Appendix. On Weak Diamonds and the Power of Ext

§0. Introduction

In [DvSh:65] K. Devlin and S. Shelah introduced a combinatorial principle Φ which they called the weak diamond. It explains some of the restrictions in theorems of the form "the limit of iteration does not add reals". See more on this in [Sh:186] and Mekler and Shelah [MkSh:274] (on consistency of uniformization properties) [Sh:208] (consistency of "ZFC+$2^{\aleph_0} < 2^{\aleph_1} < 2^{\aleph_2} + \neg\Phi_{\{\delta<\aleph_2:\mathrm{cf}(\delta)=\aleph_1\}}$") and very lately [Sh:587].

Explanation. Jensen's diamond for $\aleph_1$, denoted $\diamondsuit_{\aleph_1}$, see [Jn], can be formulated as: There exists a sequence of functions $\{g_\alpha : g_\alpha$ a function from α to α where $\alpha < \omega_1\}$ such that for every $f : \omega_1 \to \omega_1$ we have $\{\alpha < \omega_1 : f{\restriction}\alpha = g_\alpha\} \not\equiv 0 \bmod \mathcal{D}_{\aleph_1}$ (recall that $\mathcal{D}_{\aleph_1}$ is the filter on λ generated by the family of closed unbounded subsets of λ). Clearly $\diamondsuit_{\aleph_1} \to 2^{\aleph_0} = \aleph_1$. Jensen (see [DeJo]) also proved that $2^{\aleph_0} = \aleph_1 \not\Rightarrow \diamondsuit_{\aleph_1}$ (see Chapters V and VII remembering that $\diamondsuit_{\aleph_1}$ implies existence of an Aronszajn tree which is not special (even a Souslin tree)). You may ask, is there a diamond like principle which follows from $2^{\aleph_0} = \aleph_1$?

K. Devlin and S. Shelah [DvSh:65] answered this question positively, formulating a principle Φ which says:

$(*)_1$ $(\forall F : {}^{\omega_1>}2 \to 2)(\exists h : \omega_1 \to 2)(\forall \eta : \omega_1 \to 2)$

$$\{\alpha < \omega_1 : F(\eta{\restriction}\alpha) = h(\alpha)\} \not\equiv 0 \bmod \mathcal{D}_{\aleph_1}.$$