Chapter IV
Measure and Forcing

The measure-theoretic analysis of Π_1^1 sets begun in Section 6.II is continued. It is shown that every Π_1^1 set of positive measure has a hyperarithmetic member. Forcing over the hyperarithmetic hierarchy is developed in order to construct a minimal hyperdegree and to prove Louveau's separation theorem.

1. Measure-Theoretic Uniformity

The goal of this section is to show $\mathcal{M}(\omega_1^{\mathrm{CK}}, T)$ is a model of Δ_1^1 comprehension for almost all T. It follows that $\omega_1^{\mathrm{CK}} = \omega_1^T$ for almost all T. Recall the notions of ordinal rank (subsection 4.1.III) and full ordinal rank (subsection 4.4.III) for formulas of $\mathcal{L}(\omega_1^{\mathrm{CK}}, \mathcal{T})$. Define the number quantifier rank of a formula $\mathcal{F}$ to be the number of occurrences of $(\mathrm{E}x), (y), \ldots$ in $\mathcal{F}$.

Let $\mathcal{F}$ be a ranked sentence of $\mathcal{L}(\omega_1^{\mathrm{CK}}, T)$. By Lemma 4.6.II the set

$$\{T \mid \mathcal{M}(\omega_1^{\mathrm{CK}}, T) \models \mathcal{F}\}$$

is Δ_1^1, hence Borel and measurable according to subsection 6.1.II. Denote its measure by $p(\mathcal{F})$, the probability that $\mathcal{F}$ is true. The next task is to show the graph of $p(\mathcal{F})$, as $\mathcal{F}$ ranges over ranked sentences, is Π_1^1.

1.1 Proposition. *Let $\mathcal{F}(X^\alpha)$ be a ranked formula whose only free variable is X^α. Let $\mathcal{G}_i(x)$ $(i \le n)$ be formulas of rank at most α whose only free variable is x. Then*

$$\bigvee_{i \le n} \mathcal{F}(\hat{x}\mathcal{G}_i(x))$$

is logically equivalent to a sentence of full ordinal rank less than that of $(\mathrm{E}X^\alpha)\mathcal{F}(X^\alpha)$.

Proof. Put $\bigvee_{i \le n} \mathcal{F}(\hat{x}\mathcal{G}_i(x))$ into prenex normal form without changing the pattern of set quantifiers. Then contract like quantifiers as in the proof of Theorem1.5 of Chapter I. The result will have at least one less occurrence of $(\mathrm{E}X^\alpha)$ than does $(\mathrm{E}X^\alpha)\mathcal{F}(X^\alpha)$. $\square$