Chapter 5
Proof of the Second Main Theorem

Introduction

Here, as promised, we give the proof of the Second Main Theorem. (Cf. Chapter 4. The theorem is also restated at the end of this introduction.) For purposes of discussion, we recall two of the consequences of that theorem: The eigenvalues of an effectively determined self adjoint operator are computable, but the sequence of eigenvalues need not be.

How do we prove this? As might be expected, the proof is based on the spectral theorem. However, it does *not* involve an effectivization of that theorem. Nor does it involve an effectivization of some weaker version of that theorem. Rather we use certain consequences of the spectral theorem to develop an effective algorithm. This algorithm, in fact, embodies a viewpoint directly opposed to that of the spectral theorem—at least in its most standard form.

The standard form of the spectral theorem gives a decomposition of the Hilbert space H into mutually orthogonal subspaces $H_{(a_{i-1}, a_i]}$ corresponding to an arbitrary partition of the real line into intervals $(a_{i-1}, a_i]$. On these subspaces $H_{(a_{i-1}, a_i]}$ the operator T is "approximately well behaved". More precisely, (i) these subspaces are invariant under T—i.e. if x lies in the subspace, so does Tx, and (ii) the vectors x in the subspace are "approximate eigenvectors"—i.e. if $(a_{i-1}, a_i] \subseteq [\lambda - \varepsilon, \lambda + \varepsilon]$ then $\|Tx - \lambda x\| \leqslant \varepsilon \|x\|$.

It turns out that effective computations involving the spectral measure require the uniform norm, i.e. computability in the sense of Chapter 0. Thus the above decomposition—involving disjoint intervals—cannot be made effective. What we have is a classical analytic fact, the existence of such a decomposition, from which we must attempt to derive effective consequences. To do this we alter the standard spectral-theoretic decomposition in two ways.

First, we replace the disjoint intervals by intervals which overlap, after the manner of ... $[-2, 0]$, $[-1, 1]$, $[0, 2]$, $[1, 3]$, ... Second, we replace the characteristic functions of these intervals by "triangle functions" supported on them (cf. Pre-step B in Section 2). The overlapping intervals are necessary to account for the fact that a computable real number cannot be known exactly, and the triangle functions, being continuous, allow effective computations to be made. It is the necessity of using overlapping intervals and continuous functions which is at variance with the