II. Iteration of Forcing

§0. Introduction

Suppose $V_{\ell+1}$ is a generic extension of V_ℓ, for $\ell = 0, 1$. Is V_2 a generic extension of V_0? In §1 we present the possible answer, in fact if $V_{\ell+1} = V_\ell[G_\ell]$, G_0 is a subset of P generic over V_0, G_1 is a subset of $\underset{\sim}{Q}[G_0]$ generic over V_1, we can get V_2 by some subset G of $P * \underset{\sim}{Q}$ generic over V_0, and there are natural mappings between the family of possible pairs $[G_0, G_1]$ and the family of possible G's. In §2 we deal with iterations $\langle P_\ell, \underset{\sim}{Q}_\ell : \ell < \alpha \rangle$ of length an ordinal α.

This seems suitable to deal with proving the consistency of "for every x there is y such that ..." each Q_α producing a y_α for some $x_\alpha \in V^{P_\alpha}$. However V^{P_α} is not $\bigcup_{i<\alpha} V^{P_i}$, still if we speak of, say, $x \in H(\lambda)$ and $\mathrm{cf}(\alpha) \geq \lambda$ and $P_\alpha = \bigcup_{i<\alpha} P_i$, and P_α satisfies the c.c.c. (or less), then no "new" x appear in V^{P_α}, so we can "catch our tail."

An important point is what we do for limit ordinals δ. We choose $P_\delta = \bigcup_{i<\delta} P_i$ (direct limit), this is the meaning of FS (finite support iteration). An important property is (see 2.8): if each $\underset{\sim}{Q}_i$ satisfies the c.c.c. then so does P_δ.

In §3 we present MA (Martin's axiom) and prove its consistency. The axiom says inside the universe, for any c.c.c. forcing notion P we can find directed $G \subseteq P$ which are "quite generic", say not disjoint to $\mathcal{I}_i$ for $i < i^*$ if $\mathcal{I}_i \subseteq P$ is dense and $i^* < 2^{\aleph_0}$. The proof of its consistency (3.4) is by iterations as in §2 of c.c.c. forcing notions, the point being the right bookkeeping and the "catching of