Preliminaries

In this preliminary section, we first survey some basic facts from logic (and recursion theory) that are assumed to be known to the reader. Furthermore, we shall introduce the language of first order arithmetic and investigate first order definable sets of natural numbers. Finally, we shall present the beginnings of arithmetization of metamathematics by showing (or announcing) that various syntactic and some semantic logical notions can be understood as first order definable sets of natural numbers. To show that metamathematically interesting sets (like the set of all formulas, proofs, etc.) are (or can be understood as) first order definable sets of natural numbers is only the first step; the second step, more important and postponed until Chap. I, consists in investigating which first-order properties of these sets are provable in various systems of first order arithmetic. The fact that arithmetic can express its own syntax and partially its own semantics is of basic importance for the investigation of its metamathematics.

(a) Some Logic

0.1. Throughout the book, N is the set of all natural numbers (including zero). We shall denote natural numbers mainly by letters m, n, k, l, possibly indexed. The *least number principle* assures that each non-empty set of natural numbers has a least element. The *induction principle* says that if X is a set such that $0 \in X$ and X contains with each natural number n also its successor $n+1$, then $N \subseteq X$.

0.2. Our survey of logic will have a double purpose: on the one hand, we shall investigate axiomatic systems of arithmetic as first-order theories and therefore first order logic will be our main device, and, on the other hand, we shall develop our axiomatic systems as meaningful mathematical theories and shall, among other things, formalize parts of first order logic in these systems. The fact that reasonable parts of logic can be developed in first-order arithmetic is of basic importance, as we shall see in the future.