IMS Collections
Markov Processes and Related Topics: A Festschrift for Thomas G. Kurtz
Vol. 4 (2008) 41–61

DOI: 10.1214/074921708000000291

Fractional Stability of Diffusion Approximation for Random Differential Equations

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Abstract: We consider the systems of random differential equations. The coefficients of the equations depend on a small parameter. The first equation, "slow" component, Ordinary Differential Equation (ODE), has unbounded highly oscillating in space variable coefficients and random perturbations, which are described by the second equation, "fast" component, Stochastic Differential Equation (SDE) with periodic coefficients. Sufficient conditions for weak convergence as small parameter goes to zero of the solutions of the "slow" components to the certain stochastic process are given.

1. Introduction

In the paper, we consider systems of random equations with a small parameter ε. The first equation, the "slow" component, is an Ordinary Differential Equations (ODE) with unbounded highly oscillating coefficients which depend on the Markov diffusion processes with periodic coefficients, which are the "fast" component of the systems. We will study the weak convergence of probability measures, induced by the solutions of the "slow" equations to the diffusion process.

It is well known that, in the case of the Diffusion Approximation (DA), a drift coefficient of the approximating Stochastic Differential Equation (SDE), includes a derivative with respect to a space variable of the unbounded coefficients of the approximated random differential equation (see Ch. 2.2). That means, we cannot apply the DA results because of the highly oscillating character of dependency on the ε of the unbounded coefficient of the "slow" component. On the other hand, we cannot apply the limit theorem for SDEs because the "slow" component is an ODE, and consequently has no nonzero diffusion coefficient (the presence of strongly positive diffusion coefficient is a necessary condition for such kind of theorems).

The method is a combination of the results of these two directions. We choose the order of oscillation (parameter δ) in such a way that the conditions, from the DA's theorem ((A) and (AB)), allow us to get the nonnegative "candidate" to be the diffusion coefficient, and then to use the second part of the conditions (from the Limit Theorem for SDE: (B) and (C)) to obtain the limit process (see Chapters 2.2, 2.4).

The aim of this paper is to find the answers to the following questions:

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*The author is grateful to Dr. V.V. Andrievskii, Dr. J. Diestel, and Dr. R.S. Varga for useful remarks and hospitality.

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AMS 2000 subject classifications: Primary 60F17; secondary 60F99

Keywords and phrases: random dynamical systems, diffusion approximation, oscillation, fractional stability

1. Is it possible to extend DAs results to be true in the presence of high oscillation in space variable of the coefficients of the random processes? If the answer is yes, what kind of the conditions need to be added to usual conditions of DA?

2. How does the presence of oscillation in the coefficients of the random processes influence the order of convergence in DA? What is the precise order of convergence in DA and how does it depend on the order of oscillation? What is the critical case?

For the second question, we get results that depend on an apparently critical number (order of oscillation), equals to 1/2 (Theorem 3.1.1 below).

Asymptotic behavior of the solutions of the unperturbed stochastic equations with unbounded drift seems to be considered for the first time in the papers [1], [10], and [13].

For SDEs, with coefficients depending on a small parameter by irregular way without random perturbations, necessary and sufficient conditions of the weak convergence of solutions in more general situations are obtained in [11].

A different approach to the investigation of weak convergence of one-dimensional Markov processes was demonstrated in [3].

The Averaging Principle for SDEs with random perturbations and highly oscillating coefficients was considered in [7]. In that paper, the first component of the system is an SDE with a strongly positive diffusion coefficient. The asymptotic behavior of the first components, on the time intervals of the order $O(\varepsilon^{-1})$, was studied. Sufficient conditions for weak convergence of the measures, induced by the first components, were stated and the apparently critical number (order of oscillation), equals to 1/3, was obtained. In the present work, the first component is a random ODE and does not contain a nonzero diffusion coefficient. This makes the investigation of the limit behavior of the first component more difficult but, instead, we consider the DAs scheme (on the time intervals of the order $O(\varepsilon^{-2})$). The result here is Fractional Stability of the DA (note that the DA is a result of the type of the functional Central Limit Theorem).

The study of the DA was initiated by Khasminskii R. Z. [6], and developed by many authors (see, e.g. monographs [2],[15] and bibliography, and [12]). We note that the case, when the coefficients of the first equations have no high oscillations with respect to space variable, the problems of the weak convergence of solutions under the various conditions on the coefficients and random perturbations have been studied.

A generalization of Khasminskii's result [6], for a mean-zero fluctuation, stationary field, was considered in [8].

2. Conditions and preliminary results

Let (Ω, F, P) denote some probability space with filtration F_t, $t \in [0,T]$. Let E_n be a n-dimensional Euclidean space, $E_+ = [0, +\infty)$, symbol E denotes the mathematical expectation, $\dot{f}(x)$ be a derivative of the function $f(x)$, and ∇_y is the symbol of the gradient with respect to $y \in E_n$. We denote different positive constants by C, with indexes if need.

Let us consider a system of random equations, $t \in [0,T]$,

$$\xi_\varepsilon(t) = \xi_0 + \frac{1}{\varepsilon}\int_0^t g\left(\frac{\xi_\varepsilon(s)}{\varepsilon^\delta}, \eta_\varepsilon(s)\right)ds + \frac{1}{\varepsilon^\delta}\int_0^t c\left(\frac{\xi_\varepsilon(s)}{\varepsilon^\delta}, \eta_\varepsilon(s)\right)ds + \int_0^t m\left(\frac{\xi_\varepsilon(s)}{\varepsilon^\delta}, \eta_\varepsilon(s)\right)ds, \tag{2.1}$$

$$\eta_\varepsilon(t) = \eta_0 + \frac{1}{\varepsilon^2}\int_0^t b\left(\eta_\varepsilon(s)\right) ds + \frac{1}{\varepsilon}\int_0^t \sigma\left(\eta_\varepsilon(s)\right) dw_1(s). \tag{2.2}$$

Here $\{w_1(t), F_t\}$ is n-dimensional standard Wiener process. The processes $\xi_\varepsilon(t) \in E_1, \eta_\varepsilon(s) \in E_n$, the constants ξ_0, η_0 are non random; $g(x,y), c(x,y)$, and $m(x,y)$ are functions from $E_1 \times E_n$ in E_1; $b(y)$, $\sigma(y)$ are the functions from E_n in E_n and $\mathcal{L}(E_n)$, respectively; $\varepsilon > 0$ is a small parameter, and δ is a fixed number from $]0, \frac{1}{2}[$. If the equation (2.2) has a unique (in sense of law) solution, then the distribution $\eta_\varepsilon(t\varepsilon^2)$ coincides with the distribution of process $\eta(t)$, the solution of the Ito stochastic equation:

$$d\eta(t) \;=\; b(\eta(t))dt + \sigma(\eta(t))dw(t)$$

and does not depend on ε.

Let us denote, by $C^{k,l}_{x,y}(E_1, E_n)$, the class of the functions $f(x,y)$ which are k and l times continuously differentiable with respect to $x \in E_1$, and $y \in E_n$ respectively. The symbol "b" in the notation of this class ($C^{k,l}_{x,y,b}(E_1, E_n)$) indicates that these functions and their derivatives, of the stipulated order with respect to $x \in E_1$, are bounded. Let $a_{ij}(y)$ be the components of the $n \times n$ matrix $a(y) = \sigma(y)\sigma'(y)$, and let $b_i(y)$ be the components of the vector $b(y)$.

2.1. Conditions A and AB

We next introduce condition (A).

Condition (A)

A1. The functions $a_{ij}(y)$, $b_i(y) \in C^2_y(E_n)$, and are periodic of period 1;

A2. There exists a constant $\lambda_0 > 0$ such that for every $y, \zeta \in E_n$

$$a_{ij}(y)\zeta_i\zeta_j \geq \lambda_0|\zeta|^2;$$

A3. The functions, $g(x,y), c(x,y)$, and $m(x,y) \in C^{2,2}_{x,y,b}(E_1, E_n)$ are periodic of period 1 in y.

Remark 2.1. Under the condition (A) the system (2.1), (2.2) has a unique strong solution.

Let us denote by L^* the operator which is formally conjugate to the generating operator L of η_t:

$$L \;=\; \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(y)\frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{n} b_i(y)\frac{\partial}{\partial y_i}.$$

We shall denote by Y the unit torus in E_n. As it is well known (see, for example, [1]), the next problem

$$L^* p(y) \;=\; 0, \qquad \int_Y p(y)dy \;=\; 1,$$

has the unique positive periodic, of period 1, solution $p(y)$, and for a periodic of period 1 function $h(y)$ so that $\int_Y h(y)p(y)dy = 0$, the next problem

$$Ld(y) \;=\; h(y), \qquad \int_Y d(y)dy \;=\; 0 \tag{2.3}$$

has the unique periodic of period 1 solution $d(y) \in C^2(E_n)$. Lemma 4.1. ([14]) gives the estimation of the solution of the problem (2.3) by the right hand side $h(y)$.

The estimations of the solution of the problem

$$Ld(x,y) \;=\; h(x,y), \qquad \int_Y d(x,y)dy \;=\; 0 \tag{2.4}$$

and its first two derivatives with respect to the parameter x easily can be derived form Corollary 4.2.

If $\int_Y h(x,y)p(y)dy \;=\; 0$ and if $h(x,y)$ belongs to the class $C^{2,2}_{x,y,b}(E_1,E_n)$, and is periodic of period 1 with respect to $y\in E_n$, then, by Corollary 4.2, the solution of (2.4), $d(x,y)$, is an element of $C^{2,2}_{x,y,b}(E_1,E_n)$.

We are going to introduce the "balance condition."

Condition (AB)

For every $x\in E_1$,

$$\bar{g}(x) = \langle g(x,\,\cdot\,)\rangle := \int_Y g(x,y)p(y)dy = 0.$$

Let us consider the Poisson problem

$$L\psi\left(x,y\right) = -g\left(x,y\right), \qquad \int_Y \psi\left(x,y\right)dy = 0. \tag{2.5}$$

Remark 2.2. Under the Conditions (A) and (AB), problem (2.5) has the unique solution $\psi\left(x,y\right)\in C^{2,2}_{x,y,b}(E_1,E_n)$.

Let us set

$$\alpha\left(x,y\right) := \left(\sigma\left(y\right)\nabla_y\psi\left(x,y\right)\right)^2, \tag{2.6}$$

$$\beta\left(x,y\right) := c\left(x,y\right) + g\left(x,y\right)\;\frac{\partial}{\partial x}\psi\left(x,y\right), \tag{2.7}$$

and $p_\varepsilon(x,y) := p(\frac{x}{\varepsilon^\delta},y)$, and $l_\varepsilon(x) := l(\frac{x}{\varepsilon^\delta})$.

Let us consider

2.2. Model example

Let us fix $n=1$ and consider the system (2.1), (2.2) under previous assumptions, $t\in[0,T]$.

$$\xi_\varepsilon(t) = \xi_0 + \frac{1}{\varepsilon}\int_0^t D\left(\frac{\xi_\varepsilon(s)}{\varepsilon^\delta}\right)\cos(2\pi\eta_\varepsilon(s))ds,$$
$$\eta_\varepsilon(t) = \eta_0 + \frac{1}{\varepsilon}w_1(t).$$

We will investigate the limit behavior of $\xi_\varepsilon(t)$ as ϵ goes to 0.

Let the Condition (A) is satisfied. Then, $a(y) \;=\; 1,\; L \;=\; \frac{1}{2}\frac{d^2}{dy^2},$ and the problem

$$L^*p(y) \;=\; 0, \qquad \int_0^1 p(y)dy = 1$$

has the unique solution $p(y) \;=\; 1$.

Condition (AB) gives $\int_0^1\cos(2\pi y)dy \;=\; 0$.

In this case $\psi(x,y) = D(x)\Psi_1(y)$, where $\Psi_1(y) = \frac{\cos(2\pi y)}{2\pi^2}$ is unique solution of the problem

$$L\Psi_1(y) = -\cos(2\pi y), \qquad \int_0^1 \Psi_1(y)dy = 0.$$

So, $\psi(x,y) = \frac{D(x)\cos(2\pi y)}{2\pi^2}$.
Then for the process

$$\theta_\varepsilon(\xi_\varepsilon(t),\eta_\varepsilon(t)) := \xi_\varepsilon(t) + \varepsilon\psi\left(\frac{\xi_\varepsilon(t)}{\varepsilon^\delta},\eta_\varepsilon(t)\right),$$

using Ito's formula, we get

$$\theta_\varepsilon(\xi_\varepsilon(t),\eta_\varepsilon(t)) = \theta_\varepsilon(\xi_0,\eta_0) + \frac{1}{2\pi^2\varepsilon^\delta}\int_0^t \dot{D}\left(\frac{\xi_\varepsilon(s)}{\varepsilon^\delta}\right) D\left(\frac{\xi_\varepsilon(s)}{\varepsilon^\delta}\right)\cos^2(2\pi\eta_\varepsilon(s))ds$$
$$-\frac{1}{\pi}\int_0^t D\left(\frac{\xi_\varepsilon(s)}{\varepsilon^\delta}\right)\sin(2\pi\eta_\varepsilon(s))dw_1(s).$$

We will use the notation (2.6) and (2.7). Then

$$\overline{\alpha}_\varepsilon(x) = \frac{D^2\left(x/\varepsilon^\delta\right)}{(2\pi)^3}, \quad \text{and} \quad \overline{\beta}_\varepsilon(x) = \frac{D\left(x/\varepsilon^\delta\right)\dot{D}\left(x/\varepsilon^\delta\right)}{4\pi^3}.$$

The process $\xi_\varepsilon(t)$ is asymptotically "close" to the process

$$\hat{\theta}_\varepsilon(\xi_\varepsilon(t),\eta_\varepsilon(t)) = \theta_\varepsilon(\xi_0,\eta_0) + \frac{1}{\varepsilon^\delta}\int_0^t \overline{\beta}_\varepsilon(x) + \int_0^t \sqrt{\overline{\alpha}_\varepsilon(x)}dw_1(s)$$

(the Lemma 4.6). So, using the conditions of classical DA, we can not pass to the limit (the coefficients of the process $\hat{\theta}_\varepsilon(\xi_\varepsilon(t),\eta_\varepsilon(t))$ still depend on a small parameter ϵ by irregular way). It is obviously, to answer the question posed in the beginning of the example we need some additional conditions.

We will return to this example later on after introducing needed conditions .

2.3. Conditions B and C

We are taking in a mind the process $\hat{\theta}_\varepsilon(\xi_\varepsilon(t),\eta_\varepsilon(t))$ from the previous example. The next conditions are natural for investigation of the limit behavior of the SDE (compare for example with [11])

Let us introduce the next condition.

Condition (B)

B1. There exists a constant $\lambda_1 > 0$ such that for every $x \in E_1$,

$$\overline{\alpha}(x) \geq \lambda_1;$$

B2. There exists a constant $\lambda_2 > 0$ such that for every $x \in E_1$,

$$\left|\int_0^x \frac{\overline{\beta}(z)}{\overline{\alpha}(z)}dz\right| \leq \lambda_2.$$

Let us set

$$F(x) := \exp\left\{-2\int_0^x \frac{\overline{\beta}(z)}{\overline{\alpha}(z)}dz\right\}, \qquad \text{and} \quad h(x) := \int_0^x F(z)dz.$$

Condition (C)

There exist the constants $\kappa_0, \kappa_1,$ and κ_2 such that for $z \in E_1$,

C0. $\lim_{|z|\to\infty} \frac{1}{z}\int_0^z F(x)dx = \kappa_0;$
C1. $\lim_{|z|\to\infty} \frac{1}{z}\int_0^z \frac{1}{\overline{\alpha}(x)F(x)}dx = \kappa_1;$
C2. $\lim_{|z|\to\infty} \frac{1}{z}\int_0^z \frac{\overline{m}(x)}{\overline{\alpha}(x)}dx = \kappa_2.$

Remark 2.3. It follows from Lemma 4.3 and our conditions above that there exist positive constants C_1, C_2 such that

$$0 < C_1 < \kappa_0 < C_2\,; \quad 0 < C_1 < \kappa_1 < C_2\,; \quad |\kappa_2| < C_2.$$

2.4. Model example (continuation)

The Condition (B) gives:

There exist the constants $\lambda_1 > 0$ and $\lambda_2 > 0$ such that for every $x \in E_1$

$$D^2(x) \geq \lambda_1 \quad \text{and} \quad D(x) \leq \lambda_2 D(0).$$

The function $F(x)$ has the form $F(x) = \frac{D(0)}{D(x)}$. The Condition (C) implies:

There exist the constant κ_0 such that for $z \in E_1$,

$$\lim_{|z|\to\infty} \frac{1}{z}\int_0^z \frac{dx}{D(x)} = \frac{\kappa_0}{D(0)};$$

Remark 2.4.1. Under these conditions the constant from the Condition (C2) is defined by $\kappa_1 = \kappa_0 \frac{(2\pi)^3}{D^2(0)}$.
Using the Theorem 3.1.1., for the process $\xi_\varepsilon(t)$ we got the limit process

$$\xi(t) = \xi_0 + \frac{D(0)}{\kappa_0(2\pi)^{\frac{3}{2}}}w(t).$$

Let us introduce

$$f(x) := \frac{1}{\kappa_0}h(x) - x, \qquad x \in E_1, \tag{2.8}$$

Obviously,

$$\overline{L}_x f(x) = \overline{\beta}(x)\dot{f}(x) + \frac{1}{2}\overline{\alpha}(x)\ddot{f}(x) \;=\; -\overline{\beta}(x). \tag{2.9}$$

Let $\beta^0(x, y)$ denotes the function such that

$$L_x f(x) = \beta(x, y)\dot{f}(x) + \frac{1}{2}\alpha(x, y)\ddot{f}(x) = \beta^0(x, y). \tag{2.10}$$

Remark 2.4. From (2.9) and (2.10), we have $\overline{\beta^0}(x) \;=\; -\overline{\beta}(x).$

3. Main result

In this chapter we formulate and give the proof of our main theorem.

3.1. Fractional stability of diffusion approximation

Let $(C[0,T],C)$, be a space of all continuous functions on $[0,T]$ with the family of σ-algebras $C = \{C_t\}_{0\le t\le T}$. The space $C_0^\infty(E_1)$ is the space of all infinitely differentiable functions with compact support on E_1. For a fixed number $\delta \in\,]0,\frac{1}{2}[$, let us denote by $\{\mu_\delta^\varepsilon, \varepsilon > 0\}$ the family of probability measures induced by the random processes $\{\xi_\varepsilon(\cdot), \varepsilon > 0\}$ on $C([0,T])$ and by "$\Rightarrow$", the sign for the weak convergence of measures. We will prove the weak convergence

$$\mu_\delta^\varepsilon \Rightarrow \mu,$$

as ε tends to 0 for each δ, where μ is the measure corresponding to the random process

$$\xi(t) \,=\, \xi_0 \,+\, \beta_0 t \,+\, \sigma_0 w(t).$$

Here, ξ_0 is the initial condition from (2.1), β_0 and σ_0 are the certain constant coefficients, and $w(t), t \in [0,T]$, is the standard one-dimensional Wiener process.

Theorem 3.1.1. *Let conditions* $(A), (AB), (B)$, *and* (C) *be fulfilled. Then, for every* $\delta \in\,]0,\frac{1}{2}[$ *the measures* $\mu_\delta^\varepsilon \Rightarrow \mu$ *as* ε *tends to* 0. *The random process* $\xi(t)$, *which corresponds to* μ, *is defined by*

$$\xi(t) \,=\, \xi_0 \,+\, \beta_0 t \,+\, \sigma_0 w(t),$$

where

$$\beta_0 \,=\, \frac{\kappa_2}{\kappa_0\kappa_1}, \quad \sigma_0 \,=\, \frac{1}{\sqrt{\kappa_0\kappa_1}}.$$

Remark 3.1.1. The case $\delta = 0$ corresponds to the classical DAs scheme. The sufficient conditions of the weak convergence of μ_0^ε as ε tends to 0 can be simplified in this case.

3.2. Proof of Theorem 3.1.1

First (I), we will prove that, for each $\delta \in\,]0,\frac{1}{2}[$, the family of measures $\{\mu_\delta^\varepsilon,\, \varepsilon > 0\}$, is weakly compact on $C[0,T]$, and, second (II), we are going to the limit as ε tends to 0, giving the possibility to the coefficients of equation (2.1) obtain the averaging form with respect to random perturbations (at that time for new processes, which is "close" to initial one we will have the equations with positive diffusion coefficients: conditions (A), (AB), (B)) and after that we will use the condition (C) for identification of the limit process.

I. Now we fix arbitrary $\delta \in\,]0,\frac{1}{2}[$. By Ito's formula applied to $\varepsilon\psi\left(\frac{\xi_\varepsilon(t)}{\varepsilon^\delta}, \eta_\varepsilon(t)\right)$ ((2.5) and Remark 2.2.), for

$$\zeta_\varepsilon(t) := \zeta_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t)) = \xi_\varepsilon(t) + \varepsilon\psi_\varepsilon\left(\xi_\varepsilon(t), \eta_\varepsilon(t)\right),$$

using the notation (2.6) and (2.7), we obtain

$$\zeta_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t)) - \zeta_\varepsilon(\xi_0, \eta_0)$$

$$
\begin{aligned}
&= \varepsilon^{1-2\delta} \int_0^t \frac{\partial}{\partial x} \psi_\varepsilon\left(\xi_\varepsilon(s), \eta_\varepsilon(s)\right) c_\varepsilon(\xi_\varepsilon(s), \eta_\varepsilon(s)) ds \\
&\quad + \varepsilon^{1-\delta} \int_0^t \frac{\partial}{\partial x} \psi_\varepsilon\left(\xi_\varepsilon(s), \eta_\varepsilon(s)\right) m_\varepsilon(\xi_\varepsilon(s), \eta_\varepsilon(s)) ds + \int_0^t m_\varepsilon(\xi_\varepsilon(s), \eta_\varepsilon(s)) ds \\
&\quad + \frac{1}{\varepsilon^\delta} \int_0^t \beta_\varepsilon\left(\xi_\varepsilon(s), \eta_\varepsilon(s)\right) ds + \int_0^t \sqrt{\alpha_\varepsilon\left(\xi_\varepsilon(s), \eta_\varepsilon(s)\right)} dw_1(s).
\end{aligned} \tag{3.1}
$$

We will use the notations (2.9) and (2.10). By Ito's formula applied to the process $\varepsilon^\delta f\left(\frac{\zeta_\varepsilon(t)}{\varepsilon^\delta}\right)$ ((2.8)) and setting $B(x,y) := \beta(x,y) + \beta^0(x,y)$, for $\zeta_\varepsilon(t) + \varepsilon^\delta f\left(\frac{\zeta_\varepsilon(t)}{\varepsilon^\delta}\right)$ we obtain

$$
\begin{aligned}
&\zeta_\varepsilon(\xi_\varepsilon(t), \eta(t)) + \varepsilon^\delta f_\varepsilon(\zeta_\varepsilon(t)) \\
&\quad = \zeta_\varepsilon(\xi_0, \eta_0) + \varepsilon^\delta f_\varepsilon(\zeta_\varepsilon(0)) \\
&\qquad + \frac{1}{\kappa_0} \int_0^t F_\varepsilon(\zeta_\varepsilon(s)) m_\varepsilon(\xi_\varepsilon(s), \eta_\varepsilon(s)) ds + \frac{1}{\varepsilon^\delta} \int_0^t B_\varepsilon(\xi_\varepsilon(s), \eta_\varepsilon(s)) ds \\
&\qquad + \frac{1}{\varepsilon^\delta} \int_0^t \Big(\beta_\varepsilon(\xi_\varepsilon(s), \eta_\varepsilon(s)) \left(\dot f_\varepsilon(\zeta_\varepsilon(s)) - \dot f_\varepsilon(\xi_\varepsilon(s))\right) \\
&\qquad + \frac{1}{2} \alpha_\varepsilon(\xi_\varepsilon(s), \eta_\varepsilon(s)) \left(\ddot f_\varepsilon(\zeta_\varepsilon(s)) - \ddot f_\varepsilon(\xi_\varepsilon(s))\right)\Big) ds \\
&\qquad + \frac{\varepsilon^{1-2\delta}}{\kappa_0} \int_0^t F_\varepsilon(\zeta_\varepsilon(s)) \frac{\partial}{\partial x} \psi_\varepsilon(\xi_\varepsilon(s), \eta_\varepsilon(s)) c_\varepsilon(\xi_\varepsilon(s), \eta_\varepsilon(s)) ds \\
&\qquad + \frac{\varepsilon^{1-\delta}}{\kappa_0} \int_0^t F_\varepsilon(\zeta_\varepsilon(s)) \frac{\partial}{\partial x} \psi_\varepsilon(\xi_\varepsilon(s), \eta_\varepsilon(s)) m_\varepsilon(\xi_\varepsilon(s), \eta_\varepsilon(s)) ds \\
&\qquad + \frac{1}{\kappa_0} \int_0^t F_\varepsilon(\zeta_\varepsilon(s)) \sqrt{\alpha_\varepsilon\left(\xi_\varepsilon(s), \eta_\varepsilon(s)\right)} dw_1(s).
\end{aligned}
$$

Taking into account the equality $\bar B(x) = 0$ (Remark 2.4.), let us consider a function $n(x,y)$, the periodic function of period 1 with respect to y, the unique solution of

$$
Ln(x,y) = -B(x,y), \qquad \int_Y n(x,y) dy = 0,
$$

for every parameter $x \in E_1$.

Applying Ito's formula to the function $\varepsilon^{2-\delta} n_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t))$ for

$$
\lambda_\varepsilon(t) := \lambda_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t)) = \zeta_\varepsilon(t) + \varepsilon^\delta f_\varepsilon(\zeta_\varepsilon(t)) + \varepsilon^{2-\delta} n_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t))
$$

we get

$$
\begin{aligned}
\lambda_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t)) &= \lambda_\varepsilon(\xi_0, \eta_0) + \frac{1}{\kappa_0} \int_0^t F_\varepsilon(\zeta_\varepsilon(s)) m_\varepsilon(\xi_\varepsilon(s), \eta_\varepsilon(s)) ds \\
&\quad + \frac{1}{\kappa_0} \int_0^t F_\varepsilon(\zeta_\varepsilon(s)) \sigma(\eta_\varepsilon(s)) \nabla_y \psi_\varepsilon(\xi_\varepsilon(s), \eta_\varepsilon(s)) dw_1(s) \\
&\quad + \sum_{i=0}^4 \int_0^t A_i^\varepsilon(\xi_\varepsilon, \eta_\varepsilon, s) ds + \int_0^t A_5^\varepsilon(\xi_\varepsilon, \eta_\varepsilon, s) dw_1(s),
\end{aligned} \tag{3.2}
$$

where

$$
A_0^\varepsilon(\xi_\varepsilon, \eta_\varepsilon, t) = \frac{1}{\varepsilon^\delta} \left(\beta_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t)) \left(\dot f_\varepsilon(\zeta_\varepsilon(t)) - \dot f_\varepsilon(\xi_\varepsilon(t))\right)\right.
$$

$$+ \frac{1}{2}\alpha_\varepsilon(\xi_\varepsilon(t),\eta_\varepsilon(t))\left(\ddot{f}_\varepsilon(\zeta_\varepsilon(t)) - \ddot{f}_\varepsilon(\xi_\varepsilon(t))\right)\Big);$$

$$A_1^\varepsilon(\xi_\varepsilon,\eta_\varepsilon,t) = \varepsilon^{1-2\delta}\left(\frac{1}{\kappa_0}F_\varepsilon(\zeta_\varepsilon(t))\frac{\partial}{\partial x}\psi_\varepsilon(\xi_\varepsilon(t),\eta_\varepsilon(t))c_\varepsilon(\xi_\varepsilon(t),\eta_\varepsilon(t)) + \frac{\partial}{\partial x}n_\varepsilon(\xi_\varepsilon(t),\eta_\varepsilon(t))g_\varepsilon(\xi_\varepsilon(t),\eta_\varepsilon(t))\right);$$

$$A_2^\varepsilon(\xi_\varepsilon,\eta_\varepsilon,t) = \varepsilon^{2(1-\delta)}\frac{\partial}{\partial x}n_\varepsilon(\xi_\varepsilon(t),\eta_\varepsilon(t))m_\varepsilon(\xi_\varepsilon(t),\eta_\varepsilon(t));$$

$$A_3^\varepsilon(\xi_\varepsilon,\eta_\varepsilon,t) = \frac{\varepsilon^{1-\delta}}{\kappa_0}F_\varepsilon(\zeta_\varepsilon(t))\frac{\partial}{\partial x}\psi_\varepsilon(\xi_\varepsilon(t),\eta_\varepsilon(t))m_\varepsilon(\xi_\varepsilon(t),\eta_\varepsilon(t));$$

$$A_4^\varepsilon(\xi_\varepsilon,\eta_\varepsilon,t) = \varepsilon^{2-3\delta}\frac{\partial}{\partial x}n_\varepsilon(\xi_\varepsilon(t),\eta_\varepsilon(t))c_\varepsilon(\xi_\varepsilon(t),\eta_\varepsilon(t));$$

$$A_5^\varepsilon(\xi_\varepsilon,\eta_\varepsilon,t) = \varepsilon^{1-\delta}\sigma(\eta_\varepsilon(t))\nabla_y n_\varepsilon(\xi_\varepsilon(t),\eta_\varepsilon(t)).$$

Now, for every $t \in [0,T]$, our conditions and estimations of Lemma 4.3, taking into account the statement of Lemma 4.5, imply the existence of the constant $C(T) =: C$ so that

$$\sum_{i=1}^{5} E\sup_{t\in[0,T]}|A_i^\varepsilon(\xi_\varepsilon,\eta_\varepsilon,t)| \le \varepsilon^{1-2\delta}C.$$

Also the integrands of the first two integrals in (3.2) are bounded by the constant $C(T) := C$. Hence, for any fixed ε_0 so that $0 < \varepsilon \le \varepsilon_0$

$$E\sup_{s\in[0,T]}\left(\frac{1}{\kappa_0}|F_\varepsilon(\zeta_\varepsilon(s))m_\varepsilon(\xi_\varepsilon(s),\eta_\varepsilon(s))| + \frac{1}{\kappa_0}|F_\varepsilon(\zeta_\varepsilon(s))\sqrt{\alpha_\varepsilon\left(\xi_\varepsilon(s),\eta_\varepsilon(s)\right)}| + \sum_{i=0}^{5}|A_i^\varepsilon(\xi_\varepsilon,\eta_\varepsilon,s)|\right) \le C(1+C_\varepsilon),$$

where $\lim_{\varepsilon\to 0} C_\varepsilon = 0$. From this, by standard arguments ([9]), we obtain that there exists a constant $C_{\varepsilon_0}(\eta_0)$ such that for every $0 < \varepsilon < \varepsilon_0$,

$$E\sup_{t\in[0,T]}|\lambda_\varepsilon(t)|^2 \le C_{\varepsilon_0}(\eta_0)(1+|\xi_0|^2),$$

and for every $s,t\,:\; 0 \le s \le t \le T$

$$E|\lambda_\varepsilon(t)-\lambda_\varepsilon(s)|^4 \le C_{\varepsilon_0}(\eta_0)|t-s|^2.$$

Using (3.2) and Lemmas 4.6, 4.9, we can check the conditions of weak compactness ([4], Lemma 2, p.355) for the family of measures $\{\mu_\delta^\varepsilon,\, 0 < \varepsilon < \varepsilon_0\}$. Thus the set of measures, corresponding to the processes

$$\xi_\varepsilon(t) = \lambda_\varepsilon(\xi_\varepsilon(t),\eta_\varepsilon(t)) - \varepsilon\psi_\varepsilon(\xi_\varepsilon(t),\eta_\varepsilon(t)) - \varepsilon^\delta f_\varepsilon(\zeta_\varepsilon(t)) - \varepsilon^{2-\delta}n_\varepsilon(\xi_\varepsilon(t),\eta_\varepsilon(t))$$

on $C[0,T]$, is weakly compact.

II. We begin our considerations with the relationship (3.2).

Let $\phi(x) \in C_0^\infty(E_1)$, $\Phi_s(x)$ be a continuous bounded C_s-measurable functional. Applying Ito's formula to $\phi(\lambda_\varepsilon(t))$, we obtain

$$E\Phi_r(\xi_\varepsilon)[\phi(\lambda_\varepsilon(t)) - \phi(\lambda_\varepsilon(r))]$$

$$= E\Phi_r(\xi_\varepsilon)\Big\{ \int_r^t \dot{\phi}(\lambda_\varepsilon(s))\Big(\frac{1}{\kappa_0}F_\varepsilon(\zeta_\varepsilon(s))m_\varepsilon(\xi_\varepsilon(s),\eta_\varepsilon(s)) + \sum_{i=0}^{4} A_i^\varepsilon(\xi_\varepsilon,\eta_\varepsilon,s)\Big)ds$$

$$+\frac{1}{2}\int_r^t \ddot{\phi}(\lambda_\varepsilon(s))\Big(\frac{1}{\kappa_0}F_\varepsilon(\zeta_\varepsilon(s))\sqrt{\alpha_\varepsilon\Big(\xi_\varepsilon(s),\eta_\varepsilon(s)\Big)} + A_5^\varepsilon(\xi_\varepsilon,\eta_\varepsilon,s)\Big)^2 ds\Big\}. \tag{3.3}$$

Now, we denote

$$D_\varepsilon(r,t) = \int_r^t \dot{\phi}(\lambda_\varepsilon(s))\sum_{i=0}^{4} A_i^\varepsilon(\xi_\varepsilon,\eta_\varepsilon,s)ds + \int_r^t \ddot{\phi}(\lambda_\varepsilon(s))$$
$$\times\left(\frac{1}{\kappa_0}F_\varepsilon(\zeta_\varepsilon(s))\sqrt{\alpha_\varepsilon\left(\xi_\varepsilon(s),\eta_\varepsilon(s)\right)}A_5^\varepsilon(\xi_\varepsilon,\eta_\varepsilon,s) + \frac{1}{2}(A_5^\varepsilon(\xi_\varepsilon,\eta_\varepsilon,s))^2\right)ds.$$

Applying Lemma 4.10 to $k(x,y) = m(x,y) - \langle m(x,\,\cdot\,)\rangle$, $H(x) = F(x)$, and $P(x) = \dot{\phi}(x)$, we arrive at

$$E\Phi_r(\xi_\varepsilon)\int_r^t \dot{\phi}(\lambda_\varepsilon(s))F_\varepsilon(\zeta_\varepsilon(s))m_\varepsilon(\xi_\varepsilon(s),\eta_\varepsilon(s))ds$$
$$= E\Phi_r(\xi_\varepsilon)\left(\int_r^t \dot{\phi}(\lambda_\varepsilon(s))F_\varepsilon(\zeta_\varepsilon(s))\overline{m}_\varepsilon(\xi_\varepsilon(s))ds + G(\varepsilon, m_\varepsilon - \overline{m}_\varepsilon, r,t)\right).$$

In a similar way, applying Lemma 4.10 to

$$k(x,y) = \left(\sigma\left(y\right)\nabla_y\psi\left(x,y\right)\right)^2 - \left\langle\left(\sigma\left(\,\cdot\,\right)\nabla_{\cdot}\psi\left(x,\,\cdot\,\right)\right)^2\right.,$$

$H_\varepsilon(x) = F_\varepsilon^2(x)$, and $P(x) = \ddot{\phi}(x)$, we obtain

$$E\Phi_r(\xi_\varepsilon)\int_r^t \ddot{\phi}(\lambda_\varepsilon(s))F_\varepsilon^2(\zeta_\varepsilon(s))\alpha_\varepsilon(\xi_\varepsilon(s),\eta_\varepsilon(s))ds$$
$$= E\Phi_r(\xi_\varepsilon)\left(\int_r^t \ddot{\phi}(\lambda_\varepsilon(s))F_\varepsilon^2(\zeta_\varepsilon(s))\overline{\alpha}_\varepsilon(\xi_\varepsilon(s))ds + G(\varepsilon,\alpha_\varepsilon - \overline{\alpha}_\varepsilon, r,t)\right).$$

Rewriting (3.3), we arrive at

$$E\Phi_r(\xi_\varepsilon)\left[\phi(\xi_\varepsilon(t)) - \phi(\xi_\varepsilon(r)) - \int_r^t \{\dot{\phi}(\xi_\varepsilon(s))\beta_0 + \frac{1}{2}\ddot{\phi}(\xi_\varepsilon(s))\sigma_0^2\}ds\right]$$
$$= I_\varepsilon^0 + I_\varepsilon^1 + I_\varepsilon^2 + I_\varepsilon^3 + I_\varepsilon^4 + I_\varepsilon^5, \tag{3.4}$$

where

$$I_\varepsilon^0 = E\Phi_r(\xi_\varepsilon)\left(\frac{1}{\kappa_0}G(\varepsilon, m_\varepsilon - \langle m_\varepsilon\rangle, r,t) + \frac{1}{2\alpha_0^2}G(\varepsilon,\alpha_\varepsilon - \langle\alpha_\varepsilon\rangle, r,t) + D_\varepsilon(r,t)\right);$$
$$I_\varepsilon^1 = E\Phi_r(\xi_\varepsilon)(\phi(\xi_\varepsilon(t)) - \phi(\lambda_\varepsilon(t)) - \phi(\xi_\varepsilon(r)) + \phi(\lambda_\varepsilon(r)));$$
$$I_\varepsilon^2 = E\Phi_r(\xi_\varepsilon)\int_r^t\left[\left(\dot{\phi}(\lambda_\varepsilon(s)) - \dot{\phi}(\xi_\varepsilon(s))\right)\beta_0 + \frac{1}{2}\left(\ddot{\phi}(\lambda_\varepsilon(s)) - \ddot{\phi}(\xi_\varepsilon(s))\right)\sigma_0^2\right]ds;$$
$$I_\varepsilon^3 = E\Phi_r(\xi_\varepsilon)\int_r^t\Big[\dot{\phi}(\lambda_\varepsilon(s))\frac{1}{\kappa_0}F_\varepsilon(\zeta_\varepsilon(s))\left(\langle m_\varepsilon(\xi_\varepsilon(s),\cdot)\rangle - \langle m_\varepsilon(\zeta_\varepsilon(s),\cdot)\rangle\right)$$
$$+\frac{1}{2}\ddot{\phi}(\lambda_\varepsilon(s))\kappa_0^{-2}F_\varepsilon^2(\zeta_\varepsilon(s))\left(\langle\alpha_\varepsilon(\xi_\varepsilon(s),\cdot)\rangle - \langle\alpha_\varepsilon(\zeta_\varepsilon(s),\cdot)\rangle\right)\Big]ds;$$

$$I_\varepsilon^4 = E\Phi_r(\xi_\varepsilon)\int_r^t \dot{\phi}(\lambda_\varepsilon(s))\left[\frac{1}{\kappa_0}F_\varepsilon(\zeta_\varepsilon(s))\langle m_\varepsilon(\zeta_\varepsilon(s),\,\cdot\,)\rangle - \beta_0\right]ds;$$

$$I_\varepsilon^5 = \frac{1}{2}E\Phi_r(\xi_\varepsilon)\int_r^t \ddot{\phi}(\lambda_\varepsilon(s))[\kappa_0^{-2}F_\varepsilon^2(\zeta_\varepsilon(s))\langle\alpha_\varepsilon(\zeta_\varepsilon(s),\,\cdot\,)\rangle - \sigma_0^2]ds.$$

We shall prove that the limit of the right hand side of (3.4), as $\varepsilon \to 0$, is equal to zero.

For small ε, we can estimate $D_\varepsilon(r,t)$, using Condition (A1) and Lemma 4.3, as

$$E\sup_{t\in[0,T]}|D_\varepsilon(t,r)| \le \varepsilon^{1-2\delta}(1+C_\varepsilon)C_T,$$

where $\lim_{\varepsilon\to 0} C_\varepsilon = 0$. From this inequality and Lemma 4.10 we have

$$\lim_{\varepsilon\to 0} I_\varepsilon^0 = 0.$$

According to Lemma 4.6 and Lemma 4.9, we get

$$\lim_{\varepsilon\to 0} I_\varepsilon^1 = \lim_{\varepsilon\to 0} I_\varepsilon^2 = 0.$$

Using Lemma 4.3 and Lemma 4.12, we obtain

$$\lim_{\varepsilon\to 0} I_\varepsilon^3 = 0.$$

Lemma 4.15 implies

$$\lim_{\varepsilon\to 0} I_\varepsilon^4 = \lim_{\varepsilon\to 0} I_\varepsilon^5 = 0.$$

Consequently, $\lim_{\varepsilon\to 0}\sum_{i=1}^5 I_\varepsilon^i = 0$.

Let μ_δ denotes some limit point of the family $\{\mu_\delta^\varepsilon, 0<\varepsilon<\varepsilon_0\}$ and E^{μ_δ} be an expectation on this measure. Let us come to the limit in (3.4) by a subsequence $\{\varepsilon_k\}$ such that $\mu_\delta^{\varepsilon_k} \Rightarrow \mu_\delta$ as $\varepsilon_k \to 0$. We get

$$E^{\mu_\delta}\Phi_r(\xi)\left(\phi(\xi(t)) - \phi(\xi(r)) - \int_r^t \{\dot{\phi}(\xi(s))\beta_0 + \frac{1}{2}\ddot{\phi}(\xi(s))\sigma_0^2\}ds\right) = 0.$$

The coefficients do not depend on δ. That means $\mu_\delta = \mu$. Consequently, $\mu_\delta^\varepsilon \Rightarrow \mu$ as $\varepsilon \to 0$, and the limit measure coincides with the measure corresponding to the process

$$\xi(t) \;=\; \xi_0 \;+\; \frac{\kappa_2}{\kappa_0\kappa_1}t \;+\; \frac{1}{\sqrt{\kappa_0\kappa_1}}w(t).$$

4. Needed preliminary results

In this section we prove the results used for the proof of the Theorem above.

(Below we denote by ∂Y the boundary of unit cube Y in E_n)

Lemma 4.1. *Let $d = d\,(y)$ be a periodic function satisfying*

$$Ld\,(y) = h\,(y)\,, \qquad \int_Y d\,(y)\,dy = 0,$$

and Condition (A) holds.

Then,

$$\sup_Y |d| \le C\max_Y |h|\,,$$

with the constant C depending only on the prescribed quantities, such as dimension n, ellipticity constant λ_0, etc.

Proof. Let B_R denote a set in E_n which contains a cube Y. By changing $d(y)$ to $d(y) - \inf\limits_{B_R} d$, we can suppose that $\inf\limits_{B_R} d = 0$. The Theorems 9.20, 9.22 ([5]) imply

$$\sup_{E_n} d = \sup_{B_R} d \leq C(\inf_{B_R} d + \max_{B_R} |h|).$$

That means,

$$\sup_{E_n} d = osc(d) \leq C \max_{Y} |h|.$$

□

Corollary 4.2. *Under the previous assumptions, let $d(x, y)$ and $h(x, y)$ depend on a parameter x. Then the derivatives of d and h with respect to x satisfy*

$$\sup_{E_n} \left| \frac{\partial d}{\partial x} \right| \leq C \sup_{Y} \left| \frac{\partial h}{\partial x} \right|.$$

Proof. The proof follows immediately from the linearity of equation (2.4). □

The function $h(x)$ is one-to-one function and, consequently, has an inverse function denoted by $h^{-1}(x)$.

Lemma 4.3. *Let the Conditions (A) and (B) be satisfied. Then there exists a positive constant C such that*

1. $\exp\{-2\lambda_2\} \leq F(x) \leq \exp\{2\lambda_2\}$; $|\dot{F}(x)| \leq C$; $|\ddot{F}(x)| \leq C$; $|\dddot{F}(x)| \leq C$;
2. $|h(x)| \leq \exp\{2\lambda_2\}|x|$; $|\dot{h}(x)| \leq C$;
3. $|f(x)| \leq C(1 + |x|)$; $|\dot{f}(x)| \leq C$;
4. $\sup_{z \in E_1} \left| \frac{\overline{\beta}(z)}{\overline{\alpha}(z)} \right| \leq \frac{C}{\lambda_1} = C$;
5. $\sup_{x \in E_1, y \in E_n} |\beta(x, y)| \leq C$; $\quad \sup_{x \in E_1, y \in E_n} |\alpha(x, y)| \leq C$;
6. $|h^{-1}(x)| \leq \exp\{2\lambda_2\}|x|$; $\exp\{-2\lambda_2\} \leq \dot{h}^{-1}(x) \leq \exp\{2\lambda_2\}$.

Proof. The assertions 1.–5. of the lemma follow from our assumptions. Next, let us consider 6. The equality

$$\dot{h}^{-1}(x) = \frac{1}{\dot{h}(h^{-1}(x))} = \frac{1}{F(h^{-1}(x))}$$

and 1. imply second part 6. Then, using that $\dot{h}^{-1}(0) = 0$, from the previous equality, we have

$$|h^{-1}(x)| = |h^{-1}(x) - h^{-1}(0)| \leq \exp\{2\lambda_2\}|x|.$$

□

Lemma 4.4. *Let the Conditions (A) and (B) be satisfied. The processes $\xi_\varepsilon(t), \eta_\varepsilon(t)$ are the solutions of (2.1), (2.2) respectively. For every $\delta \in]0, \frac{1}{2}[$*

$$\lim_{\varepsilon \to 0} E \sup_{t \in [0,T]} \left| \frac{1}{\varepsilon^\delta} \int_{\frac{\xi_\varepsilon(t)}{\varepsilon^\delta}}^{\frac{\xi_\varepsilon(t)}{\varepsilon^\delta} + \varepsilon^{1-\delta}\psi_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t))} \frac{\overline{\beta}(z)}{\overline{\alpha}(z)} dz \right| = 0.$$

Proof. By Remark 2.2 and Lemma 4.3 (part 4.), we have

$$\lim_{\varepsilon \to 0} E \sup_{t \in [0,T]} \left| \frac{1}{\varepsilon^\delta} \int_{\frac{\xi_\varepsilon(t)}{\varepsilon^\delta}}^{\frac{\xi_\varepsilon(t)}{\varepsilon^\delta} + \varepsilon^{1-\delta}\psi_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t))} \frac{\overline{\beta}(z)}{\overline{\alpha}(z)} dz \right|$$
$$\leq C \lim_{\varepsilon \to 0} \varepsilon^{1-2\delta} E \sup_{t \in [0,T]} |\psi_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t))| = 0.$$

□

Lemma 4.5. *Let the Conditions (A) and (B) be satisfied. The processes $\xi_\varepsilon(t), \eta_\varepsilon(t)$ are the solutions of (2.1), (2.2) respectively, $\zeta_\varepsilon(t)$ is defined by (3.1). For every $\delta \in]0, \frac{1}{2}[$ and for every $t \in [0,T]$,*

$$\lim_{\varepsilon \to 0} E \sup_{t \in [0,T]} |A_0^\varepsilon(\xi_\varepsilon, \eta_\varepsilon, t)| = 0.$$

Proof. Part A. From the definition of function $f(x)$ ((2.8)) and, using Lemma 4.3 (part 2.), we have

$$\begin{aligned} |A_{01}^\varepsilon(\xi_\varepsilon(t), \zeta_\varepsilon(t))| &:= \left| \dot{f}_\varepsilon(\zeta_\varepsilon(t)) - \dot{f}_\varepsilon(\xi_\varepsilon(t)) \right| \\ &\leq \frac{C}{\kappa_0} \left| \left(\exp\left\{ -2 \int_{\frac{\xi_\varepsilon(s)}{\varepsilon^\delta}}^{\frac{\xi_\varepsilon(t)}{\varepsilon^\delta} + \varepsilon^{1-\delta}\psi_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t))} \frac{\overline{\beta}(z)}{\overline{\alpha}(z)} dz \right\} - 1 \right) \right|. \end{aligned}$$

Now, using the Lemma 4.4, we get

$$\lim_{\varepsilon \to 0} E \sup_{s \in [0,t]} \frac{1}{\varepsilon^\delta} |A_{01}^\varepsilon(\xi_\varepsilon(s), \zeta_\varepsilon(s))| \leq C \lim_{\varepsilon \to 0} \varepsilon^{1-2\delta} = 0.$$

Part B. Similarly, by the Lemma 4.3 and Lemma 4.4, using technique of part A and the condition B2, we can derive

$$\lim_{\varepsilon \to 0} E \sup_{s \in [0,t]} \frac{1}{\varepsilon^\delta} |A_{02}^\varepsilon(\xi_\varepsilon(s), \zeta_\varepsilon(s))| \leq C \lim_{\varepsilon \to 0} \varepsilon^{1-2\delta} = 0,$$

where $|A_{02}^\varepsilon(\xi_\varepsilon(t), \zeta_\varepsilon(t))| := \left| \ddot{f}_\varepsilon(\zeta_\varepsilon(t)) - \ddot{f}_\varepsilon(\xi_\varepsilon(t)) \right|$.

Part C. From parts A and B and by the estimations of Lemma 4.3, we arrive at

$$\begin{aligned} &\lim_{\varepsilon \to 0} E \sup_{t \in [0,T]} |A_0^\varepsilon(\xi_\varepsilon, \eta_\varepsilon, t)| \\ &\quad \leq C \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^\delta} E \sup_{t \in [0,T]} |A_{01}^\varepsilon(\xi_\varepsilon(s), \zeta_\varepsilon(s)) + A_{02}^\varepsilon(\xi_\varepsilon(s), \zeta_\varepsilon(s))| \leq C \lim_{\varepsilon \to 0} \varepsilon^{1-2\delta} = 0. \end{aligned}$$

□

Lemma 4.6. *Let the Conditions (A), (B), and (C0) hold. The processes $\xi_\varepsilon(t), \eta_\varepsilon(t)$ are the solutions of (2.1), (2.2) respectively. Then, for every $\delta \in]0; \frac{1}{2}[$,*

$$\lim_{\varepsilon \to 0} E \sup_{t \in [0,T]} |\varepsilon \psi_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t)) + \varepsilon^{2-\delta} n_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t))|^2 = 0.$$

Proof. From the definitions and the properties of the functions $\psi(x,y)$, (2.5), and $n(x,y)$,

$$E \sup_{t \in [0,T]} |\varepsilon \psi_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t)) + \varepsilon^{2-\delta} n_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t))|^2 \leq \lim_{\varepsilon \to 0} \varepsilon^{2(1-\delta)} C = 0.$$

follows. □

Lemma 4.7. *Let the Conditions (A) and (B) be satisfied. The processes $\xi_\varepsilon(t), \eta_\varepsilon(t)$ are the solutions of (2.1), (2.2) respectively, $\zeta_\varepsilon(t)$ is defined by (3.1). For every integer positive m and $\delta \in]0, \frac{1}{2}[$ and $\varepsilon_0 > 0$, there exist constants $C_m(\varepsilon_0)$, such that for every $\varepsilon < \varepsilon_0$,*

$$E \sup_{t \in [0,T]} |\xi_\varepsilon(t)|^m \leq C_m(\varepsilon_0, \lambda_2)(1 + C_m(\varepsilon_0, \lambda_2, \xi_0, \eta_0)).$$

Proof. We fix arbitrary $\varepsilon_0 > 0$ and, for $\varepsilon < \varepsilon_0$, apply Ito's formula to the function $\varepsilon^\delta h_\varepsilon(\zeta_\varepsilon(t)) + \kappa_0\varepsilon^{2-\delta} n_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t))$. Using Lemma 4.3 and Lemma 4.5, under Condition (B), in a standard way, as in ([9], Ch.II, Sec. 5, Corollary 12, p. 86), we can get the estimates

$$\begin{aligned} E \sup_{t\in[0,T]} &|\varepsilon^\delta h_\varepsilon(\zeta_\varepsilon(t)) + \kappa_0\varepsilon^{2-\delta} n_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t))|^m \\ &\le C_m(1 + C_\varepsilon)(1 + |\varepsilon^\delta h_\varepsilon(\zeta_0) + \kappa_0\varepsilon^{2-\delta} n_\varepsilon(\xi_0, \eta_0)|^m) \\ &\le C_m(\varepsilon_0)(1 + |\varepsilon^\delta h_\varepsilon(\zeta_0) + \kappa_0\varepsilon^{2-\delta} n_\varepsilon(\xi_0, \eta_0)|^m); \end{aligned}$$

here $\lim_{\varepsilon\to 0} C_\varepsilon = 0$, $\zeta_0 = \xi_0 + \varepsilon(\psi_\varepsilon(\xi_0, \eta_0))$.

It follows from part 6. of Lemma 4.3., that

$$\varepsilon^\delta |h_\varepsilon^{-1}(x)| \le \exp\{2\lambda_2\}|x|.$$

Let us estimate $E\sup_{t\in[0,T]} |\xi_\varepsilon(t)|^m$. We obtain

$$\begin{aligned} E \sup_{t\in[0,T]} |\xi_\varepsilon(t)|^m &= E \sup_{t\in[0,T]} |\xi_\varepsilon(t) + \varepsilon\psi_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t)) - \varepsilon\psi_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t))|^m \\ &= E \sup_{t\in[0,T]} |\varepsilon^\delta h_\varepsilon^{-1}(\varepsilon^\delta h_\varepsilon(\zeta_\varepsilon(t))) + \kappa_0\varepsilon^{2-\delta} n_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t)) \\ &\qquad - \varepsilon\psi_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t)) - \kappa_0\varepsilon^{2-\delta} n_\varepsilon(\xi_\varepsilon(t), \eta_\varepsilon(t))|. \end{aligned}$$

Combining the results of Lemma 4.3 (part 6.), and Lemma 4.6, there is ε_0 such that, for every $\varepsilon \le \varepsilon_0$, we can find constants $C(\varepsilon_0, m)$ and $C(\varepsilon_0, m, \xi_0, \eta_0)$ so that

$$E \sup_{t\in[0,T]} |\xi_\varepsilon(t)|^m \le \exp\{-2\lambda_2\}(C_m(\varepsilon_0) + C_m(\varepsilon_0, \xi_0, \eta_0)).$$

From last inequality, the statement of the lemma follows. □

Corollary 4.8. *Under the assumptions of the previous Lemma, there exist constants $C'_m(\varepsilon_0, \lambda_2)$ and $C'_m(\varepsilon_0, \xi_0, \eta_0)$ such that for every $\varepsilon < \varepsilon_0$,*

$$E \sup_{t\in[0,T]} |\zeta_\varepsilon(t)|^m \le C'_m(\varepsilon_0, \lambda_2)(1 + C'_m(\varepsilon_0, \lambda_2, \xi_0, \eta_0)).$$

Proof. The proof follows immediately from the proof of the previous Lemma. □

Lemma 4.9. *Let the Conditions (A), (B), and (C0) be hold. The process $\zeta_\varepsilon(t)$ is defined by (3.1). Then for every $\delta \in]0; \frac{1}{2}[$*

$$\lim_{\varepsilon\to 0} E \sup_{t\in[0,T]} |\varepsilon^\delta f_\varepsilon(\zeta_\varepsilon(t))|^2 = 0.$$

Proof. For every $\varepsilon > 0$ and N such that $\varepsilon^{\frac{\delta}{2}} < N < \infty$,

$$\begin{aligned} E \sup_{t\in[0,T]} |\varepsilon^\delta f_\varepsilon(\zeta_\varepsilon(t))|^2 &= E \sup_{t\in[0,T]} |\varepsilon^\delta f_\varepsilon(\zeta_\varepsilon(t))|^2 \chi\{|\zeta_\varepsilon(t)| < \varepsilon^{\frac{\delta}{2}}\} \\ &\quad + E \sup_{t\in[0,T]} |\varepsilon^\delta f_\varepsilon(\zeta_\varepsilon(t))|^2 \chi\{\varepsilon^{\frac{\delta}{2}} \le |\zeta_\varepsilon(t)| \le N\} \\ &\quad + E \sup_{t\in[0,T]} |\varepsilon^\delta f_\varepsilon(\zeta_\varepsilon(t))|^2 \chi\{|\zeta_\varepsilon(t)| > N\} \\ &= D_{11}^\varepsilon + D_{12}^\varepsilon + D_{13}^\varepsilon \end{aligned}$$

respectively. By part 3. of Lemma 4.3 we obtain

$$\lim_{\varepsilon\to 0} D_{11}^{\varepsilon} = 0.$$

Using the definition of the function $f(x)$ and the condition (C0), we have

$$\lim_{\varepsilon\to 0} D_{12}^{\varepsilon} = \lim_{\varepsilon\to 0} E \sup_{t\in[0,T]} \left| \frac{\varepsilon^{\delta}}{\kappa_0} \int_0^{\frac{\zeta_\varepsilon(t)}{\varepsilon^{\delta}}} F(z)dz - \zeta_\varepsilon(t) \right|^2 \chi\{\varepsilon^{\frac{\delta}{2}} \le |\zeta_\varepsilon(t)| \le N\} = 0.$$

Now,

$$D_{13}^{\varepsilon} \le \varepsilon^{2\delta} C \left(1 + \frac{E\sup_{t\in[0,T]} |\zeta_\varepsilon(t)|^4}{\varepsilon^{4\delta}}\right)^{\frac{1}{2}} \frac{E\sup_{t\in[0,T]} |\zeta_\varepsilon(t)|^2}{N^2}.$$

At first approaching the limit as $\varepsilon \to 0$, then letting $N \to \infty$ and using the estimation of Corollary 4.8, the statement of lemma follows. □

Lemma 4.10. *Let the functions $H(x) \in C^2_{x,b}(E_1)$ and the function $k(x,y) \in C^{2,2}_{x,y,b}(E_1, E_n)$ satisfied the condition $\bar{k}(x) = 0$. The processes $\xi_\varepsilon(t), \eta_\varepsilon(t)$ are the solutions of (2.1), (2.2) respectively, $\zeta_\varepsilon(t)$ is defined by (3.1), $\lambda_\varepsilon(t)$ is defined by (3.2). Then, for a function $P(x) \in C_0^\infty(E_1)$ and a continuous bounded C_s — measurable functional $\Phi_s(x)$, we have*

$$\lim_{\varepsilon\to 0} E\Phi_r(\xi_\varepsilon) \left(\int_r^t P(\lambda_\varepsilon(s)) H_\varepsilon(\zeta_\varepsilon(s)) k_\varepsilon(\xi_\varepsilon(s), \eta_\varepsilon(s)) ds \right) = 0.$$

Proof. Let the function $l(x,y)$ be the unique solution of the problem

$$Ll(x,y) = k(x,y), \qquad \int_Y l(x,y)dy = 0$$

for any $x \in E_1$ (x play the role of parameter). Then $l(x,y) \in C^{2,2}_{x,y,b}(E_1, E_n)$. Applying Ito's formula to the function

$$\varepsilon^2 P(\lambda_\varepsilon(s)) H_\varepsilon(\zeta_\varepsilon(s)) l\left(\frac{\xi_\varepsilon(s)}{\varepsilon^{\delta}}, \eta_\varepsilon(s) \right),$$

we get

$$\int_r^t P(\lambda_\varepsilon(s)) H_\varepsilon(\zeta_\varepsilon(s)) k_\varepsilon(\xi_\varepsilon(s), \eta_\varepsilon(s)) ds = G(\varepsilon, k, r, t),$$

where the function $G(\varepsilon, k, r, t)$ depends on the processes $\xi_\varepsilon(t), \zeta_\varepsilon(t), \lambda_\varepsilon(t)$ and $\eta_\varepsilon(t)$

Using the estimates of Lemma 4.7, under our conditions, by the standard arguments we arrive at

$$\lim_{\varepsilon\to 0} E\Phi_r(\xi_\varepsilon) G(\varepsilon, k, r, t) = 0 \le C \lim_{\varepsilon\to 0} \sqrt{\varepsilon} E\Phi_r(\xi_\varepsilon) \left(1 + \sup_{t\in[0;T]} \sum_{k=1}^{3} |\xi_\varepsilon(t)|^k \right) = 0.$$

□

Corollary 4.11. *The statement of the previous Lemma is also true for $H(x) \in C^2_x(E_1)$ such that $|H(x)| + |\dot{H}(x)| + |\ddot{H}(x)| \le C(1 + |x|)$.*

Proof. The proof follows immediately from the Corollary 4.8 and the proof of the previous Lemma 4.10. □

Lemma 4.12. *Let the Conditions (A) and (B) be satisfied. The processes $\xi_\varepsilon(t), \eta_\varepsilon(t)$ are the solutions of (2.1), (2.2) respectively. Then, for every $\delta \in]0, \frac{1}{2}[$,*

$$\lim_{\varepsilon\to 0} E \sup_{t\in[0,T]} |\,\langle m_\varepsilon(\xi_\varepsilon(t),\cdot)\rangle - \langle m_\varepsilon(\zeta_\varepsilon(t),\cdot)\rangle\,|^2 = 0.$$

and

$$\lim_{\varepsilon\to 0} E \sup_{t\in[0,T]} |\,\langle \alpha_\varepsilon(\xi_\varepsilon(t),\cdot)\rangle - \langle \alpha_\varepsilon(\zeta_\varepsilon(t),\cdot)\rangle\,|^2 = 0. \tag{4.1}$$

Proof. According to the conditions, we have

$$\lim_{\varepsilon\to 0} E \sup_{t\in[0,T]} |\,\langle m_\varepsilon\left(\xi_\varepsilon(t),\cdot\right)\rangle - \langle m_\varepsilon\left(\zeta_\varepsilon(t),\cdot\right)\rangle\,|^2$$
$$\le \lim_{\varepsilon\to 0} \varepsilon^{2(1-2\delta)} E \sup_{t\in[0,T]} \sup_{E_1} \left|\frac{\partial}{\partial x}\left\langle m\left(\frac{\xi_\varepsilon(t)}{\varepsilon^\delta},\cdot\right)\right\rangle\right|^2 |\psi_\varepsilon(\xi_\varepsilon(t),\eta_\varepsilon(t))|^2 = 0.$$

Similarly, we can prove (4.1). □

Lemma 4.13. *Let the Conditions (A) and (B) hold. The processes $\zeta_\varepsilon(t)$ is defined by (3.1), $\lambda_\varepsilon(t)$ is defined by (3.2), and let $l_\varepsilon(x)$ be such a function that*

$$E \sup_{t\in[0,T]} |l_\varepsilon(\zeta_\varepsilon(t))|^2 \le C, \tag{4.2}$$

and for every $r, t : 0 \le r \le t \le T$

$$\lim_{\varepsilon\to 0} E\left|\int_r^t l_\varepsilon(\zeta_\varepsilon(s))ds\right| = 0. \tag{4.3}$$

Then, for every $\phi(x) \in C_0^\infty(E_1)$ and $0 \le r \le t \le T$,

$$\lim_{\varepsilon\to 0} E\left|\int_r^t \phi(\lambda_\varepsilon(s)) l_\varepsilon(\zeta_\varepsilon(s))ds\right| = 0.$$

Proof. Let $\{t_i\}$ be some partition of interval $[r,t] : r \le t_1 \le t_2 \le ... \le t_n = t$ such that $|t_{i+1} - t_i| \le \eta_i$. Then

$$E\left|\int_r^t \phi(\lambda_\varepsilon(s)) l_\varepsilon(\zeta_\varepsilon(s))ds\right| \le E\sum_{i=1}^{n-1}\left|\int_{t_i}^{t_{i+1}} \big(\phi(\lambda_\varepsilon(s)) - \phi(\lambda_\varepsilon(t_i))\big) l_\varepsilon(\zeta_\varepsilon(s))ds\right|$$
$$+ \sum_{i=1}^{n-1} E\big|\phi(\lambda_\varepsilon(t_i))\big|\left|\int_{t_i}^{t_{i+1}} l_\varepsilon(\zeta_\varepsilon(s))ds\right|. \tag{4.4}$$

Let us denote first term in the right hand side (4.4) by $L(\varepsilon, n)$, then, using (4.2) and the estimation of $E|\lambda_\varepsilon(t) - \lambda_\varepsilon(s)|^4$, there exist a constant C_0 such that for every $0 < \varepsilon < \varepsilon_0$

$$L(\varepsilon,n) \le \sum_{i=1}^{n-1}\left(\int_{t_i}^{t_{i+1}} E\big|\phi(\lambda_\varepsilon(s)) - \phi(\lambda_\varepsilon(t_i))\big|^4 ds\right)^{\frac{1}{4}} \left(\int_{t_i}^{t_{i+1}} E\big|l_\varepsilon(\zeta_\varepsilon(s))\big|^{\frac{4}{3}} ds\right)^{\frac{3}{4}}$$

$$\leq C_0 \sum_{i=1}^{n-1} \left\{ \left(\int_{t_i}^{t_{i+1}} (s-t_i)^2 ds \right)^{\frac{1}{4}} (t_{i+1}-t_i)^{\frac{3}{4}} \right\} \leq C_0 (t-r) \max_i \sqrt{\eta_i}.$$

Consequently, $L(\varepsilon, n)$ can be made arbitrarily small by making the partition of the interval $[r,t]$ fine enough. By (4.3), the second term in the right hand side of (4.4) tends to 0 as $\varepsilon \to 0$. The lemma is proved. □

Lemma 4.14. *Let us define two functions*

$$\gamma(x) = 2 \int_0^x F(z) \int_0^z \frac{\kappa_0^{-1} F(y) \overline{m}(y) - \beta_0}{F(y)\overline{\alpha}(y)} dy dz,$$

$$\gamma^1(x) = 2 \int_0^x F(z) \int_0^z \frac{\kappa_0^{-2} F^2(y) \overline{\alpha}(y) - \sigma_0^2}{F(y)\overline{\alpha}(y)} dy dz.$$

Let the conditions (A), (B), and (C) be fulfilled. The process $\zeta_\varepsilon(t)$ is defined by (3.1). Then, for every $\delta \in]0; \frac{1}{2}[$,

$$\lim_{\varepsilon \to 0} E \sup_{t \in [0,T]} |\varepsilon^{2\delta} \gamma_\varepsilon(\zeta_\varepsilon(t))| = \lim_{\varepsilon \to 0} E \sup_{t \in [0,T]} |\varepsilon^{\delta} \dot{\gamma}_\varepsilon(\zeta_\varepsilon(t))|^2 = 0,$$

$$\lim_{\varepsilon \to 0} E \sup_{t \in [0,T]} |\varepsilon^{2\delta} \gamma^1_\varepsilon(\zeta_\varepsilon(t))| = \lim_{\varepsilon \to 0} E \sup_{t \in [0,T]} |\varepsilon^{\delta} \dot{\gamma}^1_\varepsilon(\zeta_\varepsilon(t))|^2 = 0. \tag{4.5}$$

Proof. For every $N : \varepsilon^{\frac{\delta}{2}} \leq N < \infty$,

$$E \sup_{t \in [0,T]} |\varepsilon^{2\delta} \gamma_\varepsilon(\zeta_\varepsilon(t))| = E \sup_{t \in [0,T]} |\varepsilon^{2\delta} \gamma_\varepsilon(\zeta_\varepsilon(t))| \Big(\chi\{|\zeta_\varepsilon(t)| < \varepsilon^{\frac{\delta}{2}}\} + \chi\{\varepsilon^{\frac{\delta}{2}} \leq |\zeta_\varepsilon(t)| \leq N\} + \chi\{|\zeta_\varepsilon(t)| > N\} \Big) = \gamma_\varepsilon^{(1)} + \gamma_\varepsilon^{(2)} + \gamma_\varepsilon^{(3)}$$

respectively.

According to Lemma 4.3, under the Conditions (A), (B), we have

$$|\gamma(x)| \leq C(1+|x|^2), \qquad \sum_{i=1}^{4} \left| \frac{d^i}{dx^i} \gamma(x) \right| \leq C(1+|x|).$$

Under the Condition (C) and by part 1. of Lemma 4.3, we obtain

$$\lim_{\varepsilon \to 0} \sup_{0<|x|\leq N} |\varepsilon^{2\delta} \gamma_\varepsilon(x)| = \lim_{\varepsilon \to 0} \sup_{0<|x|\leq N} |\varepsilon^{\delta} \dot{\gamma}_\varepsilon(x)|^2 = 0.$$

Now, we have

$$\lim_{\varepsilon \to 0} (\gamma_\varepsilon^{(1)} + \gamma_\varepsilon^{(2)}) = 0.$$

Similarly, using Corollary 4.8, we get

$$\gamma_\varepsilon^{(3)} \leq \varepsilon^{2\delta} C (1 + \varepsilon^{-4\delta} E \sup_{t \in [0,T]} |\zeta_\varepsilon(t)|^4)^{\frac{1}{2}} \frac{E \sup_{t \in [0,T]} |\zeta_\varepsilon(t)|^2}{N^2} \leq \frac{C}{N^2}.$$

Approaching the limit as $\varepsilon \to 0$ and then as $N \to \infty$,

$$\lim_{\varepsilon \to 0} E \sup_{t \in [0,T]} |\varepsilon^{2\delta} \gamma_\varepsilon(\zeta_\varepsilon(t))| = 0$$

follows. Obviously,

$$\lim_{\varepsilon \to 0} E \sup_{t \in [0,T]} |\varepsilon^{\delta} \dot{\gamma}_\varepsilon(\zeta_\varepsilon(t))|^2 = 0.$$

In a similar way, we can prove (4.5). □

Lemma 4.15. *Let the conditions (A), (B), and (C2) be fulfilled. The process $\zeta_\varepsilon(t)$ is defined by (3.1) and $\lambda_\varepsilon(t)$ is defined by (3.2). Then for every $\delta \in]0; \frac{1}{2}[$*

$$\lim_{\varepsilon\to 0} E\Phi_r(\xi_\varepsilon) \int_r^t \dot{\phi}(\lambda_\varepsilon(s)) \left(\frac{1}{\kappa_0} F_\varepsilon(\zeta_\varepsilon(s))\langle m_\varepsilon(\zeta_\varepsilon(s),\,\cdot\,)\rangle - \beta_0\right) ds \;=\; 0, \tag{4.6}$$

$$\lim_{\varepsilon\to 0} E\Phi_r(\xi_\varepsilon) \int_r^t \ddot{\phi}(\lambda_\varepsilon(s)) \left(\kappa_0^{-2} F_\varepsilon^2(\zeta_\varepsilon(s))\langle \alpha_\varepsilon(\zeta_\varepsilon(s),\,\cdot\,)\rangle - \sigma_0^2]\right) ds \;=\; 0. \tag{4.7}$$

Proof. Firstly, we will prove (4.6). According to Lemma 4.13, it is sufficient to show, for every $r, t : 0 \le r < t \le T$, that

$$\lim_{\varepsilon\to 0} E\left|\int_r^t \left[\kappa_0^{-1} F_\varepsilon(\zeta_\varepsilon(s))\langle m_\varepsilon(\zeta_\varepsilon(s),\,\cdot\,)\rangle - \beta_0\right] ds\right| = 0. \tag{4.8}$$

We note that, using the estimations for $\gamma(x)$, by virtue of Corollary 4.11 for $P(x) = 1$, $H(x) = \dot{\gamma}(x)$ ((4.16)), and $k(x,y) = \beta(x,y) - \overline{\beta}(x)$, we obtain

$$\int_r^t \dot{\gamma}_\varepsilon(\zeta_\varepsilon(s))\beta_\varepsilon(\xi_\varepsilon(s),\eta_\varepsilon(s))ds = \int_r^t \dot{\gamma}_\varepsilon(\zeta_\varepsilon(s))\overline{\beta}_\varepsilon(\xi_\varepsilon(s))ds + G(\varepsilon, \beta_\varepsilon - \overline{\beta}_\varepsilon, r, t).$$

In a similar way for $k(x,y) = \alpha(x,y) - \overline{\alpha}(x)$ and $H(x) = \ddot{\gamma}(x)$, we arrive at

$$\int_r^t \ddot{\gamma}_\varepsilon(\zeta_\varepsilon(s))\alpha_\varepsilon(\xi_\varepsilon(s),\eta_\varepsilon(s))ds = \int_r^t \ddot{\gamma}_\varepsilon(\zeta_\varepsilon(s))\overline{\alpha}_\varepsilon(\xi_\varepsilon(s))ds + G(\varepsilon, \alpha_\varepsilon - \overline{\alpha}_\varepsilon, r, t).$$

Applying Ito's formula to the function $\varepsilon^{2\delta}\gamma_\varepsilon(\zeta_\varepsilon(t))$ and taking into account the equality $\overline{L}_x\gamma(x) = \frac{1}{\kappa_0}F(x)\overline{m}(x) - \beta_0$, ((2.9)), and two previous relationships, we have

$$\begin{aligned}
&\int_0^t \left\{\frac{1}{\kappa_0} F_\varepsilon(\zeta_\varepsilon(s))\overline{m}_\varepsilon(\zeta_\varepsilon(s)) - \beta_0\right\} ds \\
&\quad = \varepsilon^{2\delta}\left(\gamma_\varepsilon(\zeta_\varepsilon(t)) - \gamma_\varepsilon(\zeta_\varepsilon(0))\right) - \varepsilon^{\delta}\int_0^t \dot{\gamma}_\varepsilon(\zeta_\varepsilon(s))m_\varepsilon(\xi_\varepsilon(s),\eta_\varepsilon(s))ds \\
&\qquad - \varepsilon\int_0^t \dot{\gamma}_\varepsilon(\zeta_\varepsilon(s))\frac{\partial}{\partial x}\psi_\varepsilon(\xi_\varepsilon(s),\eta_\varepsilon(s))m_\varepsilon(\xi_\varepsilon(s),\eta_\varepsilon(s))ds \\
&\qquad - \varepsilon^{1-\delta}\int_0^t \dot{\gamma}_\varepsilon(\zeta_\varepsilon(s))\frac{\partial}{\partial x}\psi_\varepsilon(\xi_\varepsilon(s),\eta_\varepsilon(s))c_\varepsilon(\xi_\varepsilon(s),\eta_\varepsilon(s))ds \\
&\qquad - \varepsilon^{\delta}\int_0^t \dot{\gamma}_\varepsilon(\zeta_\varepsilon(s))\alpha_\varepsilon^{\frac{1}{2}}(\xi_\varepsilon(s),\eta_\varepsilon(s))dw_1(s) \\
&\qquad - G(\varepsilon, \beta_\varepsilon - \overline{\beta}_\varepsilon, 0, t) - \frac{1}{2}G(\varepsilon, \alpha_\varepsilon - \overline{\alpha}_\varepsilon, 0, t).
\end{aligned}$$

Using this, Lemma 4.14 and Corollary 4.8, we obtain (4.8) and, consequently, (4.6).

In a similar way, we can prove (4.7) This result can be obtained by an application Ito's formula to $\varepsilon^{\delta}\gamma_\varepsilon^1(\zeta_\varepsilon(t))$. After that, we use Lemma 4.14 and Corollary 4.8. □

5. Examples

In this section the result of the theorem is applied to classes of random processes.

5.1. *Example 1*

Let $\eta_\varepsilon(t)$ be a one-dimensional process (the solution of (2.2), $n = 1$) and the Condition (A) is satisfied.

Then, $L = \frac{1}{2}a(y)\frac{d^2}{dy^2} + b(y)\frac{d}{dy}$, with $a(y) = \sigma^2(y)$ and

$$p(y) = \frac{1}{C_0 a(y) R(y)},$$

where $R(y) = exp\left(-\int_0^y \frac{2b(z)}{a(z)}dz\right)$, and $C_0 = \int_0^1 \frac{1}{a(z)R(z)}dz$. Condition (AB) gives $\int_0^1 g(x,y)p(y)dy = 0$. Let us define $M(x,y) = \int_0^y R(z)\int_0^z \frac{g(x,k)}{a(k)R(k)}dkdz$

The problem (2.5) has unique solution

$$\psi(x,y) = -M(x,y) + C_1(x)\int_0^y R(z)dz + C_2(x),$$

where the functions $C_1(x)$, $C_2(x)$ can be defined by the periodicity condition $\psi(x,0) = \psi(x,1)$, $C_1(x) = \frac{M(x,1)}{\int_0^1 R(z)dz}$, and from the condition in (2.5), $C_2(x) = \int_0^1 M(x,y)dy - \frac{M(x,1)}{\int_0^1 R(z)dz}\int_0^1\int_0^y R(z)dzdy.$

Then

$$\overline{\alpha}(x) = \int_0^1 \left(\sigma(y)\frac{\partial}{\partial y}\psi(x,y)\right)^2 p(y)dy,$$

$$\overline{\beta}(x) = \int_0^1 \left(c(x,y) + g(x,y)\ \frac{\partial}{\partial x}\psi(x,y)\right)p(y)dy,\ \overline{m}(x) = \int_0^1 m(x,y)p(y)dy.$$

Let Condition (B) be satisfied. Then $F(x) = \exp\left\{-2\int_0^x \frac{\overline{\beta}(z)}{\overline{\alpha}(z)}dz\right\}$. If Condition (C) is satisfied, then the limit process is

$$\xi(t) = \xi_0 + \frac{\kappa_2}{\kappa_0\kappa_1}t + \frac{1}{\kappa_0\kappa_1}w(t).$$

In this case, all auxiliary functions have the explicit form.

5.2. *Example 2*

We again consider the case $n = 1$, under previous assumptions.
Let us simplify the problem in such a way, that we can check the result by applying the Diffusion Approximation Theorem.

The system has the form, $t \in [0,T]$,

$$\xi_\varepsilon(t) = \xi_0 + \frac{1}{\varepsilon}\int_0^t g_1(\eta_\varepsilon(s))ds + \int_0^t m_1(\eta_\varepsilon(s))ds,$$

$$\eta_\varepsilon(t) = \eta_0 + \frac{1}{\varepsilon^2}\int_0^t b(\eta_\varepsilon(s))\,ds + \frac{1}{\varepsilon}\int_0^t \sigma(\eta_\varepsilon(s))\,dw_1(s).$$

We have $\psi(x,y) = \psi_1(y)$. In this case $\beta(x,y) = g_1(y)\ \frac{\partial}{\partial x}\psi(y) = 0$, and

$$\overline{\alpha} = \int_0^1 \left(\sigma(y)\frac{d}{dy}\psi_1(y)\right)^2 p(y)dy, \quad \overline{m}_1 = \int_0^1 m_1(y)p(y)dy.$$

Then $F(x) = 1$. Under Condition (C) the limit process for $\xi_\varepsilon(t)$, as ε tends to 0, is

$$\xi(t) = \xi_0 + \overline{m}_1 t + \sqrt{\overline{\alpha}} w(t).$$

To get this result, we can reduce the conditions of the theorem. For example, we do not need the conditions (B) and (C).

5.3. Example 3

Let us again consider the case $n = 1$ under previous assumptions. The system has the form, $t \in [0, T]$,

$$\xi_\varepsilon(t) = \xi_0 + \frac{1}{\varepsilon}\int_0^t \left(G_\varepsilon(\xi_\varepsilon(s))\sin(2\pi\eta_\varepsilon(s)) + D_\varepsilon(\xi_\varepsilon(s))\cos(2\pi\eta_\varepsilon(s))\right) ds,$$
$$\eta_\varepsilon(t) = \eta_0 + \frac{1}{\varepsilon} w_1(t).$$

Under our conditions, $p(y) = 1$, and $\psi(x, y) = \frac{1}{2\pi^2}(G(x)\sin(2\pi y) + D(x)\cos(2\pi y))$.
Then

$$\overline{\alpha}(x) = \frac{G^2(x) + D^2(x)}{(2\pi)^3}, \quad \overline{\beta}(x) = \frac{G(x)\dot{G}(x) + D(x)\dot{D}(x)}{4\pi^3}.$$

We can write the Conditions (B) and (C) precisely and note these conditions by (B') and C'.

Condition (B')

$B'1$. There exists a constant $\lambda_1 > 0$ such that for every $x \in E_1$,

$$G^2(x) + D^2(x) \geq \lambda_1.$$

$B'2$. There exists a constant $\lambda_2 > 0$ such that for every $x \in E_1$,

$$G^2(x) + D^2(x) \leq \lambda_2(G^2(0) + D^2(0)).$$

Condition (C')

There exist the constant κ_0 such that for $z \in E_1$,

$C'0$. $\lim_{|z|\to\infty} \frac{1}{z}\int_0^z \frac{dx}{\sqrt{G^2(x)+D^2(x)}} = \frac{\kappa_0}{\sqrt{G^2(0)+D^2(0)}}$;

Remark 5.3.1. Under these conditions the constant from the Condition $(C'1)$ is defined by $\kappa_1 = \kappa_0 \frac{(2\pi)^3}{(G^2(0)+D^2(0))}$.

For asymptotic behavior of $\xi_\varepsilon(t)$we get the statement:

Theorem 5.3.1. *Let conditions $(A), (AB), (B')$, and (C') be fulfilled. Then for every $\delta \in \,]0, \frac{1}{2}[$ the measures $\mu_\delta^\varepsilon \Rightarrow \mu$ as ε tends to 0. The random process $\xi(t)$, which corresponds to μ, is defined by*

$$\xi(t) = \xi_0 + \frac{\sqrt{G^2(0) + D^2(0)}}{\kappa_0 (2\pi)^{\frac{3}{2}}} w(t).$$

Acknowledgements

It was the author's pleasure to present the result at the Conference in honor of Tom Kurtz who has made a valuable contribution to Stochastic Approximation Theory.

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