Chapter IV

Models of Fragments of Arithmetic

Introduction. The present chapter is devoted to the study of models of arithmetic, i.e. structures M for the language L of arithmetic such that $M \models T$, where T is either PA or a fragment of PA. Models are useful as means for showing unprovability of a formula φ in T (by exhibiting a model M for $T + \neg\varphi$) as well as for showing provability (by proving that $(T + \neg\varphi)$ has no model). In particular, we shall prove some conservation results by model-theoretical methods.

Model theory of arithmetic is a rather broad field and we shall not try to be exhaustive; our aim will be to present selected typical results and techniques. Section 1 (Some basic constructions) makes the reader familiar with non-standard models and some techniques of their construction; the main results are hierarchy results (concerning theories $I\Sigma_n$, $B\Sigma_n$, $P\Sigma_n$), theorems concerning existence of elementary extensions and the Paris-Friedman conservation result. Section 2 (Cuts in models of arithmetic with top) introduces and studies cuts (roughly: initial segments of models having no greatest element) and various kinds of cuts: in particular, k-extendable cuts and their relationship to models of $B\Sigma_{k+1}$. Furthermore, the section contains a proof of the theorem saying that each non-standard model of $I\Sigma_1$ is isomorphic to a proper cut of itself. Section 3 (Provably recursive functions and the method of indicators) presents two characterizations of a $I\Sigma_k$-provably recursive function; a corollary is the fact that $I\Sigma_1$-provably recursive functions coincide with primitive recursive functions. We rely heavily on the method of indicators, a formalization of ordinals in $I\Sigma_1$, k-extendable (or k-restrainable) cuts and the notion of α-large sets. Finally, there is a short Sect. 4, dealing with a formalization of some parts of model theory of fragments; we formalize in $I\Sigma_1$. In particular, we give in $I\Sigma_1$ a model-theoretical proof of Paris-Friedman's conservation theorem and get various corollaries, among them the fact that $I\Sigma_{k+1}$ proves the consistency of $B\Sigma^{\bullet}_{k+1}$.

Our main omission is the absence of the theory of recursively saturated models. We apologize for this and refer to bibliographical remarks for references. But we hope that the present chapter will give the reader satisfactory insight into the world of models of arithmetic and their possible uses.