IV. On Oracle-c.c., the Lifting Problem of the Measure Algebra, and "$\mathcal{P}(\omega)$/finite Has No Non-trivial Automorphism"

§0. Introduction

We present here the oracle chain condition and two applications: the lifting problem for the measure algebra, and the automorphism group of $\mathcal{P}(\omega)$/finite.

Let $\mathcal{B}$ be the family of the Borel subsets of $(0, 1)$ (i.e. sets of reals which are > 0 but < 1). Let I_{mz} be the family of $A \in \mathcal{B}$ which have Lebesgue measure zero. Clearly I_{mz} is an ideal. The lifting problem is: "Can the natural homomorphism from $\mathcal{B}$ to $\mathcal{B}/I_{mz}$ be lifted ($\equiv$ split), i.e. does it have a right inverse? Equivalently, define on $\mathcal{B}$ an equivalence relation: $A_1, A_2 \in \mathcal{B}$ are equivalent if $(A_1 \setminus A_2) \cup (A_2 \setminus A_1)$ has Lebesgue measure zero: is there a set of representatives which forms a Boolean algebra? If CH holds the answer is positive (see Oxtoby [Ox]). (This holds for any $\aleph_1$-complete ideal). We will show, in §4, that a negative answer is also consistent with ZFC.

Since the problem of splitting the measure algebra is simpler, we will consider it first, but in the introduction we use the second problem to describe the main idea of our technique.

It is well known that if CH holds then $\mathcal{P}(\omega)$/finite is a saturated (model-theoretically) atomless Boolean algebra of power $\aleph_1$, hence has $2^{2^{\aleph_0}}$-many automorphisms, as any isomorphism from one countable subalgebra to another