}\left|\xi_s\right|^2 ds + \int_0^t \xi_s \cdot dW_s\right).$$

We easily see that

$$X_t^* = x\mathcal{E}_t.$$

Note that, the returns of the processes X^* and Y are the *same*, i.e., $\frac{dX_t^*}{X_t^*} = \frac{dY_t}{Y_t}$ and, thus, $X_t^* = \frac{x}{y}Y_t$.

The optimal asset allocation is given by

$$\pi_t^* = B_t X_t^* \sigma_t^+ \delta_t$$

and the optimal performance level by

$$U_t\left(X_t^*\right) = -\exp\left(-\frac{x}{Y_t} - \int_0^t \frac{1}{2}\left|\xi_s\right|^2 ds + \int_0^t \xi_s \cdot dW_s\right).$$

Observe that, contrary to what we have observed in traditional backward exponential utility problems, the optimal portfolio is a *linear functional of the wealth* and not independent of it.

Let us, next, assume that $\mathbf{1}\cdot\sigma_t^{+}\left(\lambda_t+\xi_t\right)=1$. We, then, have

$$\mathbf{1}\cdot\frac{\pi_t^*}{B_t}=X_t^* \qquad \text{and} \qquad \pi_t^{0,*}=0. \tag{5.2}$$

Hence, the optimal allocation π^* puts *zero* amount in the riskless asset and invests *all* wealth in the risky assets, according to the weights specified by the vector $\sigma^{+}\left(\lambda+\xi\right).$

Note, also, that for an arbitrary vector ν_t with $\mathbf{1}\cdot\sigma_t^{+}\nu_t\neq 0$, the vector

$$\xi_t=\frac{1-\mathbf{1}\cdot\sigma_t^{+}\lambda_t}{\mathbf{1}\cdot\sigma_t^{+}\nu_t}\nu_t$$

satisfies the above constraint since $\mathbf{1}\cdot\sigma_t^{+}\left(\lambda_t+\frac{1-\mathbf{1}\cdot\sigma_t^{+}\lambda_t}{\mathbf{1}\cdot\sigma_t^{+}\nu_t}\nu_t\right)=1$. It is, then, natural to ask whether we can generate optimal portfolios that allocate *arbitrary*, but constant, fractions of wealth to the different accounts. The answer is affirmative. Indeed, for $p\in\mathcal{R}$, we set,

$$\mathbf{1}\cdot\sigma_t^{+}\left(\lambda_t+\xi_t\right)=p$$

a.e. and $t\in[0,T]$. Then, the total investment in the risky assets and the allocation in the riskless bond are given, respectively, by

$$\mathbf{1}\cdot\frac{\pi_t^*}{B_t}=pX_t^* \qquad \text{and} \qquad \frac{\pi_t^0}{B_t}=\left(1-p\right)X_t^*.$$

6. Conclusions and extensions

We introduced a new concept in stochastic optimization, namely, the one of forward performance. A forward performance is an adapted process that is a martingale at an optimum and a supermartingale otherwise. In addition, in contrast to the traditional control approach, it is prespecified today and not at the end of the horizon. This removes the horizon dependence and enables us to define the optimal solution for all future times.

As an application, we study a portfolio choice problem in incomplete markets. The model is general and the forward performance is obtained under minimal assumptions on the underlying dynamics. It is constructed by combining appropriately chosen deterministic and stochastic market inputs. The deterministic input depends on the investor's preferences while the stochastic input incorporates exclusively information from the market changes. An interesting class of policies was discovered yielding, among others, two extreme situations. In one of them, the investor allocates zero wealth in the risky assets while in the other the situation is totally reversed.

Working with forward performance criteria, instead of the classical (backward) ones, seems to give more intuitive and tractable solutions for both the performance process and the optimal policies. It is worth observing that the classical solutions can be thought as special, but rather limited, cases of forward solutions.

Interesting questions arise. They are related, among others, to necessary and sufficient conditions for the solution to remain log-affine (cf. (4.9)). A more challenging question is to solve the problem for arbitrary initial data. While the power and logarithmic case appear sufficiently tractable, the general case, currently under study (see [7]) poses several difficulties, related among others to existence of solutions to inverse problems of fast diffusion type.

In a different direction, one could try to price claims using forward performance criteria. This has been done by the authors for incomplete binomial models and for diffusion models with stochastic volatility (see, respectively, [4] and [8], and [6]). The emerging forward indifference prices do not coincide with their traditional counterparts and have more intuitive structural representation properties.

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