VIII. κ-pic and Not Adding Reals

§0. Introduction

In the first section we show that we can iterate $\aleph_2$-complete forcing, and $\aleph_1$-complete forcing which satisfy the $\aleph_2$-c.c. in a strong sense.

In the second section we deal with a strong version of the $\aleph_2$-c.c. called $\aleph_2$-pic. It is useful for proving that for CS iteration of length ω_2 of proper forcing notions, the limit still satisfies the $\aleph_2$-c.c. This in turn will be used in order to get universes with $2^{\aleph_1} > 2^{\aleph_0} = \aleph_2$.

In the third section we deal again with the axioms; starting with a model of ZFC (not assuming the existence of large cardinals) we phrase the axioms we can get. There are four cases according to whether $2^{\aleph_0}$ is $\aleph_1$ or $\aleph_2$, and $2^{\aleph_1}$ is $\aleph_2$ or larger [our knowledge on the case $2^{\aleph_0} \geq \aleph_3$ is slim].

In the fourth section we return to the problem of when a CS iteration of proper forcing preserves "not adding reals". We weaken "each $\underset{\sim}{Q}_i$ (a P_i-name) is $\mathbb{D}$-complete for some $\mathbb{D}$ a $(\lambda, 1, \kappa)$-system", by replacing "each $\mathbb{D}_x$ is an $\aleph_1$-complete filter" or even just "each $\mathbb{D}_x$ is a filter" by "each $\mathbb{D}_x$ is a family of sets, the intersection of e.g. any two is nonempty". So we can deduce ZFC+CH $\nvdash \Phi^3_{\aleph_1}$. We also try to formulate the property preserved by iteration weaker than this completeness. See references in the relevant sections.