Chapter IX
Splitting, Density and Beyond

This last chapter on α-recursion theory focuses on priority arguments more difficult than those of Chapters VII and VIII. Shore's splitting theorem relies heavily on his method of Σ_2 blocking. His density theorem, the first instance of α-infinite injury, requires further fine structure results, and consequently its proof is not entirely dynamic. Its nondynamic features support consideration of recursion theory on inadmissible structures, the concluding topic of the chapter.

1. Shore's Splitting Theorem

Let A be a regular α-recursively enumerable set, not α-recursive. The object is to split A into two sets, B_0 and B_1, so that each is of lower α-degree than A. Thus

$$A = B_0 \cup B_1,\ B_0 \cap B_1 = \varnothing, \quad \text{and} \quad A \nleq_\alpha B_i (i < 2).$$

Superficially the strategy is the same as that followed by Sacks 1963b when $\alpha = \omega$. In the ω-case the splitting theorem makes stronger use of Σ_2 replacement than the Friedman–Muchnik theorem does. In general terms the former is a full-blown Σ_2 recursion while the latter is tame in the sense of Theorem 4.4.VIII. In specific terms the difference arises from the urgency of splitting. At stage σ some x is enumerated in A. That x must be put in either B_0 or B_1 immediately. The force of the positive requirements is so great that numerous negative requirements are unavoidably injured.

The negative requirements are indexed by ordinals less than α^*:

$$\text{Req } 2\varepsilon: \quad A \neq \{\varepsilon\}^{B_0},$$

$$\text{Req } 2\varepsilon + 1: \quad B \neq \{\varepsilon\}^{B_1}.$$

$\{\varepsilon\}$ means $\{f^{-1}\varepsilon\}$ for some one-one α-recursive f from α into α^*.
Req u has higher priority than req v if $u < v$. Thus req 0 is never injured. As in the ω-case, if A and $\{\varepsilon\}^{B_i}$ agree on an initial segment of α at stage σ, then $\{\varepsilon\}^{B_i}$ is commmited to preservation on that initial segment, if the priorities allow it. The preservations associated with req 0 must be bounded in both time and space, since otherwise A would be α-recursive. To compute $A(z)$, unfold the construction until