Chapter XII
Selection and k-Sections

Selection theorems, unlike so many results in recursion theory, have the virtue of being positive. They reveal uniformities not immediately apparent to the most discerning recursion theorist. They are often based on farfetched computations that converge slowly. A typical selection theorem addresses the following question. Suppose A is E-recursively enumerable in b, and $A \cap x$ is nonempty. Is there a uniform method for computing an element of $A \cap x$ from b, x? More generally, since x may not be effectively wellorderable, is there a uniform procedure for computing a nonempty $y \subseteq A \cap x$ from b, x? As a rule y is obtained by computing an ordinal θ such that the set of all computations from b of length at most θ suffices to enumerate some elements of $A \cap x$.

Let Selection (x) denote the following principle: there exists a partial E-recursive function f such that for all $e \leq \omega$ and all b,

$$(1) \qquad (\mathrm{E}z)_{z \in x}[\{e\}(z, b)\downarrow] \leftrightarrow f(e, b)\downarrow \quad \& \quad (\mathrm{E}z)_{z \in x}[|\{e\}(z, b)| \leq f(e, b)].$$

Gandy selection, Theorem 4.1.X, is equivalent to Selection (ω).

Selection (2^ω) and Selection (ω_1) are false. If the first were true, then $E(2^\omega)$ would be Σ_1 admissible, but the existence of Moschovakis witnesses in $E(2^\omega)$ gives rise to a $\Sigma_1^{E(2^\omega)}$ map from $\omega \times 2^\omega$ onto $E(2^\omega)$. Connections between selection and admissibility are made in Section 2 below.

Restricting the enumeration parameter b of (1) can make a difference, and even more so the introduction of a *special* parameter p into f. For example, Grilliot selection, established in Section 1, implies (1) when

$$x = 2^\omega, \quad b \in 2^{2^\omega} \quad \text{and} \quad p = 2^{2^\omega}.$$

(The $f(e, b)$ of (1) is replaced by $f(e, b, p)$.)

To sum up prematurely, a selection theorem involves a fixed set x and a collection C of procedures for enumerating elements of x. C is defined in terms of Gödel numbers e, enumeration parameters b, and additional predicates R. A typical member of C is $\{e\}^R(z, b)$, where z ranges over x. The selection theorem asserts the existence of a uniform method of computing a nonempty $y \subseteq x$ such that $\{c\}^R(z, b)\downarrow$ for all $z \in y$. y is completed from x, b, p; R, where p is a special parameter independent of b and R.