Chapter VIII
Priority Arguments

In this chapter some standard results of classical recursion theory are lifted to every Σ_1 admissible ordinal α by techniques hinted at in earlier chapters. Since the classical proofs use the Σ_2 admissibility of $L(\omega)$, the proofs to come may be regarded as instances of the austere art of making Σ_1 admissibility do the work of Σ_2. The initial technique relies strongly on the L-ness of L, the ability of L to support downward Skolem–Löwenheim arguments. The later technique depends on combinatoric consequences of Σ_1 admissibility. Its dynamic nature makes it applicable to Σ_1 admissible structures for which hull-collapsing arguments fail.

*1. α-Finite Injury via α**

In this section and the next it will be shown that there exist α-recursively enumerable sets A and B such that neither is α-recursive in the other. The method extends that applied in Section 5.VII, to construct a hyperregular, non-α-recursive, α-recursively enumerable set. The injury sets become more complex. Back in Chapter VII each negative requirement was injured at most once for the sake of each positive requirement of higher priority. Such simplicity is rare in the present chapter. Consequently the α-recursive projection of α into α^*, which arranges that each negative requirement be opposed by less than-α^* positive requirements of higher priority, does not always compel the injury sets to be α-finite. In Section 2 below, when $\alpha^* = \alpha$ and there is a greatest α-cardinal, it will be necessary to project α downward by means of a carefully chosen Σ_2^α function.

1.1 Strategy. Define $\{p^{-1}\varepsilon\}^B$ for each $\varepsilon < \alpha^*$ as in the beginning of the proof of Theorem 5.5.VII. Thus $A \leq_{w\alpha} B$ iff $A = \{p^{-1}\varepsilon\}^B$ for some $\varepsilon < \alpha^*$. The requirements on A and B are as follows.

Requirement 2ε: If $\{p^{-1}\varepsilon\}^B$ is a total function, then $A \neq \{p^{-1}\varepsilon\}^B$.

Requirement $2\varepsilon + 1$: Same as req 2ε with A and B interchanged.

Let $\{Z_\varepsilon \mid \varepsilon < \alpha^*\}$ be a collection of simultaneously α-recursive, pairwise disjoint, unbounded subsets of α. The strategy for satisfying req 2ε consists of finding an $x \in Z_\varepsilon$ such that:

$$\text{(1)} \qquad \text{if } \{p^{-1}\dot{\varepsilon}\}^B(x) = 0, \text{ then } x \in A.$$