Chapter II
The Hyperarithmetic Hierarchy

The hyperarithmetic sets are defined by iterating the Turing jump through the recursive ordinals, and are shown to equal the Δ_1^1 sets. The equality is important for two reasons. First, it reveals that Δ_1^1 is more constructive than it appears to be. Second, it allows properties of Δ_1^1 sets to be proved by induction, since hyperarithmetic sets fall into a hierarchy and can be assigned ordinal ranks less than ω_1^{CK}.

Hyperarithmetic reducibility, hyperdegrees and the hyperjump are defined.

1. Hyperarithmetic Implies Δ_1^1

The H-sets are defined after some properties of the Turing jump are reviewed. A set is defined to be hyperarithmetic if it is recursive in some H-set. Then an effective transfinite recursion produces an effective method for passing from the index of an H-set X to a Δ_1^1 index for X.

1.1 ***H*-Sets.** Let c_X be the characteristic function of the set X. Y is said to be Turing reducible to (or recursive in) Z if

$$(1) \qquad (\mathrm{E}e)[c_Y=\{e\}^{c_Z}].$$

$\{e\}$ is sketched in Chapter I, subsection 1.1. Formula (1) is often rendered as $X \leq_T Y$.

The Turing jump of X is denoted by X'_1 and is defined by

$$\{e \mid \{(e)_0\}^X((e)_1) \text{ is defined}\}.$$

X' can be regarded as the effective disjoint union of all sets recursively enumerable in X.

The following elementary facts about Turing reducibility and jump are proved in Rogers 1967.

(2) X' is not recursive in X.