Chapter VIII
Morasses and the Cardinal Transfer Theorem

By now it should be quite clear how it is that $V = L$ is of use when it is necessary to carry out intricate constructions by recursion. The uniform structure of L, and in particular the Condensation Lemma, enables us to take care of many "future" possibilities in a relatively small number of steps. For instance, in the construction of a Souslin tree, we need to take care of a collection of ω_2 potential uncountable antichains in ω_1 steps. As we shall see in this chapter, the uniformity of L enables us to do much more than this. In certain circumstances it is possible to construct structures of cardinality ω_2 in ω_1 steps. The idea is to use a sort of two-staged condensation principle, simultaneously approximating the final structure of size ω_2 by means of structures of size ω_1, and approximating each of these approximations by countable structures. In order to make this work, what is necessary is to investigate the way in which the two parts of such an approximation procedure must (and can) fit together. The essential combinatorial structure of L which is required is called a "morass". There is no need to stop there. We can go on to develop three-stage "morasses" which enable us to get up to ω_3 using only countable structures, and so on. In fact the subject of morasses is a vast area on its own, and would require an entire book of its own for a complete coverage. What we shall do in this chapter is look at the very simplest kind of morass, the one that gets us up to ω_2, in full detail, and then give little more than a glance at what comes after. In order both to motivate and illustrate the definition and use of a morass we take the problem which itself led to the development of morass theory, the Cardinal Transfer Problem of Model Theory.

1. Cardinal Transfer Theorems

Cardinals Transfer Theorems are generalised Löwenheim-Skolem Theorems. In its simplest form, the Löwenheim-Skolem Theorem says that if $\mathscr{A}$ is a model of a countable, first-order language, K, then there is a countable K-structure $\mathscr{B}$ such that $\mathscr{B} \equiv \mathscr{A}$. (More generally, given any infinite cardinal κ, there is a K-structure $\mathscr{B}$ of cardinality κ such that $\mathscr{B} \equiv \mathscr{A}$. Here and throughout it will be assumed that all structures considered are infinite, and that all first-order theories involved admit infinite models. This will exclude trivial special cases.) Now suppose that