Chapter II

Fragments and Combinatorics

Introduction. In the present chapter we shall elaborate proofs of various combinatorial principles in suitable fragments of arithmetic. In general, infinite principles deal with graphs, functions etc. on infinite sets, finite principles relate similarly to finite sets. We prove both some infinite and some finite principles; furthermore, we show some infinite principles to be equivalent to certain collection principles and some finite principles to be equivalent to certain consistency statements. Sections 1 and 2 deal with strengthenings of the infinite and finite Ramsey theorem (they will be formulated at the beginning of Sect. 1), in particular with various forms and instances of Paris-Harrington principle. This principle is very famous since it has been the first example of an arithmetical statement that has a clear combinatorial meaning, is true (in N) and is unprovable in PA.

Instances of Paris-Harrington principle will form a hierarchy of formulas, n-th of them will be proved in $I\Sigma_n$ ($n \geq 1$). As said above, in this chapter we deal with concrete proofs, not with unprovability; but unprovability results immediately follow from the results of this chapter using Gödel's incompleteness theorems (elaborated in Chap. III). We shall mention this on corresponding places in this chapter: $(n+1)$-th instance is unprovable in $I\Sigma_n$.

In Sect. 3 we shall deal with ordinals in fragments, introduce the notion of α-large sets (α an ordinal) and investigate another hierarchy of combinatorial statements, related to the first one. Results of this section will be used in Chap. IV for a characterization of functions provably recursive in $I\Sigma_n$ ($n \geq 1$).

1. Ramsey's Theorems and Fragments

(a) Statement of Results

1.1 First we shall recall Ramsey's theorems in an informal formulation. If X is a set of natural numbers, then $[X]^u$ is the set of all u-element subsets