Chapter IX
Silver Machines

1. Silver Machines

Silver machines are a device for avoiding the use of the fine structure theory in proving results such as $\Box_\kappa$. The idea is as follows. In proving, say $\Box_\kappa$, as we did in Chapter IV, the main tool was the hierarchy of skolem functions $h_{\varrho\frac{n}{a},A\frac{n}{a}}$. Of course, these functions, and the properties of them that we made use of, were obtained by our fine structure theory. But the fine structure theory itself was not used in the proof of $\Box_\kappa$. Any hierarchy of functions with similar properties would suffice. As we shall see, it is possible to construct such a functioinal hierarchy without using the fine structure theory. The idea is as follows.

We shall say that an ordinal α is *-definable from a class X of ordinals iff there is an $\mathscr{L}$-formula $\varphi(v_0, \ldots, v_n)$ and elements $\beta_1, \ldots, \beta_n, \gamma$ of X such that α is the unique ordinal for which

$$\vDash_{L_\gamma} \varphi(\mathring{\alpha}, \mathring{\beta}_1, \ldots, \mathring{\beta}_n).$$

The idea behind the machine concept is this. Suppose we were to define a *-*skolem function* for L as a function h such that $\text{dom}(h) \subseteq \omega \times \text{On}^{<\omega}$, $\text{ran}(h) \subseteq \text{On}$, and whenever α is *-definable from $X \subseteq \text{On}$, then $\alpha \in h''(\omega \times X^{<\omega})$, where we use $X^{<\omega}$ to denote $\bigcup_{n<\omega} X^n$. In order to construct, say, a $\Box$-sequence, we might then go on to define a hierarchy of (set) functions convering to h, possessing some kind of condensation property. And to a point, this is the idea behind the definition of a Silver machine. But there are some differences. For instance, we shall not work with a single skolem function h but rather an infinite family of functions h_i, $i < \omega$. Although h_i will, in some sense, correspond to the function $h(i, -)$ of the above sketch, the index i will not be the Gödel number of a formula as was the case with the skolem functions of the fine structure theory, and for different indices i the functions h_i may be quite different in structure. (Hence there is no point in trying to combine them into one function.)

One remark concerning the use of the word "machine". This stems from the motivation which led Silver to develop the concept in the first place. "Silver hierarchy" would be a more suitable term for the structure we shall develop here (which is not quite the same as the original), but we shall, of course, stick to the established usage.