4. Fine Structure of Uncountably Categorical Theories

In the preceding chapter, for T a countable uncountably categorical theory, we solved problems concerning the number of models of T in a fixed cardinality. However, this study leaves many unanswered questions about uncountably categorical theories, and raises others. Here are a few such questions.

- In [Vau61] Vaught asked if an uncountably categorical theory can be finitely axiomatizable. (It was through Zil'ber's work on this problem that geometrical stability theory, the area in which the subject matter of this chapter belongs, was born.)
- Can we isolate a broad class of uncountably categorical theories which have a strongly minimal formula (or at least a formula of Morley rank 1) over $\emptyset$? (While working on the Baldwin-Lachlan Theorem we recognized that an easier proof would be possible in such theories.)
- Are there strongly minimal sets which are radically different from the examples given in Example 3.1.1?

What is surprising is that work on each of these questions has given insight into the others. The issues underlying this connection are the following imprecisely worded problems concerning the definable relations in models of uncountably categorical theories. Recall that algebraic closure restricted to the subsets of strongly minimal set defines a pregeometry.

(1) Find a natural and meaningful dividing line between "simple" pregeometries and "complex" pregeometries among those which occur as the pregeometry on a strongly minimal set.
(2) Prove that whenever the pregeometry on a strongly minimal set is simple, the Morley rank dependence relation on tuples is also simple in a meaningful way.

In order to formulate the properties which will meet these requirements we need the notion of M^{eq} (M a model), which is developed in the next section. In the expansion M^{eq} we have not only the elements of M but elements which act as names for the definable relations in M. This expansion is used in most of model theory today.