CHAPTER 8

Appendix : Elements of Calculus

1. What should not be ignored on limits in $\mathbb{R}$

Calculus is fundamental to Probability Theory and Statistics. Especially, the notions on limits in $\mathbb{R}$ are extremely important. The current section allows the reader to revise these notions and to complete his knowledge on this subject through exercises whose solutions are given in detail s.

Definition: $\ell \in \overline{\mathbb{R}}$ is an accumulation point of a sequence $(x_n)_{n\geq 0}$ of real numbers finite or infinite, in $\overline{\mathbb{R}}$, if and only if there exists a sub sequence $(x_{n(k)})_{k\geq 0}$ of $(x_n)_{n\geq 0}$ such that $x_{n(k)}$ converges to ℓ, as $k \to +\infty$.

Exercise 1: Set $y_n = \inf_{p\geq n} x_p$ and $z_n = \sup_{p\geq n} x_p$ for all $n \geq 0$. Show that :

(1) $\forall n \geq 0, y_n \leq x_n \leq z_n$

(2) Justify the existence of the limit of y_n called limit inferior of the sequence $(x_n)_{n\geq 0}$, denoted by $\liminf x_n$ or $\underline{\lim}\ x_n$, and that it is equal to the following

$$\underline{\lim}\ x_n = \liminf x_n = \sup_{n\geq 0} \inf_{p\geq n} x_p.$$

(3) Justify the existence of the limit of z_n called limit superior of the sequence $(x_n)_{n\geq 0}$ denoted by $\limsup x_n$ or $\overline{\lim}\ x_n$, and that it is equal

$$\overline{\lim}\ x_n = \limsup x_n = \inf_{n\geq 0} \sup_{p\geq n} x_p.$$

(4) Establish that

$$-\liminf x_n = \limsup(-x_n) \quad \text{and} \quad -\limsup x_n = \liminf(-x_n).$$