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Absorption Time Distribution for an Asymmetric Random Walk

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Abstract: Consider the random walk on the set of nonnegative integers that takes two steps to the left (just one step from state 1) with probability $p \in [1/3, 1)$ and one step to the right with probability $1-p$. State 0 is absorbing and the initial state is a fixed positive integer j_0. Here we find the distribution of the absorption time. The absorption time is the duration of (or the number of coups in) the well-known Labouchere betting system. As a consequence of this, we obtain in the fair case ($p = 1/2$) the asymptotic behavior of the Labouchere bettor's conditional expected deficit after n coups, given that the system has not yet been completed.

1. Introduction

Fix a positive integer j_0, and let $\{X_n\}_{n\geq 0}$ be the random walk in $\mathbf{Z}_+$ with initial state $X_0 = j_0$, one-step transition probabilities

$$P(j,k) := \begin{cases} p & \text{if } k = (j-2)^+, \\ q & \text{if } k = j+1, \\ 0 & \text{otherwise,} \end{cases} \qquad j \geq 1, \tag{1}$$

where $1/3 \leq p < 1$ and $q := 1-p$, and absorption at state 0. We are interested in the distribution of the absorption time

$$N := \min\{n \geq 1 : X_n = 0\}. \tag{2}$$

Since $-2p + q \leq 0$, N is finite with probability 1. Probabilities and expectations involving N will be subscripted to indicate their dependence on j_0.

The random walk $\{X_n\}_{n\geq 0}$ arises in connection with the Labouchere system (also known as the cancellation system), one of the two or three best-known betting systems. It was popularized by British journalist and Member of Parliament Henry Du Pré Labouchere (1831–1912), who attributed it to French mathematician and philosopher Marie Jean Antoine Nicolas de Caritat, Marquis de Condorcet (1743–1794) (Thorold 1913, p. 66).

The system is applied to games of repeated coups that pay even money. The gambler's bet size at each coup is determined by an ordered list of positive integers kept on his score sheet and updated after each coup. Given such a list, the gambler's bet size at the next coup is the sum of the extreme terms on the list. (This is the same as the sum of the first and last terms on the list, except when there is only one term.) Following the resolution of this bet, the list is updated as follows: After a win, the extreme terms are cancelled. After a loss, the amount just lost is appended

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to the list as a new last term. The system is begun with an initial list, the most popular choice for which is $1, 2, 3, 4$; however, Labouchere himself used $3, 4, 5, 6, 7$ (Thorold 1913, p. 66). The initial list, together with the sequence of wins and losses, determines all bet sizes.

Notice that the sum of the terms on the list plus the gambler's cumulative profit remains constant in time (see Section 4 for details). Therefore, once the list becomes empty, betting is stopped and the gambler's cumulative profit is the sum of the terms on the initial list. Notice also that X_n represents the length of the list after n independent coups, each with win probability p, so N is the duration of the system, that is, the number of coups required to complete it. Of course, we are making the unrealistic assumptions that the gambler has unlimited resources and that there is no maximum betting limit.

Downton (1980) found a recursive formula for the distribution of N in the case $j_0 = 4$. It is easy to generalize his result to arbitrary j_0.

Theorem 1 (Downton). *With $l_n := \lceil (2n + 1 - j_0)/3 \rceil^+$ for all $n \geq 0$, define a modified Pascal triangle (depending on j_0) recursively by $c(0, l) := \delta_{0,l}$, where $\delta_{0,l}$ is the Kronecker delta, and*[1]

$$c(n, l) := \begin{cases} c(n-1, l-1) + c(n-1, l) & \text{if } l_n \leq l \leq n, \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

for all $n \geq 1$. Then

$$P_{j_0}(N = n) = c(n-1, l_{n-1}) p^{n-l_{n-1}} q^{l_{n-1}} \tag{4}$$

if $n \geq \lfloor (j_0 + 1)/2 \rfloor$ and $(n + j_0 - 1)/3 \notin \mathbf{Z}$, and $P_{j_0}(N = n) = 0$ otherwise.

Proof. Let us redefine $c(n, l)$ to be the number of the $\binom{n}{l}$ permutations of l losses and $n - l$ wins for which the Labouchere bettor has not yet completed the system. For $c(n, l)$ to be positive, we must have $0 \leq l \leq n$ and $j_0 + l - 2(n - l) \geq 1$, hence $l_n \leq l \leq n$. To establish (3) it suffices to consider whether the nth coup results in a loss or a win. Then (4) follows by noting that, if $N = n$, then the first $n - 1$ coups must have l_{n-1} losses (the minimal number) and $n - 1 - l_{n-1}$ wins, and the nth coup must result in a win. Finally, one can check that

$$j_0 + l_{n-1} - 2(n - 1 - l_{n-1}) = \begin{cases} 3 & \text{if } n + j_0 - 1 \equiv 0 \pmod 3, \\ 1 & \text{if } n + j_0 - 1 \equiv 1 \pmod 3, \\ 2 & \text{if } n + j_0 - 1 \equiv 2 \pmod 3, \end{cases} \tag{5}$$

so the system can be completed at the nth coup if and only if $(n+j_0-1)/3 \notin \mathbf{Z}$. $\square$

Downton's theorem is useful for numerical computation. It also gives the upper bound

$$P_{j_0}(N = n) \leq \binom{n-1}{l_{n-1}} p^{n-l_{n-1}} q^{l_{n-1}} \tag{6}$$

under the conditions of (4). For example, taking $j_0 = 1$ for convenience, Downton's bound gives

$$P_1(N = 3m + 1) \leq \binom{3m}{m} p^{m+1} q^{2m} \tag{7}$$

[1] There is a minor error in Downton's formulation: He replaced n by $n + 1$ in the equation of our (3) without doing so in the inequalities of our (3). His tables are nevertheless correct.

and

$$P_1(N = 3m + 2) \leq \binom{3m+1}{m} p^{m+1} q^{2m+1} \tag{8}$$

for all $m \geq 0$.

As we will see in Section 2, the actual state of affairs is

$$P_1(N = 3m + 1) = \frac{1}{2m+1} \binom{3m}{m} p^{m+1} q^{2m} \tag{9}$$

and

$$P_1(N = 3m + 2) = \frac{1}{m+1} \binom{3m+1}{m} p^{m+1} q^{2m+1} \tag{10}$$

for all $m \geq 0$. Equations (9) and (10) allow us to determine (in Section 3) the asymptotic behavior of $P_1(N \geq n + 1)$ as $n \to \infty$, and this leads (in Section 4) to the asymptotic behavior of the Labouchere bettor's conditional expected deficit after n coups, given that the system has not yet been completed, at least if the game is fair ($p = 1/2$). Of course, we do not restrict our attention to the case $j_0 = 1$.

Downton (1980) observed that "no probability analysis specific to the [Labouchere] system appears to have been made." He also remarked that "the probability structure of the size of the bets in the system remains an unsolved problem." More than 25 years later, the problem is still open. The present note is a start, however, which we hope will encourage further investigation.

2. The absorption time distribution

A direct derivation of the distribution of N is not entirely straightforward. Let $Y_1, Y_2, \ldots$ be i.i.d. with $P(Y_1 = -2) = p$ and $P(Y_1 = 1) = q$. Define $N_1 := \min\{n \geq 1 : 1 + Y_1 + \cdots + Y_n \in \{-1, 0\}\}$, and note that N_1 is distributed as the P_1-distribution of N. With $g(u) := E[u^{Y_1}] = pu^{-2} + qu$ ($u \neq 0$), apply the optional stopping theorem to the martingale $M_n := u^{1+Y_1+\cdots+Y_n} g(u)^{-n}$ ($n \geq 0$) stopped at time N_1. Using Cardano's formula to solve the cubic equation $v = 1/g(u)$ or $qvu^3 - u^2 + pv = 0$, one can show that the P_1-probability generating function of N is

$$\begin{aligned} E_1[v^N] = {} & 2\,\frac{1 - \cos((1/3)\cos^{-1}(1 - (27/2)pq^2v^3))}{3qv} \\ & - \frac{[1 - \cos((1/3)\cos^{-1}(1 - (27/2)pq^2v^3))]^2}{(3qv)^2} \\ & + 3\,\frac{\sin^2((1/3)\cos^{-1}(1 - (27/2)pq^2v^3))}{(3qv)^2} \end{aligned} \tag{11}$$

for $0 < v < 1$. With some difficulty this can then be written as a power series in v (the singularity at $v = 0$ is removable).

Fortunately, the hard work has already been done in the combinatorics literature. The results we need follow easily from a generalization of the ballot theorem stated by Barbier (1887) and proved by Aeppli (1924) in his Ph.D. thesis under G. Pólya. The ballot theorem itself is due to Bertrand (1887). See Takacs (1997) for a survey of this topic. Let a_m be the number of paths, with steps $(1, 0)$ and $(0, 1)$ (i.e., east and north), from $(0, 0)$ to $(2m, m)$ that never rise above (but may touch) the line

$y = x/2$. Let b_m be the number of paths, with steps $(1,0)$ and $(0,1)$, from $(0,0)$ to $(2m, m)$ that never rise above (but may touch) the line $y = (x+1)/2$. Then

$$a_m = \frac{1}{2m+1}\binom{3m}{m} = \frac{1}{3m+1}\binom{3m+1}{m} \tag{12}$$

and

$$b_m = \frac{1}{m+1}\binom{3m+1}{m}. \tag{13}$$

To find the P_1-probability of the event $\{N = 3m+1\}$, notice that, for this event to occur, the first $3m$ coups must result in exactly $2m$ losses and m wins with the cumulative number of wins never being more than half of the cumulative number of losses, for otherwise absorption would have occurred earlier. Finally, coup $3m+1$ must result in a win. We conclude from (12) that (9) holds for all $m \geq 0$.

To find the P_1-probability of the event $\{N = 3m+2\}$, notice that, for this event to occur, the first coup must result in a loss, the next $3m$ coups must result in exactly $2m$ losses and m wins with the cumulative number of wins never being more than half of the cumulative number of losses (including the first one), for otherwise absorption would have occurred earlier. Finally, coup $3m+2$ must result in a win. We conclude from (13) that (10) holds for all $m \geq 0$.

Notice also that $P_1(N = 3m+3) = 0$ for all $m \geq 0$ because of the periodicity of the random walk.

The distribution of N for arbitrary j_0 can be found recursively using the Markov identity

$$P_{j_0}(N = n+1) = pP_{j_0-2}(N = n) + qP_{j_0+1}(N = n), \tag{14}$$

or equivalently

$$P_{j_0+1}(N = n) = q^{-1}P_{j_0}(N = n+1) - pq^{-1}P_{j_0-2}(N = n), \tag{15}$$

for all $n \geq 1$, where $P_0(N = n) := 0$ and $P_{-1}(N = n) := 0$. The results for $j_0 = 1, 2, \ldots, 9$ are

$$\begin{aligned}
P_1(N = 3m+1) &= a_m p^{m+1} q^{2m} \\
P_2(N = 3m+3) &= a_{m+1} p^{m+2} q^{2m+1} \\
P_3(N = 3m+2) &= a_{m+1} p^{m+2} q^{2m} \\
P_4(N = 3m+4) &= (a_{m+2} - a_{m+1}) p^{m+3} q^{2m+1} \\
P_5(N = 3m+3) &= (a_{m+2} - 2a_{m+1}) p^{m+3} q^{2m} \\
P_6(N = 3m+5) &= (a_{m+3} - 3a_{m+2}) p^{m+4} q^{2m+1} \\
P_7(N = 3m+4) &= (a_{m+3} - 4a_{m+2} + a_{m+1}) p^{m+4} q^{2m} \\
P_8(N = 3m+6) &= (a_{m+4} - 5a_{m+3} + 3a_{m+2}) p^{m+5} q^{2m+1} \\
P_9(N = 3m+5) &= (a_{m+4} - 6a_{m+3} + 6a_{m+2}) p^{m+5} q^{2m}
\end{aligned} \tag{16}$$

for all $m \geq 0$,

$$\begin{aligned}
P_1(N = 3m+2) &= b_m p^{m+1} q^{2m+1} \\
P_2(N = 3m+1) &= b_m p^{m+1} q^{2m} \\
P_3(N = 3m+3) &= b_{m+1} p^{m+2} q^{2m+1} \\
P_4(N = 3m+2) &= (b_{m+1} - b_m) p^{m+2} q^{2m} \\
P_5(N = 3m+4) &= (b_{m+2} - 2b_{m+1}) p^{m+3} q^{2m+1}
\end{aligned} \tag{17}$$

$$P_6(N = 3m + 3) = (b_{m+2} - 3b_{m+1})p^{m+3}q^{2m}$$
$$P_7(N = 3m + 5) = (b_{m+3} - 4b_{m+2} + b_{m+1})p^{m+4}q^{2m+1}$$
$$P_8(N = 3m + 4) = (b_{m+3} - 5b_{m+2} + 3b_{m+1})p^{m+4}q^{2m}$$
$$P_9(N = 3m + 6) = (b_{m+4} - 6b_{m+3} + 6b_{m+2})p^{m+5}q^{2m+1}$$

for all $m \geq 0$, and

$$\begin{aligned} P_1(N = 3m + 3) &= 0 \\ P_2(N = 3m + 2) &= 0 \\ P_3(N = 3m + 4) &= 0 \\ P_4(N = 3m + 3) &= 0 \\ P_5(N = 3m + 5) &= 0 \\ P_6(N = 3m + 4) &= 0 \\ P_7(N = 3m + 6) &= 0 \\ P_8(N = 3m + 5) &= 0 \\ P_9(N = 3m + 7) &= 0 \end{aligned} \tag{18}$$

for all $m \geq 0$.

From these special cases we can easily conjecture and prove the general result.

Theorem 2. *For each $m \geq 0$,*

$$\begin{aligned} &P_{j_0}(N = 3m + 3\lfloor j_0/2 \rfloor - j_0 + 2) \\ &\quad = \sum_{i=0}^{\lceil j_0/3 \rceil - 1} (-1)^i \binom{j_0 - 1 - 2i}{i} a_{m+\lfloor j_0/2 \rfloor - i} \\ &\qquad\qquad \cdot p^{m+\lfloor j_0/2 \rfloor + 1} q^{2m+1-(j_0 - 2\lfloor j_0/2 \rfloor)}, \end{aligned} \tag{19}$$

$$\begin{aligned} &P_{j_0}(N = 3m + 3\lfloor (j_0 - 1)/2 \rfloor - (j_0 - 1) + 2) \\ &\quad = \sum_{i=0}^{\lceil j_0/3 \rceil - 1} (-1)^i \binom{j_0 - 1 - 2i}{i} b_{m+\lfloor (j_0-1)/2 \rfloor - i} \\ &\qquad\qquad \cdot p^{m+\lfloor (j_0-1)/2 \rfloor + 1} q^{2m+1-\{(j_0-1) - 2\lfloor (j_0-1)/2 \rfloor\}}, \end{aligned} \tag{20}$$

and

$$P_{j_0}(N = 3m + 3\lfloor (j_0 - 1)/2 \rfloor - (j_0 - 1) + 3) = 0. \tag{21}$$

Of course, $P_{j_0}(N \geq \lfloor (j_0 + 1)/2 \rfloor) = 1$.

Remark. The theorem can be derived directly from the combinatorics literature without reference to probability. The result needed is a formula of Niederhausen (2002, middle of p. 9)[2]; he attributed the formula to Koroljuk (1955).

Proof. The proof of each of the equations, (19)–(21), proceeds by complete induction on j_0 using (14), the case $j_0 = 1$ having been already established (not to mention the cases $j_0 = 2, 3, \ldots, 9$). To avoid the awkward floor and ceiling functions, one can consider six cases, $j_0 = 3i_0 + 1$, $j_0 = 3i_0 + 2$, or $j_0 = 3i_0 + 3$, each with i_0 an even or odd nonnegative integer. The details are straightforward but tedious. □

[2] His $\lfloor d \rfloor$ should be $\lceil d \rceil - 1$.

3. Asymptotic tail behavior

First we notice that

$$\frac{a_m}{a_{m-1}} = \frac{3m(3m-1)(3m-2)}{m(2m)(2m+1)} < \frac{27}{4}, \qquad m \geq 1, \tag{22}$$

$$\frac{b_m}{b_{m-1}} = \frac{(3m+1)(3m)(3m-1)}{(m+1)(2m)(2m+1)} < \frac{27}{4}, \qquad m \geq 1, \tag{23}$$

and

$$\lim_{m\to\infty} \frac{a_m}{a_{m-1}} = \lim_{m\to\infty} \frac{b_m}{b_{m-1}} = \frac{27}{4}. \tag{24}$$

Since $a_0 = b_0 = 1$, we obtain $a_m < (27/4)^m$ and $b_m < (27/4)^m$ for each $m \geq 1$. More precisely, using Stirling's formula,

$$a_m = \frac{m+1}{3m+1} b_m \sim \frac{\sqrt{3}}{4\sqrt{\pi}} m^{-3/2} \left(\frac{27}{4}\right)^m. \tag{25}$$

Since $p \in [1/3, 1)$, we have $pq^2 \leq 4/27$ with strict inequality if $p > 1/3$, hence

$$\rho := \frac{27}{4} pq^2 \leq 1 \quad (\rho < 1 \text{ if } p > 1/3). \tag{26}$$

Consequently,

$$P_1(N = 3m+1) = a_m p^{m+1} q^{2m} \sim \frac{\sqrt{3}\, p}{4\sqrt{\pi}} m^{-3/2} \rho^m \tag{27}$$

and

$$P_1(N = 3m+2) = b_m p^{m+1} q^{2m+1} \sim \frac{3\sqrt{3}\, pq}{4\sqrt{\pi}} m^{-3/2} \rho^m. \tag{28}$$

Now assume that $p > 1/3$. It follows that, with the convention that empty products are 1,

$$\begin{aligned} P_1(N \geq 3m+1) &= P_1(N = 3m+1) \sum_{n=0}^{\infty} \prod_{l=m+1}^{m+n} \frac{P_1(N = 3l+1)}{P_1(N = 3l-2)} \\ &\qquad + P_1(N = 3m+2) \sum_{n=0}^{\infty} \prod_{l=m+1}^{m+n} \frac{P_1(N = 3l+2)}{P_1(N = 3l-1)} \\ &= P_1(N = 3m+1) \sum_{n=0}^{\infty} \left[\prod_{l=m+1}^{m+n} \frac{a_l}{a_{l-1}} \right] (pq^2)^n \\ &\qquad + P_1(N = 3m+2) \sum_{n=0}^{\infty} \left[\prod_{l=m+1}^{m+n} \frac{b_l}{b_{l-1}} \right] (pq^2)^n \\ &\sim P_1(N = 3m+1) \sum_{n=0}^{\infty} \rho^n + P_1(N = 3m+2) \sum_{n=0}^{\infty} \rho^n \\ &\sim C_{1,1}\, m^{-3/2} \rho^m, \end{aligned} \tag{29}$$

where $C_{1,1} := \sqrt{3}\, p(1+3q)/[4\sqrt{\pi}(1-\rho)]$. Similarly,

$$P_1(N \geq 3m+2) \sim C_{1,2}\, m^{-3/2} \rho^m, \tag{30}$$

where $C_{1,2} := \sqrt{3}\,p(\rho+3q)/[4\sqrt{\pi}(1-\rho)]$, and

$$P_1(N \geq 3m+3) = P_1(N \geq 3m+4) \sim C_{1,3}\, m^{-3/2}\rho^m, \tag{31}$$

where $C_{1,3} := \rho C_{1,1}$.

We claim that the same type of asymptotic decay holds for the tail of the distribution of N for arbitrary j_0. Only the multiplicative constants differ.

Theorem 3. *Assume that $p > 1/3$. As $m \to \infty$,*

$$P_{j_0}(N \geq 3m+1) \sim D_{j_0,1}\, m^{-3/2}\rho^m, \tag{32}$$

$$P_{j_0}(N \geq 3m+2) \sim D_{j_0,2}\, m^{-3/2}\rho^m, \tag{33}$$

$$P_{j_0}(N \geq 3m+3) \sim D_{j_0,3}\, m^{-3/2}\rho^m, \tag{34}$$

for suitable constants $D_{j_0,1}, D_{j_0,2}, D_{j_0,3}$ to be defined below.

Proof. Using Theorem 2 and arguing as above, we obtain

$$P_{j_0}(N = 3m + 3\lfloor j_0/2\rfloor - j_0 + 2) \sim A_{j_0} m^{-3/2}\rho^m, \tag{35}$$

$$P_{j_0}(N = 3m + 3\lfloor (j_0-1)/2\rfloor - (j_0-1) + 2) \sim B_{j_0} m^{-3/2}\rho^m, \tag{36}$$

where

$$A_{j_0} := \frac{\sqrt{3}}{4\sqrt{\pi}} \sum_{i=0}^{\lceil j_0/3\rceil - 1} (-1)^i \binom{j_0-1-2i}{i} \left(\frac{27}{4}\right)^{\lfloor j_0/2\rfloor - i} \cdot p^{\lfloor j_0/2\rfloor+1} q^{1-(j_0-2\lfloor j_0/2\rfloor)}, \tag{37}$$

$$B_{j_0} := \frac{3\sqrt{3}}{4\sqrt{\pi}} \sum_{i=0}^{\lceil j_0/3\rceil - 1} (-1)^i \binom{j_0-1-2i}{i} \left(\frac{27}{4}\right)^{\lfloor (j_0-1)/2\rfloor - i} \cdot p^{\lfloor (j_0-1)/2\rfloor+1} q^{1-\{(j_0-1)-2\lfloor (j_0-1)/2\rfloor\}}. \tag{38}$$

It follows that

$$P_{j_0}(N \geq 3m + \lfloor (j_0+1)/2\rfloor) \sim C_{j_0,1}\, m^{-3/2}\rho^m, \tag{39}$$

$$P_{j_0}(N \geq 3m + \lfloor (j_0+1)/2\rfloor + 1) \sim C_{j_0,2}\, m^{-3/2}\rho^m, \tag{40}$$

$$P_{j_0}(N \geq 3m + \lfloor (j_0+1)/2\rfloor + 2) \sim C_{j_0,3}\, m^{-3/2}\rho^m, \tag{41}$$

where

$$C_{j_0,1} := (1-\rho)^{-1}(A_{j_0} + B_{j_0}), \tag{42}$$

$$C_{j_0,2} := \begin{cases} (1-\rho)^{-1}(\rho A_{j_0} + B_{j_0}) & \text{if } j_0 \text{ is odd,} \\ (1-\rho)^{-1}(A_{j_0} + \rho B_{j_0}) & \text{if } j_0 \text{ is even,} \end{cases} \tag{43}$$

$$C_{j_0,3} := \begin{cases} (1-\rho)^{-1}(\rho A_{j_0} + \rho B_{j_0}) & \text{if } j_0 \text{ is odd,} \\ (1-\rho)^{-1}(A_{j_0} + \rho B_{j_0}) & \text{if } j_0 \text{ is even,} \end{cases} \tag{44}$$

Finally, if we define

$$D_{j_0,1} := C_{j_0,1}, \qquad D_{j_0,2} := C_{j_0,2}, \qquad D_{j_0,3} := C_{j_0,3} \tag{45}$$

for $j_0 = 1, 2$,

$$D_{j_0,1} := \rho^{-1} C_{j_0,3}, \qquad D_{j_0,2} := C_{j_0,1}, \qquad D_{j_0,3} := C_{j_0,2} \tag{46}$$

for $j_0 = 3, 4$,

$$D_{j_0,1} := \rho^{-1} C_{j_0,2}, \qquad D_{j_0,2} := \rho^{-1} C_{j_0,3}, \qquad D_{j_0,3} := C_{j_0,1} \tag{47}$$

for $j_0 = 5, 6$, and

$$D_{j_0,i} := \rho^{-\lfloor (j_0-1)/6 \rfloor} D_{j_0 - 6\lfloor (j_0-1)/6 \rfloor, i} \tag{48}$$

for all $j_0 \geq 7$ and $i = 1, 2, 3$, then (32)–(34) hold. □

It will be convenient to restate Theorem 3 in a condensed form. Let us define

$$D_{j_0}(n) := \begin{cases} 3^{3/2} D_{j_0,1} & \text{if } n \equiv 0 \pmod 3, \\ 3^{3/2} \rho^{-1/3} D_{j_0,2} & \text{if } n \equiv 1 \pmod 3, \\ 3^{3/2} \rho^{-2/3} D_{j_0,3} & \text{if } n \equiv 2 \pmod 3, \end{cases} \tag{49}$$

so that $\{D_{j_0}(n)\}_{n \geq 0}$ is a sequence that repeatedly cycles through three specific constants.

Corollary 4. *Assume that $p > 1/3$. As $n \to \infty$,*

$$P_{j_0}(N \geq n+1) \sim D_{j_0}(n) n^{-3/2} (\rho^{1/3})^n. \tag{50}$$

Proof. By Theorem 3,

$$\begin{aligned} P_{j_0}(N \geq n+1) \\ \sim \begin{cases} D_{j_0,1}(n/3)^{-3/2} \rho^{n/3} & \text{if } n \equiv 0 \pmod 3, \\ D_{j_0,2}((n-1)/3)^{-3/2} \rho^{(n-1)/3} & \text{if } n \equiv 1 \pmod 3, \\ D_{j_0,3}((n-2)/3)^{-3/2} \rho^{(n-2)/3} & \text{if } n \equiv 2 \pmod 3, \end{cases} \end{aligned} \tag{51}$$

and the conclusion follows from (49). □

4. Application to gambling

Recall the Labouchere system as explained in Section 1. Let $\xi_1, \xi_2, \ldots$ be i.i.d. with common distribution

$$P(\xi_1 = 1) = p \quad \text{and} \quad P(\xi_1 = -1) = q, \tag{52}$$

with ξ_n representing the profit per unit bet at the nth coup.

Let B_n be the amount bet at the nth coup, let F_n be the gambler's cumulative profit after the nth coup, and let S_n be the sum of the terms on the gambler's list after the nth coup. We have already noted that $F_n + S_n$ does not depend on n. Indeed, if $1 \leq n \leq N$, then

$$F_n - F_{n-1} = B_n \xi_n \quad \text{and} \quad S_n - S_{n-1} = -B_n \xi_n, \tag{53}$$

implying that $F_n + S_n = F_{n-1} + S_{n-1}$. Since $F_0 = 0$, it follows that

$$F_n = S_0 - S_n, \qquad 1 \leq n \leq N. \tag{54}$$

In particular, since $p \geq 1/3$ by assumption, we have $N < \infty$ a.s., hence $S_N = 0$ a.s., that is,

$$P(F_N = S_0) = 1. \tag{55}$$

This equation says that, with probability 1, the gambler wins an amount equal to the sum of the terms on the initial list. Under our unrealistic assumptions that the gambler has unlimited resources and that there is no maximum betting limit, the Labouchere system is an infallible one.

Let us assume for now that $1/3 \le p \le 1/2$, so that the game is fair or subfair. It follows that $\{F_{n\wedge N}\}_{n\ge 0}$ is a supermartingale. By (55), it is not the case that $E[F_N] \le 0 = E[F_0]$, that is, the conclusion of the optional stopping theorem fails. More specifically, $\{F_{n\wedge N}\}_{n\ge 0}$ must fail to be uniformly integrable. But

$$|F_{n\wedge N}| \le \sum_{l=1}^{N} |F_l - F_{l-1}| = \sum_{l=1}^{N} B_l, \tag{56}$$

and in fact

$$|F_{n\wedge N}| \le \max_{0\le l\le N} |F_l| \le 2S_0 + \max_{0\le l\le N}(-F_l), \tag{57}$$

where the last inequality uses $|F_l| = F_l^+ + F_l^- = 2F_l^+ + (F_l^- - F_l^+) \le 2S_0 + (-F_l)$. Thus, the quantities on the right sides of (56) and (57) must fail to be integrable. In particular, the total amount bet by the Labouchere bettor has infinite expectation, as does his maximum deficit. This surprising result is due to Grimmett and Stirzaker (2001, Problem 12.9.15 and Solution). They also raised the question of whether the maximum bet size has infinite expectation as well, but this question is currently unresolved.

We return to the original assumption that $1/3 \le p < 1$.

Theorem 5. *If $p = 1/2$, then*

$$-E[F_n \mid N \ge n+1] = S_0\{P_{j_0}(N \ge n+1)^{-1} - 1\} \tag{58}$$

for all $n \ge 1$. If $1/3 \le p < 1/2$, then (58) holds with the $=$ sign replaced by $\ge$. If $1/2 < p < 1$, then (58) holds with the $=$ sign replaced by $\le$. In any case in which $p > 1/3$, as $n \to \infty$,

$$S_0\{P_{j_0}(N \ge n+1)^{-1} - 1\} \sim S_0 D_{j_0}(n)^{-1} n^{3/2} (\rho^{-1/3})^n, \tag{59}$$

where $D_{j_0}(n)$ is as in (49).

Remark. In words, the Labouchere bettor's conditional expected deficit after n coups at a fair game ($p = 1/2$), given that the system has not yet been completed, grows like a constant times $n^{3/2}(\rho^{-1/3})^n$, where $\rho^{-1/3} = 2^{5/3}/3 \approx 1.058267$. This geometric rate may be smaller than expected, but the factor $n^{3/2}$ should not be overlooked. Indeed, it dominates the factor $(2^{5/3}/3)^n$ for $2 \le n \le 128$. The right side of (58), as well as the multiplicative constant in (59), depends on the initial list only through the sum (S_0) and number (j_0) of its terms.

Proof. If $p = 1/2$, then $\{F_{n\wedge N}\}_{n\ge 0}$ is a martingale, and therefore

$$0 = E[F_0] = E[F_{n\wedge N}] = E[F_N\, 1_{\{N\le n\}}] + E[F_n\, 1_{\{N\ge n+1\}}], \tag{60}$$

hence

$$-E[F_n\, 1_{\{N\ge n+1\}}] = S_0\{1 - P_{j_0}(N \ge n+1)\}, \tag{61}$$

and the first conclusion follows from this. If $p < 1/2$ (resp., $p > 1/2$), then $\{F_{n\wedge N}\}_{n\ge 0}$ is a supermartingale (resp., submartingale), and the second $=$ sign in (60) is replaced by $\ge$ (resp., $\le$).

The asymptotic result is a consequence of Corollary 4. □

For example, assume an initial list of $1, 2, 3, 4$. Given that the system is still incomplete after 128 coups, the Labouchere bettor's conditional expected deficit is 461,933.96 units if $p = 1/2$. It is *at least* 142,204.88 units if $p = 18/37$. These figures were calculated from (58) using Theorem 2. If we apply the asymptotic formula (59), the corresponding figures are 360,566.66 and 108,272.40, respectively. The reason for the larger numbers in the fair case than in the subfair one is that we are conditioning on a less likely event.

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