2. Constructing Models with Special Properties

Much of the richness of model theory is attributable to the weakness of first-order logic. In the last chapter it was proved that a first-order theory cannot express, for example, that every model has cardinality $\aleph_0$. In fact, by the Löwenheim-Skolem Theorems an elementary class (in a countable language) containing an infinite model contains models in every infinite cardinality. Thus, a basic property expressible in first-order logic and true in an infinite model is true in a great variety of models. Besides the Löwenheim-Skolem Theorems, the Omitting Types Theorem is an example of such a result. Given a nonisolated type p in a countable theory there is a countable model $\mathcal{M}$ realizing p and a countable model $\mathcal{N}$ omitting p. The theme of this chapter is to continue this program of constructing models of a given theory with widely varying properties. In the first two sections we define several kinds of models (of complete theories having infinite models) distinguished by the elementary embeddings they admit and the types they realize or omit.

2.1 Prime and Atomic Models

The first special kind of model to be considered is one which is, intuitively, the "smallest" model of the theory.

Definition 2.1.1. *Given a theory T, we call $\mathcal{M} \models T$ a* prime *model of T if, for any $\mathcal{N} \models T$, $\mathcal{M}$ can be elementarily embedded into $\mathcal{N}$.*

For example, if T is the theory of algebraically closed fields of characteristic 0, the algebraic closure of the rationals, $\tilde{\mathbb{Q}}$, forms a prime model of T. (Since T has quantifier elimination, any model of T contains a copy of $\tilde{\mathbb{Q}}$ as an elementary submodel.) While the definition of a prime model makes sense for any theory only a complete theory can have a prime model (see the exercises). We will see, moreover, that the uniqueness of prime models and a useful condition sufficient for their existence can only be proved for countable complete theories. As a first observation, if $\mathcal{M}$ is a prime model of the countable theory T and $p \in S_n(\emptyset)$ is realized in $\mathcal{M}$ then p must be isolated. (Suppose to the contrary that p is nonisolated and realized in $\mathcal{M}$ by the n-tuple $\bar{a}$. By the Omitting Types Theorem, T has a countable model