VII. Axioms and Their Application

§0. Introduction

In the first section we introduce the κ-e.c.c. (κ-extra chain condition). We prove that if we have an iteration of length $\leq \kappa$ of $(< \omega_1)$-proper forcing notions which do not add reals, and if, moreover each forcing used is $\mathbb{D}$-complete for some simple $\aleph_1$-completeness system $\mathbb{D}$, *then* the limit satisfies the κ-c.c. . This helps us e.g. in iterations of length ω_2 of forcings among which none add reals, but each adds many subsets of $\aleph_1$.

In the second section we deal with forcing axioms; essentially our knowledge is good when we want $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ and reasonable when we want $2^{\aleph_0} = \aleph_1$ and even $2^{\aleph_0} = \aleph_2$. In the third section we discuss applications of the forcing axiom which is consistent with CH as just mentioned. In the fourth section we discuss the forcing axiom which is consistent with $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$, and in the fifth section we give an example of a CS iteration collapsing $\aleph_1$ only in the limit. See relevant references in the sections.

§1. On the κ-Chain Condition, When Reals Are Not Added

When we prove various consistency results by iterating proper forcings, we often have to check that the $\aleph_2$-chain condition holds.