Chapter 1
Further Topics in Computable Analysis

Introduction

This chapter starts where Chapter 0 leaves off, but involves a transition to more difficult topics. It begins with differentiation and the classes C^n, $1 \leqslant n \leqslant \infty$, and progresses through the computable theory of analytic functions. It concludes with a general theorem on "translation invariant operators" which subsumes several of our previous results. We continue to use the "Chapter 0" notion of computability for continuous functions.

Naively, one might suppose that since the derivative, the operations of complex analysis, etc. are given by formulas, they should be computable. Of course, as we know, this is not necessarily the case. In fact, it is not always easy to guess what the results should be. Consider, for example, differentiation, the topic of Section 1. If a computable function possesses a continuous derivative, is the derivative necessarily computable? The answer turns out to be no. This raises the question: are there regularity conditions which we can impose on the initial computable function (e.g. being C^n for some fixed n, or C^∞) which insure the computability of the derivative. Here the answer is yes. The problem is to find the right conditions.

Theorem 1, essentially due to Myhill [1971], asserts that there exists a computable C^1 function whose derivative is not computable. Theorem 2, due to the authors [1983a], asserts that, on the other hand, if the initial function is C^2, then the derivative is computable. Thus the cutoff point must lie somewhere between C^1 and C^2. Where is it located? We give a slight strengthening of Myhill's example, which shows that the function f can be twice differentiable (but not continuously so) and still give a noncomputable f'. Hence the cutoff is pinned between "twice differentiable" (Theorem 1) and "twice continuously differentiable" (Theorem 2).

A curious result which emerges from Theorems 1 and 2 is the following: There exists a function on $[0, 1]$ which is effectively continuous at each point of $[0, 1]$ but not effectively uniformly continuous on $[0, 1]$ (Corollary 2b). Thus the classical theorem that a continuous function on a compact set is uniformly continuous does not effectivize.

An immediate consequence of Theorem 2 (Corollary 2a) asserts that if a computable function f is C^∞, then each derivative $f^{(n)}$ is computable. This raises the question: Is the sequence of derivatives $\{f^{(n)}\}$ computable, effectively in n? The answer is no, as we show in Theorem 3 (cf. Pour-El, Richards [1983a]).