6. Superstable Theories

In Section 5.6 we defined the notion of a basis of a type p (relative to a set) and posed several questions on the behavior of the corresponding dimension function (including its well-definedness). We proved in Section 5.6.3 that on the class of weight 1 types dimension is well-defined, and nonorthogonality is the same as domination equivalence. In Section 5.6.4 we showed that in a superstable theory every type is domination equivalent to a finite product of weight 1 types. For a full-featured dimension theory, though, we need an additional property (additivity) which may fail for a weight 1 type (see Remark 5.6.7). In the second section of this chapter we develop the theory of a class of weight 1 types called the regular types in an arbitrary superstable theory. A regular type in a superstable theory satisfies the additivity property missing for weight 1 types (Proposition 6.3.2) and every weight 1 type is domination equivalent to a regular type.

This well-behaved dimension theory is at the heart of the solution of such problems as Morley's Conjecture for countable first-order theories (mentioned in the Preface). Regular types will be used to characterize the models of a "bounded" t.t. theory in Section 7.1.1.

Before turning to regular types we develop two notions of rank which are used in virtually every study of superstable theories.

6.1 More Ranks

Many of the properties proved for t.t. theories relied heavily on the existence of Morley rank; i.e., the fact that every type has Morley rank $< \infty$. The family of $\Delta-$ranks served to define the forking dependence relation but, because it is a family of ranks instead of a single rank, it is missing many of properties of an ordinal-valued rank. Here we define two ranks which exist in superstable theories and provide a sharper measure of the complexity of formulas and types with respect to the forking relation.

Throughout the section we assume any mentioned theory to be stable.

Definition 6.1.1. *(i) In a stable theory we define the rank $U(-)$ on complete types by the following recursion. For p a complete type and α an ordinal,*