X. On Semi-Proper Forcing

§0. Introduction

We weaken the notion of proper to semiproper, so that some important properties (the most important is not collapsing $\aleph_1$, being preserved by some iterations) still hold for this weaker notion. But the class of semiproper forcing will also include some forcings which change the cofinality of a regular cardinal $> \aleph_1$ to $\aleph_0$. We will also describe how to iterate such forcings preserving semiproperness. So, using the right iterations, we can iterate such forcings without collapsing $\aleph_1$. As a result, we solve the following problems of Friedman, Magidor and Abraham respectively, by proving (modulo suitable large cardinals) the consistency of the following with G.C.H.:

(1) for every $S \subseteq \aleph_2, S$ or $\aleph_2 \setminus S$ contains a closed copy of ω_1,

(2) there is a normal precipitous filter D on $\aleph_2$, $\{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_0\} \in D$,

(3) for every $A \subseteq \aleph_2$, $\{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_0, \delta$ is regular in $L[\delta \cap A]\}$ is stationary.

However, the countable support iteration does not work, so we introduce the revised countable support. Though it is harder to define, it satisfies more of the properties we intuitively assume iterations satisfy and is applicable for the purpose of this chapter.